On Gressman's

Affine Invariant Measures

Lecture 6

The next question that we try to answer is the second part of (1) – that is, we look at criteria that let us establish whether $\nu_{\mathcal{A}} \equiv 0$ or not, for a given submanifold Σ.

Let us use

$$L^k f(t) := \bigwedge_{\substack{\underline{\alpha} \in \mathbb{N}^d: \\ 1 \leq |\underline{\alpha}| \leq k}} \frac{1}{\underline{\alpha}!} \partial^{\underline{\alpha}} f(t)$$

to denote the wedge product of all complete blocks up to order k.
The discussion before, recall, has shown that if

$$m = \# \text{ of incomplete block (order } k_n)$$

then

$$\left(\frac{d\nu_{\mathcal{A}}}{dt}\right)^{\frac{Q}{d}} = \inf_{M \in SL(\mathbb{R}^d)} \left(\sum_{\substack{\underline{\beta}_1, \dots, \underline{\beta}_m \in \mathbb{N}^d: \\ |\underline{\beta}_1| = \dots = |\underline{\beta}_m| = k_n}} \frac{(k_n!)^m}{\underline{\beta}_1! \cdots \underline{\beta}_m!} \right.$$

$$\left. \times \Big| \underbrace{L^{k_n - 1} f(t)}_{\text{(complete blocks)}} \wedge \underbrace{(M^T \partial)^{\underline{\beta}_1} f(t) \wedge \dots \wedge (M^T \partial)^{\underline{\beta}_m} f(t)}_{\text{(incomplete block)}} \Big|^2 \right)^{1/2}.$$

Looking at the expression in the absolute value we see that we have effectively produced a "new" tensor on $T\Sigma$, which is related to $\mathcal{A}_t^f$ but has fewer variables.

More precisely, we have introduced

$$\tilde{A}^f_t \big(\underbrace{\underbrace{X_{1,1}, \dots, X_{1,k_n}}, \dots, \underbrace{X_{m,1}, \dots, X_{m,k_n}}}_{m \text{ blocks of } k_n \text{ each}} \big)$$

$$:= L^{k_n - 1} f(t) \wedge \left(X_{1,1} \cdots X_{1,k_n} \right) f(t) \wedge \dots \wedge \left(X_{m,1} \cdots X_{m,k_n} f(t) \right).$$

This expression is now <u>homogeneous</u> of degree k_n in each of the m blocks $(X_{j,1}, \dots, X_{j,k_n})$ above – we say $\tilde{A}^f_t$ is <u>multi-homogeneous</u>.

One can also observe that $\tilde{A}^f_t$ enjoys some symmetries:

- the order of $X_{j,1}, \dots, X_{j,k_n}$ doesn't matter
- it is alternating in the blocks of k_n tangent vectors (because the wedge product is, or the determinant if you prefer).

This means that

$$\tilde{A}^f_t \in \Lambda^m \, \mathrm{Sym}^{k_n}(\mathbb{R}^d) =: V,$$

but for us it will only matter that V is a vector space: the coordinates of $\tilde{A}^f_t$ are given in the standard basis by

$$\tilde{A}^f_t \left(\partial_{j^1_1}, \dots, \partial_{j^1_{k_n}}, \dots, \partial_{j^m_1}, \dots, \partial_{j^m_{k_n}} \right).$$

On this vector space V we have a norm, as given by the expressions seen before: more precisely, it is simply the ℓ^2 norm

$$\|\tilde{A}\| := \left(\sum_{\underline{j}_1 \in \{1,\dots,d\}^{k_n}} \cdots \sum_{\underline{j}_m \in \{1,\dots,d\}^{k_n}} \left| \tilde{A}\left(\partial_{j_1^{(1)}}, \dots, \partial_{j_1^{(k_n)}}, \dots, \partial_{j_m^{(1)}}, \dots, \partial_{j_m^{(k_n)}}\right) \right|^2 \right)^{1/2}.$$

With ρ_M the usual action of $SL(\mathbb{R}^d)$, we have therefore

$$\left(\frac{dv_A}{dt}\right)^{\frac{Q}{d}} = \inf_{M \in SL(\mathbb{R}^d)} \|\rho_M \tilde{A}_t^f\|$$

[this is just to reassure ourselves that we haven't really changed anything.]

Gressman shows that expressions like the RHS in the last one are related to certain <u>polynomial invariants</u> that are important for Geometric Invariant Theory.

Consider polynomials on V, that is, polynomials in the coordinates of elements of V: for $\tilde{A} \in V$, $P(\tilde{A})$ is a polynomial expression in the quantities

$$\tilde{A}\left(\partial_{j_1^{(1)}}, \dots, \partial_{j_1^{(k_n)}}, \dots, \partial_{j_m^{(1)}}, \dots, \partial_{j_m^{(k_n)}}\right).$$

Using the action ρ_M we can define an action on the polynomials:

$$(\tilde{\rho}_M P)(\tilde{A}) := P(\rho_{M^T} \tilde{A})$$

(the ρ_{M^T} is just so it's an action on the left).

This allows us to define a special class of polynomials: the $SL(\mathbb{R}^d)$-invariant polynomials. That is the polynomials P on V such that

$$\tilde{\rho}_M P = P \qquad \forall M \in SL(\mathbb{R}^d).$$

We will let

$$P_1, \dots, P_N$$

be a minimal list of homogeneous and $SL(\mathbb{R}^d)$-invariant polynomials that generates the whole algebra of $SL(\mathbb{R}^d)$-invariant polynomials.

Remark: That such a list is finite was proven by Hilbert.

We can now state Gressman's observation.

> Lemma: If $P_1, \dots, P_N$ are the generators as above, then for every $\tilde{A} \in V$ we have
>
> $$\inf_{M \in SL(\mathbb{R}^d)} \| \rho_M \tilde{A} \| \sim_{d,n} \max_{j=1,\dots,N} |P_j(\tilde{A})|^{\frac{1}{\deg P_j}}.$$

Proof:
To show that LHS $\gtrsim$ RHS is easy. Indeed, since P_j is homogeneous we have by scaling

$$|P_j(\tilde{A})|^{\frac{1}{\deg P_j}} \leq \|\tilde{A}\| \cdot \underbrace{\sup_{\|D\|=1} |P_j(D)|^{\frac{1}{\deg P_j}}}_{=: \|P_j\|_\infty^{1/\deg P_j}}.$$

and therefore

$$|P_j(\tilde{A})|^{\frac{1}{\deg P_j}} \leq \|\tilde{A}\| \overbrace{\left[\max_{j'=1,\dots,N} \|P_{j'}\|_\infty^{\frac{1}{\deg P_{j'}}}\right]}^{=:\, C_{d,n}}$$

and taking $\rho_M \tilde{A}$ in place of $\tilde{A}$ we have by invariance of P_j that after taking the infimum

$$\frac{1}{C_{d,n}} |P_j(\tilde{A})| \leq \inf_{M \in SL(\mathbb{R}^d)} \|\rho_M \tilde{A}\| .$$

This is true for all $j = 1, \dots, N$, so taking the maximum on the LHS we conclude the $\gtrsim$ part of the inequality.

For the $\lesssim$ part, we argue by contradiction. Assume there is a sequence $(\tilde{A}_k)_k \subset V$ such that

$$\text{i)} \quad \inf_{M \in SL(\mathbb{R}^d)} \|\rho_M \tilde{A}_k\| = 1 \quad \forall k \ ,$$

$$\text{ii)} \quad \max_j |P_j(\tilde{A}_k)|^{\frac{1}{\deg P_j}} \longrightarrow 0$$

as $k \to \infty$.

By definition we can find for every k a matrix $M_k \in SL(\mathbb{R}^d)$ such that

$$\inf_{M \in SL(\mathbb{R}^d)} \|\rho_M \tilde{A}_k\| = 1 \leq \|\rho_{M_k} \tilde{A}_k\| \leq 1 + \frac{1}{k} \ ;$$

the sequence $(\rho_{M_k} \tilde{A}_k)_k$ admits then a converging subsequence (by compactness): For some $(k_j)_j \subset \mathbb{N}$,

$$\rho_{M_{k_j}} \tilde{A}_{k_j} \longrightarrow \tilde{A} \quad \text{in } (V, \|\cdot\|).$$

By the above, we must have $\|\tilde{A}\| = 1$.

However, by (ii) and by continuity we must have

$$P_j(\tilde{A}) = 0 \qquad \text{for all } j = 1, \dots, N.$$

The simplest form of the Hilbert–Mumford criterion says that

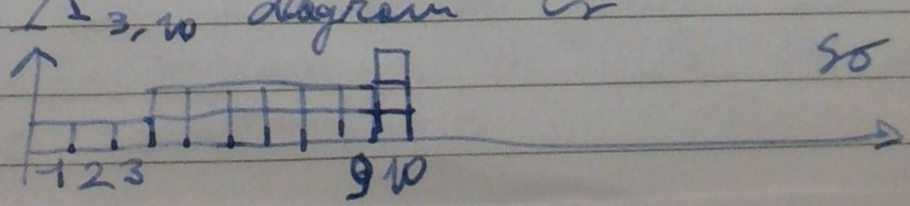
> The only common zero of all the homogeneous $SL(\mathbb{R}^d)$-invariant polynomials is 0.

Thus we have $\|\tilde{A}\| = 1$ and $\tilde{A} = 0$, a contradiction, and the $\leq$ part of the inequality is proved. ∎

The lemma provides a very precise characterisation of these infima we are interested in; however, this characterisation is <u>rarely effective</u>, because finding a minimal set of generators of the $SL(\mathbb{R}^d)$-invariant polynomials is generally an intractable problem (these sets of generators can easily become very large.)

There are however some cases in which sets of generators are known, and they can be used to produce examples.

<u>Example</u>: Consider the case of 3-surfaces in $\mathbb{R}^{10}$ $(d = 3,\ n = 10)$.
The $\Delta_{3,10}$ diagram is



so $m = 1$,
$Q = 18$

Thus

$$\tilde{\mathcal{A}}^f_t = \tilde{\mathcal{A}}^f_t(X_1, X_2, X_3)$$

$$= \underbrace{\Big(\bigwedge_{\substack{\underline{\alpha} \in \mathbb{N}^3: \\ 1 \leq |\underline{\alpha}| \leq 2}} \frac{1}{\underline{\alpha}!} \partial^{\alpha} f(t) \Big) \wedge \big(X_1 X_2 X_3 f(t)\big)}_{= L^2 f(t)}$$

and so $\tilde{\mathcal{A}}^f_t \in \mathrm{Sym}^3(\mathbb{R}^d) = V$.

It is known that the $SL(\mathbb{R}^3)$-invariant polynomials over V are generated by two homogeneous polynomials

$$S \text{ of } \deg = 4, \qquad T \text{ of } \deg = 6,$$

called the Aronhold's Invariants.

Thus one has

$$\left(\frac{d\nu_{\mathcal{A}}}{dt}\right)^{\frac{18}{3}} \sim |S(\tilde{\mathcal{A}}^f_t)|^{\frac{1}{4}} + |T(\tilde{\mathcal{A}}^f_t)|^{\frac{1}{6}}.$$

For example, if the surface is

$$\Big((t^{\underline{\alpha}})_{\substack{\underline{\alpha} \in \mathbb{N}^3: \\ 1 \leq |\underline{\alpha}| \leq 2}}, \; t_1^3 + t_2^3 + t_3^3 \Big)$$

then $S = 0$ and $T = 1$.

[The reader is encouraged to look up S and T: the latter is a monster with around 100 terms.]

Morally, one could say there are two ways for a 3-surface in $\mathbb{R}^{10}$ to be well-curved: $S \neq 0$ and $T \neq 0$.

The next example ties things up with what we have seen before.

<u>Example</u>:

Consider a hypersurface given in graph form
$$f(t_1, \dots, t_{n-1}) = (t_1, \dots, t_{n-1}, \phi(t_1, \dots, t_{n-1}))$$

We have seen before what $\Lambda_{n-1,n}$ looks like: $Q = n+1$, in particular.

The $\tilde{A}^f_t$ tensor is simply
$$\tilde{A}^f_{\underline{t}}(X_1, X_2) = L^1 f(\underline{t}) \wedge X_1 X_2 f(\underline{t}) = X_1 X_2 \phi(\underline{t}),$$
thus $\tilde{A}^f_t \in \mathrm{Sym}^2(\mathbb{R}^{n-1})$, the space of ~~sym~~ symmetric matrices:
$$\tilde{A}^f_t(\partial_i, \partial_j) = \partial_i \partial_j \phi \Rightarrow \text{the matrix is the Hessian } \nabla^2\phi.$$

The polynomials invariant with respect to $SL(\mathbb{R}^{n-1})$ are generated by a unique polynomial: the <u>determinant</u> of the corresponding matrix,
$$\det\left(\left(\tilde{A}(\partial_i, \partial_j)\right)_{ij} \right)$$

[Notice indeed $\rho_M \tilde{A} = M^T A M$, where $\tilde{A}(\partial_i, \partial_j) = A_{ij}$, so $\det(M^T A M) = (\det M)^2 \det A = \det A$.]

Thus by the lemma $\left(\frac{d\nu_A}{dt}\right)^{\frac{n+1}{n-1}} \sim \left|\det(\nabla^2\phi)\right|^{\frac{1}{n-1}}$

(and we know this actually holds with $=$.)