On the Structure of n-Lie Algebras

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Contents

Introduction

In this work we will study a generalization of the concept of Lie algebras. In order to give the reader an idea how this generalization arises we are going to motivate our study by first recalling the concept of Lie algebras. A Lie algebra is by definition a vector space L with a bilinear product $[x, y]$, such that for all $x, y, z \in L$

L1) $[y, x] = -[x, y]$,

L2) $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ (Jacobi's identity).

Clearly the Jacobi identity is equivalent to the condition that the left multiplication $\text{ad}x : L \to L$ defined by $\text{ad}x(y) = [x, y]$ is a derivation of L. Therefore we can reformulate the definition of Lie algebras as follows: An algebra L is a Lie algebra if

L1') its product is alternating, and

L2') every left multiplication is a derivation.

We also recall the concept of triple systems. A triple system is simply a vector space T with a trilinear product $p : T \times T \times T \to T$. In 1985 Faulkner [3] discussed a new class of triple systems which he called alternating triple systems. He was led to this type of triple systems by studying which trilinear identities are satisfied by a nearly simple triple system over an algebraically closed field K of characteristic 0, i.e. by a triple system over K such that

1) its left multiplications are derivations,

2) its derivation algebra acts irreducibly on it.

To be more concrete, let T be an arbitrary triple system with product p. An element $u = \sum \alpha_\sigma \sigma$ in the group algebra $K[S_3]$, is viewed as a trilinear identity if $u(p)(x_1, x_2, x_3) = \sum \alpha_\sigma p(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma_3}) = 0$. Clearly the trilinear identities satisfied by p form a left ideal I_p in $K[S_3]$. It is proved in [3] that if T is a nontrivial, nearly simple triple system, then I_p contains at least one of the left ideals corresponding to six types of triple systems which are called: left-skew, left-symmetric, Anti-Lie, Lie, Jacobi and alternating triple system (see [3,Thm 9]). This reduces the study of the infinite family of inequivalent identities to the six types above.

Furthermore Faulkner has proved that the alternating triple system on the space of quaternions with triple product $x\overline{y}z - z\overline{y}x$ is the only nontrivial, alternating, nearly simple triple system up to isomorphism.

Recall that an alternating triple system is by definition a triple sytem such that

AT1) its product p is alternating and

AT2) its left multiplications $z \to p(x, y, z)$ are derivations.

As we have seen above, Lie algebras and alternating triple systems are characterized by the properties that their products are alternating and their left multiplications are derivations. We can formally require these two properties for an n-linear product on a vector space. Then we have naturally generalized the concept of Lie algebras and alternating triple systems. Now, a vector space V with an n-linear map $p : \times^n V \to V$ is called an n-Lie algebra if

NL1) p is alternating and

NL2) all left multiplications $y \to p(x_1, \cdots, x_{n-1}, y)$ are derivations.

This definition of an n-Lie algebra was originally given by Filippov in [5]. In that paper there are also determined the $(n+1)$-dimensional n-Lie algebras and it is shown that every $(n+1)$-dimensional n-Lie algebras is isomorphic to one of the following n-Lie algebras on K^{n+1} given by the multiplication table:

$$[e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_n] = \alpha_i e_i, \quad i = 1, \cdots, n+1, \qquad (*)$$

where $\{e_1, \cdots, e_{n+1}\}$ is a base of K^{n+1} and $\alpha_i \in K$, $i = 1, \cdots, n+1$. Note that if K is algebraically closed, then all α_i can be chosen as 1 or 0.

Let us recall the definitions of solvability and semisimplicity given in [5]. We define recursively:

$$V^{(0)} := V, \quad V^{(s+1)} := [V^{(s)}, \cdots, V^{(s)}], s \in \mathbb{N}_0.$$

Then an n-Lie algebra is called solvable if $V^{(s)} = \{0\}$ for some s. An ideal of an n-Lie algebra is called solvable if it is solvable as an n-Lie algebra. As in case of Lie algebras a finite dimensional n-Lie algebra has a unique maximal solvable ideal called the radical of the given n-Lie algebra. If an n-Lie algebra possesses no nonzero solvable ideals, then it is called semisimple. In case $n = 2$ the above definitions agree with the usual definitions for Lie algebras. Therefore the concept of solvability and the concept of semisimplicity of n-Lie algebras generalize those of Lie algebras.

In this work we investigate the structure of finite dimensional n-Lie algebras over an algebraically closed field of characteristic 0. We are mainly interested in classifying n-Lie algebras. Now, at first we are concerned with the problem of finding the simple representatives of n-Lie algebras and of determining whether a semisimple n-Lie algebra is a direct sum of its simple ideals. These problems have a well-known beautiful answer in the case of Lie algebras. Here we will give a complete answer too. In fact, we shall prove that a finite dimensional n-Lie algebra

V over a field of characteristic 0 is semisimple if and only if it is a direct sum of simple ideals (see Theorem 2.7). In order to prove this we will show that the Lie algebra Der(V) of derivations of V is semisimple. Then an application of Weyl's theorem on complete reducibility yields the result.

In the third chapter we classify the finite dimensional simple n-Lie algebras over an algebraically closed field K of characteristic 0 and over the real numbers. As in the case of 3-Lie algebras studied in Faulkner [3] we shall show that for $n > 2$ all finite dimensional simple n-Lie algebras over K are isomorphic to each other (see Theorem 3.9). In the proof of this result we will heavily use the representation theory of finite dimensional semisimple Lie algebras. In particular, we need some informations about the decomposition of the n-fold wedge product of an irreducible module of a semisimple Lie algebra.

It turns out that every finite dimensional simple n-Lie algebra over K is of dimension $n + 1$. So we get a realization by the multiplication table ($*$) mentioned above, where all coefficients α_i may be chosen to be 1. Another realization can be given as follows. It generalizes the vector product on K^3 to K^{n+1} (see Example 1.1.1). Let b be a nondegenerate symmetric bilinear form and f be a nonzero determinant form on K^{n+1}. For $v_1, \cdots, v_n \in K^{n+1}$ let $[v_1, \cdots, v_n]$ be the unique element in K^{n+1} such that for all $x \in K^{n+1}$ the identity

$$b([v_1, \cdots, v_n], x) = f(v_1, \cdots, v_n, x)$$

holds. Then K^{n+1} is an n-Lie algebra with product $[v_1, \cdots, v_n]$.

Every real simple n-Lie algebra is isomorphic to an n-Lie algebra $(\mathbb{R}^{n+1}, b, f)$ or a realification of a complex simple n-Lie algebra.

In the fourth chapter we will prove an analog of Levi decompositions of finite dimensional Lie algebras for finite dimensional n-Lie algebras. The result here is that each finite dimensional n-Lie algebra over an algebraically closed field of characteristic 0 has a semisimple subalgebra such that the given n-Lie algebra is the direct sum of its radical and this subalgebra (see Theorem 5.1). Note that one has proved the existence of a Levi decomposition for a Lie algebra by means of cohomology. Unfortunately, we do not have a concept of cohomology for n-Lie algebras. But our results for the structure of semisimple n-Lie algebras provide us with an alternative way to construct a Levi decomposition.

We will begin our study in the first two chapters by introducing some basic concepts for n-Lie algebras. Moreover, we discuss some examples and describe the $(n + 1)$-dimensional simple n-Lie algebras. After reviewing the definitions of k-solvability and k-semisimplicity of Kasymov [10] we study the structure of finite dimensional semisimple respectively reductive n-Lie algebras. A reductive n-Lie algebra is by definition an n-Lie algebra whose centre agrees with its radical. We shall show that a finite dimensional n-Lie algebra over a field of characteristic 0 is reductive if and only if it is a direct sum of its centre and the semisimple subalgebra $[V, \cdots, V]$.

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Chapter 1

n-Lie algebras

Throughout this chapter K will be a field of characteristic different from 2 except where otherwise noted.

1.1 Definitions and Examples

In this section we shall introduce some basic concepts for n-Lie algebras and discuss some examples.

Let us begin with the well-known Lie algebra $\mathbb{R}^3$ with the vector product $[x, y]$ which can be realized by means of a scalarproduct b and a determinant form f in the following way: $b([x,y],z) = f(x,y,z), \quad x,y,z \in \mathbb{R}^3$. This Lie algebra is isomorphic to $su(2)$ whose complexification is isomorphic to $sl(2,\mathbb{C})$ which plays an important role in the theory of finite dimensional complex semisimple Lie algebras. We generalize the vector product structure to K^n, $n \geq 3$, in the following way.

<u>**Example 1.1.1**</u>: Let $n \in \mathbb{N}$, $n \geq 2$. Let f be a nonzero determinant form and b a nondegenerate symmetric bilinear form on K^{n+1}. Given any fixed $v_i \in K^{n+1}$, $i \in \underline{n}$, there exists a unique element $[v_1, \cdots, v_n] \in K^{n+1}$ depending only on $v_1, \cdots, v_n$ such that for all $x \in K^{n+1}$: $f(v_1, \cdots, v_n, x) = b([v_1, \cdots, v_n], x)$. Obviously the map $(v_1, \cdots, v_n) \to [v_1, \cdots, v_n]$ is n-linear and alternating. Moreover we find the following property which one might think of as a generalization of the Jacobi identity:

$$[u_1, \cdots, u_{n-1}, [v_1, \cdots, v_n]] = \sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], v_{i+1}, \cdots, v_n], \quad (1.1)$$

where $u_i \in K^{n+1}$, $i \in \underline{n-1}$ and $v_j \in K^{n+1}$, $j \in \underline{n}$.

In order to verify this identity let us define for fixed $u_i \in K^{n+1}$, $i \in \underline{n-1}$, an endomorphism $\mathrm{ad}(u_1, \cdots, u_{n-1})$ of K^{n+1} via $\mathrm{ad}(u_1, \cdots, u_n)v := [u_1, \cdots, u_n, v]$, $v \in K^{n+1}$. Then (1.1) can be written equivalently as

$$\mathrm{ad}(u_1, \cdots, u_{n-1})[v_1, \cdots, v_n] = \sum_{i=1}^{n} [v_1, \cdots, \mathrm{ad}(u_1, \cdots, u_{n-1})v_i, \cdots, v_n]. \quad (1.2)$$

To prove (1.2) we need a simple fact: Let T be an endomorphism of K^{n+1}, then for all $v_i \in K^{n+1}$, $i \in \underline{n+1}$:

$$\sum_{i=1}^{n+1} f(v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_{n+1}) = \mathrm{tr}(T) \cdot f(v_1, \cdots, v_{n+1}). \tag{1.3}$$

By definition of $[v_1, \cdots, v_n]$, identity (1.3) is equivalent to

$$\begin{aligned}&\sum_{i=1}^{n} b([v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_n], v_{n+1})\\ &+b([v_1, \cdots, v_n], Tv_{n+1}) = \mathrm{tr}(T) \cdot b([v_1, \cdots, v_n], v_{n+1}).\end{aligned} \tag{1.4}$$

If $T \in so(K^{n+1}, b)$, that is, T is an endomorphism of K^{n+1} such that $b(Tu, v) + b(u, Tv) = 0$, then (1.4) becomes

$$\sum_{i=1}^{n} b([v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_n], v_{n+1}) = b(T[v_1, \cdots, v_n], v_{n+1})$$

because of $\mathrm{tr}(T) = 0$. Since v_{n+1} is arbitrary in K^{n+1} and b is nondegenerate, we conclude that for all $T \in so(K^{n+1}, b)$ equation (1.4) is equivalent to

$$T[v_1, \cdots, v_n] = \sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_n]. \tag{1.5}$$

Thanks to (1.5) it remains to show that $\mathrm{ad}(u_1, \cdots, u_n) \in so(K^{n+1}, b)$ in order to get (1.2). Indeed, since b is symmetric and f is alternating, we have

$$\begin{aligned} b(\mathrm{ad}(u_1, \cdots, u_{n-1})v, w) &= b([u_1, \cdots, u_{n-1}, v], w)\\ &= f(u_1, \cdots, u_{n-1}, v, w)\\ &= -f(u_1, \cdots, u_{n-1}, w, v)\\ &= -b([u_1, \cdots, u_{n-1}, w], v)\\ &= -b(\mathrm{ad}(u_1, \cdots, u_{n-1})w, v)\\ &= -b(v, \mathrm{ad}(u_1, \cdots, u_{n-1})w). \end{aligned}$$

In this work we shall study the algebraic structure which arises in the above example. We describe this kind of structure abstractly in a few axioms.

Definition of n-Lie algebra:

Let $n \in \mathbb{N}$, $n \geq 2$. A vector space V over K together with a map $(v_1, \cdots, v_n) \to [v_1, \cdots, v_n]$ of $\times^n V$ into V is called an n-Lie algebra if the following properties are satisfied:

1) the map is n-linear,

2) the map is alternating,

3) identity (1.1) holds for all u_i, $v_j \in V$, $i \in \underline{n-1}$, $j \in \underline{n}$.

We call identity (1.1) the generalized Jacobi identity (G.J.I.) and the endomorphisms $\mathrm{ad}(u_1, \cdots, u_{n-1})$, defined by $\mathrm{ad}(u_1, \cdots, u_{n-1})v := [u_1, \cdots, u_{n-1}, v]$, left multiplications. With the help of left multiplications the G.J.I. can also be rewritten as

$$\begin{aligned}&[\mathrm{ad}(u_1, \cdots, u_{n-1}), \mathrm{ad}(v_1, \cdots, v_{n-1})]\\&= \sum_{i=1}^{n-1} \mathrm{ad}(v_1, \cdots, v_{i-1}, \mathrm{ad}(u_1, \cdots, u_{n-1})v_i, v_{i+1}, \cdots, v_{n-1}),\end{aligned} \tag{1.6}$$

that is, the Lie product of two left multiplications is the sum of some left multiplications. One can prove that (1.1) is also equivalent to

$$\begin{aligned}&\mathrm{ad}(u_1, \cdots, u_{n-2}, [v_1, \cdots, v_n])\\&= \sum_{i=1}^{n} (-1)^{n-i} \mathrm{ad}(v_1, \cdots, \widehat{v_i}, \cdots, v_n) \mathrm{ad}(u_1, \cdots, u_{n-2}, v_i).\end{aligned} \tag{1.7}$$

Therefore we get four equivalent identities (1.1), (1.2), (1.6) and (1.7).

We remark that in case $n = 2$ this definition agrees with that of a Lie algebra. Therefore the concept of n-Lie algebras generalizes that of Lie algebras.

Example 1.1 shows that K^{n+1} carries the structure of an n-Lie algebra with respect to the product defined by means of a nondegenerate symmetric bilinear form b and a nonzero determinant form f on V. This n-Lie algebra will be denoted by (K^{n+1}, b, f).

Let us look at some more examples.

Example 1.1.2: Let A be an associative commutative algebra over K. Let D_i, $i \in \underline{n}$, be derivations of A satisfying $D_i D_j = D_j D_i$, for all i, $j \in \underline{n}$. We put for $a_i \in A$, $i \in \underline{n}$,

$$[a_1, \cdots, a_n] := \det(D_i a_j)_{(i,j) \in \underline{n} \times \underline{n}}.$$

Then the vector space A equipped with this product becomes an n-Lie algebra.

Proof: Suppose that D is a derivation of A of the form $D = \sum_{i=1}^{n} x_i D_i$, then $[D_j, D] = D_j D - D D_j = \sum_{i=1}^{n} (D_j x_i) D_i$. For this D we have the identity

$$D[a_1, \cdots, a_n] - \sum_{j=1}^{n} [a_1, \cdots, D a_j, \cdots, a_n] = -\left(\sum_{i=1}^{n} D_i x_i\right) \cdot [a_1, \cdots, a_n]. \tag{1.8}$$

In fact, for all $a_i \in A$, $i \in \underline{n}$,

$$
\begin{aligned}
& D[a_1,\cdots,a_n]-\sum_{j=1}^{n}[a_1,\cdots,Da_j,\cdots,a_n] \\
&= D\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_n \end{vmatrix} \\
&\quad -\sum_{j=1}^{n}\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1Da_j & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & D_2Da_j & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_nDa_j & \cdots & D_na_n \end{vmatrix} \\
&= \sum_{j=1}^{n}\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & DD_1a_j & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & DD_2a_j & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & DD_na_j & \cdots & D_na_n \end{vmatrix} \\
&\quad -\sum_{j=1}^{n}\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1Da_j & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & D_2Da_j & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_nDa_j & \cdots & D_na_n \end{vmatrix} \\
&= -\sum_{j=1}^{n}\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & [D_1,D]a_j & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & [D_2,D]a_j & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & [D_n,D]a_j & \cdots & D_na_n \end{vmatrix} \\
&= -\sum_{j=1}^{n}\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & \sum_{i=1}^{n}(D_1x_i)(D_ia_j) & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & \sum_{i=1}^{n}(D_2x_i)(D_ia_j) & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & \sum_{i=1}^{n}(D_nx_i)(D_ia_j) & \cdots & D_na_n \end{vmatrix} \\
&= -\left(\sum_{i=1}^{n} D_ix_i\right)\begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_n \\ D_2a_1 & D_2a_2 & \cdots & D_2a_n \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_n \end{vmatrix} \\
&= -\left(\sum_{i=1}^{n} D_ix_i\right)\cdot[a_1,\cdots,a_n].
\end{aligned}
$$

Let us now consider the left multiplications. We claim that (1.8) is true for any left multiplication $D = \mathrm{ad}(a_1,\cdots,a_{n-1})$ because we can write $D = \sum_{i=1}^{n} x_iD_i$, where

x_i is given by

$$x_i = (-1)^{n+i} \begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{D_ia_1} & \widehat{D_ia_2} & \cdots & \widehat{D_ia_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_{n-1} \end{vmatrix}.$$

If $\sum_{i=1}^n D_ix_i$ is 0, then G.J.I. results from identity (1.8). Thus it suffices to show that $\sum_{i=1}^n Dx_i = 0$. In fact,

$$\begin{aligned} \sum_{i=1}^n D_ix_i &= \sum_{i=1}^n \sum_{j=1, j\neq i}^n (-1)^{n+i} \begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{D_ia_1} & \widehat{D_ia_2} & \cdots & \widehat{D_ia_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ D_iD_ja_1 & D_iD_ja_2 & \cdots & D_iD_ja_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_{n-1} \end{vmatrix} \\ &= \sum_{1\leq i<j\leq n} (-1)^{n+i} \begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{D_ia_1} & \widehat{D_ia_2} & \cdots & \widehat{D_ia_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ D_iD_ja_1 & D_iD_ja_2 & \cdots & D_iD_ja_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_{n-1} \end{vmatrix} \\ &\quad + \sum_{1\leq i<j\leq n} (-1)^{n+j} \begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ D_iD_ja_1 & D_iD_ja_2 & \cdots & D_iD_ja_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{D_ja_1} & \widehat{D_ja_2} & \cdots & \widehat{D_ja_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_{n-1} \end{vmatrix}. \end{aligned}$$

By moving the i-th row to the $(j-1)$-th row in the determinant in the second sum without changing the odering of the remaining rows, we get the following expression:

$$(-1)^{j-i-1} \begin{vmatrix} D_1a_1 & D_1a_2 & \cdots & D_1a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{D_ia_1} & \widehat{D_ia_2} & \cdots & \widehat{D_ia_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ D_iD_ja_1 & D_iD_ja_2 & \cdots & D_iD_ja_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ D_na_1 & D_na_2 & \cdots & D_na_{n-1} \end{vmatrix}.$$

Hence each summand in the first sum cancels with the corresponding one in the second sum and therefore $\sum_{i=1}^{n} Dx_i = 0$. □

As a concrete example take A to be the real algebra $C^\infty(\mathbb{R}^n)$ of C^∞-functions on $\mathbb{R}^n$ and $D_i = \frac{\partial}{\partial x_i}$, $i \in \underline{n}$, where $n \geq 2$. Then we get an n-Lie algebra on $C^\infty(\mathbb{R}^n)$ with the product $[g_1, \cdots, g_n] := \left|\frac{\partial g_j}{\partial x_i}\right|$.

Example 1.1 and 1.2 can also be found in Filippov [5].

Example 1.1.3: Let V be an n-dimensional vector space over K, f a nonzero determinant form on V and $0 \neq v_0 \in V$. Then V becomes an n-Lie algebra relative to the product $[v_1, \cdots, v_n] := f(v_1, \cdots, v_n)\, v_0$.

In fact, by using Cramer's rule

$$f(v_1, \cdots, v_n)v_0 = \sum_{i=1}^{n} f(v_1, \cdots, v_{i-1}, v_0, v_{i+1}, \cdots, v_n)v_i,$$

we get

$$\begin{aligned}
&\sum_{i=1}^{n}[v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], v_{i+1}, \cdots, v_n] \\
&= \sum_{i=1}^{n} f(u_1, \cdots, u_{n-1}, v_i)[v_1, \cdots, v_{i-1}, v_0, v_{i+1}, \cdots, v_n] \\
&= \sum_{i=1}^{n} f(u_1, \cdots, u_{n-1}, v_i) f(v_1, \cdots, v_{i-1}, v_0, v_{i+1} \cdots, v_n)v_0 \\
&= f(u_1, \cdots, u_{n-1}, \sum_{i=1}^{n} f(v_1, \cdots, v_{i-1}, v_0, v_{i+1} \cdots, v_n)v_i)v_0 \\
&= f(u_1, \cdots, u_{n-1}, f(v_1, \cdots, v_n)v_0)v_0 \\
&= f(u_1, \cdots, u_{n-1}, [v_1, \cdots, v_n])v_0 \\
&= [u_1, \cdots, u_{n-1}, [v_1, \cdots, v_n]],
\end{aligned}$$

which gives the G.J.I..

Remark 1.1.1: Let V be an n-Lie algebra ($n > 2$) with product $[v_1, \cdots, v_n]$. For any fixed $v_0 \in V$ we may define an $(n-1)$-Lie algebra structure on the underlying vector space V by $[v_1, \cdots, v_{n-1}] := [v_1, \cdots, v_{n-1}, v_0]$. Although very easy to prove, it demonstrates us how to construct an $(n-1)$-Lie algebra from a given n-Lie algebra.

Definition: *Let V_1, V_2 be n-Lie algebras over K. A linear map $\tau : V_1 \to V_2$ is called*

1) a homomorphism if for all $v_i \in V$, $i \in \underline{n} : \tau[v_1, \cdots, v_n] = [\tau(v_1), \cdots, \tau(v_n)]$;

2) an isomorphism if τ *is bijective in addition;*

3) an automorphism if $V_1 = V_2$ *and* τ *is an isomorphism.*

For the sake of convenience we introduce the following notation. Let V be an n-Lie algebra and U_i, $i \in \underline{n}$, be subspaces of V. We shall denote by $[U_1, \cdots, U_n]$ the subspace of V which is spanned by the elements of the form $[u_1, \cdots, u_n]$, $u_i \in U_i$.

Definition: *Let* V *be an* n*-Lie algebra.*

1) A subspace U *of* V *is called an* n*-Lie subalgebra if* $[U, \cdots, U] \subseteq U$.

2) A subspace I *of* V *is called an ideal of* V *if* $[I, V, \cdots, V] \subseteq I$.

3) V *is called simple if* $[V, \cdots, V] \neq \{0\}$ *and has no ideals except itself and* $\{0\}$.

Let V be an n-Lie algebra. Then

$$C(V) := \{u \in V \mid [v_1, \cdots, v_{n-1}, u] = 0, \ \forall v_i \in V, \ i \in \underline{n-1}\},$$

the centre of V, is an ideal of V. In the following we give more examples of ideals and subalgebras.

Example 1.1.4: The n-Lie algebra V in Example 1.1.3 has Kv_0 as an ideal. It is clear that the nonzero ideals of V are exactly the subspaces of V which include Kv_0.

Example 1.1.5: The subspace $\mathbb{R}[x_1, \cdots, x_n]$ of polynomial functions in the n-Lie algebra $C^\infty(\mathbb{R}^n)$ is closed with respect to the n-Lie product, therefore an n-Lie subalgebra of $C^\infty(\mathbb{R}^n)$. Moreover one can verify that for each i the set of all polynomial functions of the form $x_i f$, $f \in \mathbb{R}[x_1, \cdots, x_n]$ is a subalgebra of $\mathbb{R}[x_1, \cdots, x_n]$.

In the following proposition we list some simple properties of ideals. We shall give no proofs.

Proposition 1.1.1: *Let* V *be an* n*-Lie algebra over* K *and* I, J *be two ideals of* V. *Then the following are valid.*

1) $I + J$ *is an ideal of* V.

2) *The vector space V/I is an n-Lie algebra relative to the product*

$$[\pi v_1, \cdots, \pi v_n] := \pi[v_1, \cdots, v_n],$$

where $\pi : V \to V/I$ denotes the canonical map.

3) *Let $\psi : V \to V'$ be an n-Lie algebra homomorphism.*

   (a) *If I' is an ideal of V', then $\psi^{-1}I$ is an ideal of V, in particular, $Ker\psi$ is an ideal of V. Moreover ψ induces an isomorphism $\tilde{\psi} : V/Ker\psi \to Im\psi$. If I is any ideal of V included in $Ker\psi$, then there exists a unique homomorphism $\phi : V/I \to V'$ such that $\psi = \phi\pi$.*

   (b) *If ψ is in addition surjective, then the image of any ideal of V under ψ is an ideal of V'.*

4) *If $J \subseteq I$, then I/J is an ideal of V/J and $(V/J)/(I/J)$ is naturally isomorphic to V/I.*

5) $(I + J)/J \cong I/(I \cap J)$.

Definition: *Let V be an n-Lie algebra over K. We call an endomorphism D of V a derivation if for all $v_i \in V$, $i \in \underline{n}$:*

$$D[v_1, \cdots, v_n] = \sum_{i=1}^{n}[v_1, \cdots, v_{i-1}, Dv_i, v_{i+1}, \cdots, v_n]. \tag{1.9}$$

In consequence of identity (1.2) each left multiplication is a derivation. We refer to an endomorphism D of V as an inner derivation if it may be written as a sum of some left multiplications. We denote by $Der(V)$ ($Inder(V)$) the set of all (inner) derivations of V. One can verify

Proposition 1.1.2: *Relative to the Lie bracket $Der(V)$ is a Lie algebra and $Inder(V)$ is an ideal of $Der(V)$.*

Let V be an arbitrary n-Lie algebra and let U_i, $i \in \underline{n-1}$, be subspaces of V. We denote by $\mathrm{ad}(U_1, \cdots, U_{n-1})$ the subspace of $Inder(V)$ spanned by the left multiplications $\mathrm{ad}(u_1, \cdots, u_{n-1})$, $u_i \in U_i$. If U_i, $i \in \underline{n-1}$, are ideals of V, then $\mathrm{ad}(U_1, \cdots, U_{n-1})$ is an ideal of $Inder(V)$.

Let V be an arbitrary n-Lie algebra. Then the Lie algebras $Der(V)$ and $Inder(V)$ operate in a natural way on V, so we have a representation of them on V, or equivalently, V is a $Der(V)$- and an $Inder(V)$-module. If I is an ideal of V, that is, $[I, V, \cdots, V] \subseteq I$, then I is an $Inder(V)$-submodule of V. Conversely if

$I \subseteq V$ is a $Der(V)$- or an $Inder(V)$-submodule, then $\text{ad}(u_1, \cdots, u_{n-1})I \subseteq I$ for all $u_1, \cdots, u_{n-1} \in V$, therefore I is an ideal of V.

We close this section with the following theorem.

Theorem 1.1.3: *Let V a simple n-Lie algebra over K, then $Der(V)$ and $Inder(V)$ act irreducibly on V. If K is of characteristic 0 and V is finite dimensional in addition, then*

1) all derivations of V are inner,

2) $Der(V)$ is semisimple.

For its proof we need

Lemma 1.1.4: *Let V be an n-Lie algebra over K, char$K = 0$, with $C(V) = \{0\}$ or $[V, \cdots, V] = V$. If D is a derivation of V which commutes with all inner derivations of V, then $D = 0$.*

Proof: Let $v_1, \cdots, v_n \in V$. For each $i \in \underline{n}$ we have by the assumption,

$$\begin{aligned} D[v_1, \cdots, v_n] &= (-1)^{n-i} D \ \text{ad}(v_1, \cdots, \widehat{v_i}, \cdots, v_n)v_i \\ &= (-1)^{n-i} \text{ad}(v_1, \cdots, \widehat{v_i}, \cdots, v_n)Dv_i \\ &= [v_1, \cdots, v_{i-1}, Dv_i, v_{i+1}, \cdots, v_n], \end{aligned}$$

that is, $D[v_1, \cdots, v_n] = [v_1, \cdots, v_{i-1}, Dv_i, v_{i+1}, \cdots, v_n]$. From (1.9) we get

$$D[v_1, \cdots, v_n] = n \, D[v_1, \cdots, v_n].$$

Since $n \geq 2$, $D[v_1, \cdots, v_n] = 0$ for all $v_i \in V$, $i \in \underline{n}$. If $[V, \cdots, V] = V$, then $Dv = 0$ for all $v \in V$ or $D = 0$. Since $[Dv_1, \cdots, v_n] = D[v_1, v_2, \cdots, v_n] = 0$, $D(V) \subseteq C(V)$. Therefore $D = 0$ if $C(V) = \{0\}$. □

Proof of Theorem 1.1.3:

Set $L' := Der(V)$, $L := Inder(V)$. The first assertion is obvious, since an L'- or L-submodule of V is also an ideal of the n-Lie algebra V. Now let K be of characteristic 0. Since L' operates faithfully and irreducibly on V, it is a reductive Lie algebra (see Theorem A2) , that is, L can be represented as the direct sum of its centre Z and the semisimple Lie algebra $[L', L']$. But each element in Z commutes with those of L and $C(V) = \{0\}$, hence $Z = \{0\}$ by Lemma 1.1.4. Thus L' is semisimple.

Let L_1 be the ideal of L' such that $L' = L \oplus L_1$. Since $[L_1, L] = \{0\}$, it follows once again from Lemma 1.1.4 that $L_1 = \{0\}$. Thus $L' = L$. □

1.2 Simple n-Lie Algebras of Dimension $n+1$

We are concerned here with $(n+1)$-dimensional simple n-Lie algebras. It turns out that each $(n+1)$-dimensional simple n-Lie algebra can be realized as (K^{n+1}, b, f) for some nondegenerate symmetric bilinear form b and some nonzero determinant form f.

Let V be a simple n-Lie algebra over K of dimension $n+1$ with product $[v_1, \cdots, v_n]$. Since $[V, \cdots, V] = V$, the n-Lie product gives an isomorphism $\tau : \wedge^n V \to V$ with $\tau(v_1 \wedge \cdots \wedge v_n) = [v_1, \cdots, v_n]$. Let f be an arbitrary nonzero determinant form on V. We define an isomorphism $\sigma : \wedge^n V \to V^*$ by $\sigma(v_1 \wedge \cdots \wedge v_n)(v_{n+1}) = f(v_1, \cdots, v_{n+1})$, for $v_i \in V$, $i \in \underline{n+1}$. For $u, v \in V$, let $b(u, v) := (\sigma\tau^{-1}(u))(v)$. The bilinear form b has the following properties:

1) b is nondegenerate,

2) for all $v_i \in V$, $i \in \underline{n+1} : b([v_1, \cdots, v_n], v_{n+1}) = f(v_1, \cdots, v_{n+1})$,

3) b is symmetric.

Proof. 1) If $b(u, v) = 0$ for all $v \in V$, then $\sigma\tau^{-1}(u) = 0$ which implies $u = 0$, since σ and τ are isomorphisms. Hence b is nondegenerate.

2) For all $v_i \in V$, $i \in \underline{n+1}$:

$$\begin{aligned} b([v_1, \cdots, v_n], v_{n+1}) &= (\sigma\tau^{-1}([v_1, \cdots v_n]))(v_{n+1}) \\ &= \sigma(v_1 \wedge \cdots \wedge v_n)(v_{n+1}) \\ &= f(v_1, \cdots, v_{n+1}). \end{aligned}$$

3) Let u_i, $v_i \in V$, $i \in \underline{n}$. By (1.3), the G.J.I and 2) we get

$$\begin{aligned} & b([u_1, \cdots, u_n], [v_1, \cdots, v_n]) \\ &= f(u_1, \cdots, u_n, [v_1, \cdots, v_n]) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\ &\quad - \sum_{i=1}^{n} f(u_1, \cdots, u_{i-1}, [v_1, \cdots, v_{n-1}, u_i], u_{i+1}, \cdots, u_n, v_n) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\ &\quad - \sum_{i=1}^{n} b([u_1, \cdots, u_{i-1}, [v_1, \cdots, v_{n-1}, u_i], u_{i+1}, \cdots, u_n], v_n) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\ &\quad - b([v_1, \cdots, v_{n-1}, [u_1, \cdots, u_n]], v_n) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) - \end{aligned}$$

$$\begin{aligned}
&f(v_1, \cdots, v_{n-1}, [u_1, \cdots, u_n], v_n) \\
&= \operatorname{tr}(\operatorname{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\
&\quad + f(v_1, \cdots, v_n, [u_1, \cdots, u_n]) \\
&= \operatorname{tr}(\operatorname{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\
&\quad + b([v_1, \cdots, v_n], [u_1, \cdots, u_n]).
\end{aligned}$$

Therefore

$$\begin{aligned}
&b([u_1, \cdots, u_n], [v_1, \cdots, v_n]) - b([v_1, \cdots, v_n], [u_1, \cdots, u_n]) \\
&= \operatorname{tr}(\operatorname{ad}(v_1, \cdots, v_{n-1})) f(u_1, \cdots, u_n, v_n). \qquad (1.10)
\end{aligned}$$

Suppose that $v_{n-1} \neq 0$. Let $v_n := v_{n-1}$ and let $\{u_1, \cdots, u_n, v_n\}$ be a base of V. Then we get from (1.10) that $\operatorname{tr}(\operatorname{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) = 0$, which leads to $\operatorname{tr}(\operatorname{ad}(v_1, \cdots, v_{n-1})) = 0$. Now (1.10) becomes

$$b([u_1, \cdots, u_n], [v_1, \cdots, v_n]) = b([v_1, \cdots, v_n], [u_1, \cdots, v_n]).$$

Since V is simple, $[V, \cdots, V] = V$, the symmetry of b follows. □

Proposition 1.2.1: *Let V be an $(n+1)$-dimensional n-Lie algebra. If V is simple, then $V = (K^{n+1}, b, f)$ for some nondegenerate symmetric bilinear form b and a nonzero determinant form f.*

The converse is also true.

Proposition 1.2.2: *The n-Lie algebra (K^{n+1}, b, f) is simple.*

Proof: Set $V := (K^{n+1}, b, f)$. Let I be a nonzero proper ideal of V. Set $I^\perp := \{v \in V \mid b(v, u) = 0, \forall u \in I\}$. Then $I^\perp \neq \{0\}$. Let $v_n \in I$ and $v_{n+1} \in I^\perp$ be nonzero elements. If v_n and v_{n+1} are not proportional to each other, then we can choose $v_1, \cdots, v_{n-1} \in V$ so, that $\{v_1, \cdots, v_{n+1}\}$ is a base of V. Since $[v_1, \cdots, v_n] \in I$, we have $f(v_1, \cdots, v_{n+1}) = b([v_1, \cdots, v_n], v_{n+1}) = 0$, which is a contradiction to $f \neq 0$. Thus $I = I^\perp$ and is one dimensional. It follows that $\dim(K^{n+1}) = \dim(I) + \dim(I^\perp) = 2$ and $n = 1$. Hence V possesses no nonzero proper ideal, i.e. V is simple. □

In the following we study when two n-Lie algebras of the form (K^{n+1}, b, f) are isomorphic. Since the n-Lie algebra $(K^{n+1}, b, \alpha f)$ is the same as the n-Lie algebra $(K^{n+1}, \alpha^{-1} b, f)$ for all $\alpha \in K$, $\alpha \neq 0$, we can fix f and assume that $f(e_1, \cdots, e_{n+1}) = 1$, where $\{e_1, \cdots, e_{n+1}\}$ is the canonical base of K^{n+1}.

Let $V_i := (K^{n+1}, b_i, f)$, $i = 1, 2$, be two n-Lie algebras defined as in Example 1.1.1. The n-Lie product $[v_1, \cdots, v_n]_i$ of V_i satisfies the identity:

$$b_i([v_1, \cdots, v_n]_i, v_{n+1}) = f(v_1, \cdots, v_{n+1}).$$

Let τ be a vector space automorphism of K^{n+1}. Then τ is an isomorphism from V_1 onto V_2 if and only if

$$b_2(\tau[v_1, \cdots, v_n]_1, \tau v_{n+1}) = b_2([\tau v_1, \cdots, \tau v_n]_2, \tau v_{n+1}) \tag{1.11}$$

for all $v_i \in V_1$, $i \in \underline{n+1}$. For the right side of the identity we have

$$\begin{aligned} b_2([\tau v_1, \cdots, \tau v_n]_2, \tau v_{n+1}) &= f(\tau v_1, \cdots, \tau v_{n+1}) \\ &= \det\tau \cdot f(v_1, \cdots, v_{n+1}) \\ &= \det\tau \cdot b_1([v_1, \cdots, v_n]_1, v_{n+1}). \end{aligned}$$

Thus (1.11) can be written as

$$b_2(\tau[v_1, \cdots, v_n]_1, \tau v_{n+1}) = \det\tau \cdot b_1([v_1, \cdots, v_n]_1, v_{n+1}).$$

Since V_1 is simple by Proposition 1.1.2, $[V_1, \cdots, V_1]_1 = V_1$. Thus τ is an isomorphism from V_1 onto V_2 if and only if τ satisfies the identity

$$b_2(\tau u, \tau v) = \det\tau \cdot b_1(u, v) \tag{1.12}$$

for all $u, v \in K^{n+1}$. Since for each bilinear form b there exists a base of K^{n+1} relative to which the associated matrix of b has diagonal form and each diagonal matrix multiplied by a scalar remains diagonal, any $(n+1)$-dimensional simple n-Lie algebra is isomorphic to one of the n-Lie algebras (K^{n+1}, b, f), where b runs through the set of the bilinear forms whose associated matrix relative to the canonical base is $\mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$, where $\alpha_i \in K$, $i \in \underline{n+1}$ and $\alpha_1 \cdots \alpha_{n+1} \neq 0$.

If K is algebraically closed, there exists a vector space automorphism σ of K^{n+1} such that $b_2(\sigma u, \sigma v) = b_1(u, v)$. Set $\tau := (\det\sigma)^{-\frac{1}{n-1}}\sigma$. Then τ is clearly also an automorphism of K^{n+1} and fulfills identity (1.12). Indeed, because of $\det\tau = (\det\sigma)^{-\frac{n+1}{n-1}}\det\sigma = (\det\sigma)^{-\frac{2}{n-1}}$ we have

$$\begin{aligned} b_2(\tau u, \tau v) &= (\det\sigma)^{-\frac{2}{n-1}} \cdot b_2(\sigma u, \sigma v) \\ &= (\det\sigma)^{-\frac{2}{n-1}} \cdot b_1(u, v) \\ &= \det\tau \cdot b_1(u, v). \end{aligned}$$

Hence V_1 is isomorphic to V_2 and we have shown

Proposition 1.2.3: *(K^{n+1}, b_1, f) is isomorphic to (K^{n+1}, b_2, f) if and only if there exists an isomorphism τ of K^{n+1} with property (1.12). If K is algebraically closed, then all n-Lie algebras of the form (K^{n+1}, b, f) are isomorphic to each other.*

Remark 1.2.1: If K is algebraically closed, then we mean by the vector product on K^{n+1} $(n > 2)$ the n-Lie algebra (K^{n+1}, b, f), where the determinant form f is normalized by $f(e_1, \cdots, e_n) = 1$ ($\{e_1, \cdots, e_{n+1}\}$ is the canonical base of K^{n+1}) and b is the bilinear form whose associated matrix relative to $\{e_1, \cdots, e_{n+1}\}$ of K^{n+1} is given by

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

according as n is odd or even, for some appropriate unit matrix I.

In the following we want to determine the group $Aut(V)$ of automorphisms of the vector product $V = (K^{n+1}, b, f)$. By Proposition 1.2.3 an automorphism τ of the vector space K^{n+1} is an automorphism of the vector product if and only if

$$b(\tau v, \tau w) = \det\tau \cdot b(v, w), \tag{1.13}$$

As a result of this we have that if τ is an automorphism of V, then $(\det \tau)^{n-1} = 1$ by choosing a base of V. Let $\Omega := \{\alpha \in K \mid \alpha^{n-1} = 1\}$.

Let $\tau \in Aut(V)$ and set $\sigma = \tau/\sqrt{\det \tau}$. Clearly $b(\sigma v, \sigma w) = b(v, w)$, that is, $\sigma \in O(K^{n+1}, b)$, while $\det\sigma = \det\tau/(\sqrt{\det\tau})^{n+1} = \pm 1$, according as $\det\sigma$ is a square in Ω or not. Therefore

$$\begin{aligned} Aut(V) \subseteq\; & \{\alpha\, \sigma \mid \alpha \in \Omega,\ \sigma \in SO(K^{n+1}, b)\} \cup \\ & \{\pm\sqrt{\beta}\, \sigma \mid \beta \in \Omega \setminus \Omega^2,\ \sigma \in O(K^{n+1}, b),\ \det\tau = -1\}. \end{aligned} \tag{1.14}$$

Since the multiplication by an element in Ω belongs to $Aut(V)$ and $SO(K^{n+1}, b) \subseteq Aut(V)$ by (1.13), the first set on the right side of (1.14) is included in $Aut(V)$. If n is even, every element in Ω is a square, then

$$Aut(V) = \{\alpha\, \sigma \mid \alpha \in \Omega,\ \sigma \in SO(K^{n+1}, b)\}. \tag{1.15}$$

Now let n be odd. For any non-square $\beta \in \Omega$ and any $\sigma \in O(K^{n+1}, b)$ with $\det\sigma = -1$, the endomorphism $\sqrt{\beta}\sigma$ is an automorphism of V. Indeed, because of $\det(\sqrt{\beta}\sigma) = \sqrt{\beta}^{n+1}\det\sigma = (-\beta)(-1) = \beta$ we have

$$b(\sqrt{\beta}\sigma u, \sqrt{\beta}\sigma v) = \beta\, b(\sigma u, \sigma v) = \beta\, b(u, v) = \det(\sqrt{\beta}\sigma)\, b(u, v).$$

According to (1.13), $\sqrt{\beta}\sigma$ is an element in $Aut(V)$. Therefore we have proved

$$\begin{aligned} Aut(V) =\; & \{\alpha\, \sigma \mid \alpha \in \Omega,\ \sigma \in SO(K^{n+1}, b)\} \cup \\ & \{\pm\sqrt{\beta}\, \sigma \mid \beta \notin \Omega^2,\ \sigma \in O(K^{n+1}, b),\ \det\tau = -1\}. \end{aligned} \tag{1.16}$$

Remark 1.2.2: If $K = \mathbb{C}$, then we obtain from (1.15) and (1.16) that $Der(V) = so(K^{n+1}, b)$, the set of endomorphisms D of K^{n+1} with $b(Du, v) + b(u, Dv) = 0$. In fact, for any field K of characteristic 0 the Lie algebra of derivations of (K^{n+1}, b, f) is $so(K^{n+1}, b)$. This will be proved in the following.

Let K be an arbitrary field of characteristic 0. Let $V = (K^{n+1}, b, f)$. Let us describe $Der(V)$ and $Inder(V)$. By Proposition 1.2.2 and Theorem 1.1.3, $Der(V) = Inder(V)$. Hence we need to determine the inner derivations only.

We have seen in Example 1.1.1 that the elements of $so(K^{n+1}, b)$ are derivations of V (see (1.5)). Now let D be a derivation of V. For $v_1, \cdots, v_{n+1} \in V$ we get according to (1.4) and (1.9) for all $i \in \underline{n}$

$$\begin{aligned} b(D[v_1, \cdots, v_n], v_{n+1}) &= \sum_{i=1}^{n} b([v_1, \cdots, De_i, \cdots, v_n], v_{n+1}) \\ &= \operatorname{tr}(D)\, b([v_1, \cdots, v_n], v_{n+1}) - b([v_1, \cdots, v_n], D\, v_{n+1}). \end{aligned}$$

Since V is simple, $[V, \cdots, V] = V$, we get $b(Du, v) + b(u, Dv) = \operatorname{tr}(D)\, b(u, v)$. Since $\operatorname{tr}(D) = 0$ (see the proof of 3) at the beginn of this section), it follows that $b(Du, v) + b(u, Dv) = 0$, that is, D is an element in $so(K^{n+1}, b)$. Hence $Inder(V) = so(K^{n+1}, b)$.

For later use we summarize the main results in this section.

Theorem 1.2.4: *Let K be an algebraically closed field. Then there is only one simple n-Lie algebra of dimension $n + 1$ up to isomorphism. A realization of this n-Lie algebra is the vector product (K^{n+1}, b, f). If $\operatorname{char} K = 0$, then the Lie algebra of its derivations is $so(K^{n+1}, b)$ and all derivations are inner.*

Chapter 2

Solvability of n-Lie algebras

In this chapter we are going to study the structure of n-Lie algebras. We shall introduce the concepts of k-solvability and k-semisimplicity for n-Lie algebras. Then we give a characterization of n-semisimple n Lie algebras. The reductive n-Lie algebras will also be studied. Throughout K will be a field of characteristic different from 2.

Let us recall the definition of Kasymov [10] for solvability.

Let V be an n-Lie algebra over K. For a given ideal I of V and a given $k \in \underline{n}$ we define inductively a sequence of ideals $I^{(s,k)}$, $s \in \mathbb{N}_0$, of V:

$$I^{(0,k)} := I, \quad I^{(s+1,k)} := [\underbrace{I^{(s,k)}, \cdots, I^{(s,k)}}_{k}, V, \cdots, V].$$

Clearly we have $I^{(s+1,k)} \subseteq I^{(s,k)}$.

<u>**Definition of k-solvability**</u>:

1) *An ideal I of V is called a k-solvable ideal of V if for some $s \in \mathbb{N}$: $I^{(s,k)} = \{0\}$.*

2) *V is called a k-solvable n-Lie algebra if V is k-solvable as an ideal of itself.*

Let V be an n-Lie algebra and I an ideal of V. According to the definition, if I is a k-solvable ideal of V, then I is a k-solvable subalgebra of V. We shall see in Example 2.1 below that the converse is not true except in case $k = n$. In the following we consider some extrem cases. If $k = n$, the definition agrees with that of Filippov [5]; if $n = 2$, then the 2-solvability is the same as the solvability for Lie algebras. To understand the 1-solvability we define the so-called upper central series. Let $C_0(V) := \{0\}$, $C_{s+1} = \{v \in V \,|\, [V, \cdots, V, v] \subseteq C_s\}$ for $s \geq 0$. Clearly $C_s(V)$ has the following properties:

1) *$C_1(V)$ is the centre $C(V)$ of V.*

2) *For each $s \in \mathbb{N}_0$, $C_s(V)$ is a 1-solvable ideal of V.*

3) *For each $s \in \mathbb{N}_0$, $C_s(V) \subseteq C_{s+1}(V)$.*

4) *If I is a 1-solvable ideal of V with $I^{(s,1)} = \{0\}$ with s minimal, then $I^{(i,1)} \subseteq C_{s-i}(V)$ for all $0 \leq i \leq s$.*

To answer the question which connections there exist among the k-solvabilities of ideals we prove the inclusion $I^{(s,k')} \subseteq I^{(s,k)}$ for $k \leq k'$. The case $s = 0$ is clear, since $I^{(0,k)} = I^{(0,k')} = I$. Suppose now that the inclusion is true for s, we proceed to $s+1$. By definition,

$$\begin{aligned} I^{(s+1,k')} &= [\underbrace{I^{(s,k')}, \cdots, I^{(s,k')}}_{k'}, V, \cdots, V] \\ &\subseteq [\underbrace{I^{(s,k)}, \cdots, I^{(s,k)}}_{k'}, V, \cdots, V] \\ &\subseteq [\underbrace{I^{(s,k)}, \cdots, I^{(s,k)}}_{k}, V, \cdots, V] \\ &= I^{(s+1,k)}. \end{aligned}$$

If I is a k-solvable ideal, then $I^{(s,k)} = \{0\}$ for some $s \in \mathbb{N}$, therefore $I^{(s,k')} = \{0\}$, that is, I is a k'-solvable ideal by definition. Analogously we can show that a k-solvable subalgebra of V is also a k'-solvable subalgebra of V.

Proposition 2.1: *Let V be an n-Lie algebra and $1 \leq k \leq k' \leq n$. A k-solvable ideal (subalgebra) of V is also a k'-solvable ideal (subalgebra) of V. In particular, the k-solvability implies n-solvability for all k.*

Example 2.1: One can show that an $(n+1)$-dimensional space V with base $\{e_1, \cdots, e_{n+1}\}$ and multiplication defined by the formulas $[e_1, \cdots, \widehat{e_i}, \cdots, e_{n+1}] = \alpha_i e_i$, $\alpha_i \in K$, $i \in \underline{n+1}$, is an n-Lie algebra for any choice of the constants α_i (cf. Filippov [5]). Now suppose $\alpha_1 \cdots \alpha_k \neq 0$, $\alpha_{k+1} = \cdots = \alpha_{n+1} = 0$, $2 \leq k \leq n$. Then the ideal $I := V^{(1,k)} = [V, \cdots, V]$ has base $\{e_1, \cdots, e_k\}$ and $V^{(2,k)} = [\underbrace{I, \cdots, I}_{k}, V, \cdots, V] = \{0\}$, hence V is a k-solvable n-Lie algebra and I a k-solvable ideal of V. Since $[I, \cdots, I] = \{0\}$, I is l-solvable subalgebra of V for all $1 \leq l \leq n$. Further we can check that $V^{(2,k-1)} = [\underbrace{I, \cdots, I}_{k-1}, V, \cdots, V] = I$, hence V is not $(k-1)$-solvable and I is not a $(k-1)$-solvable ideal of V (I is also not l-solvable ideal of V for all $l \leq k-1$ by Proposition 2.1).

Indeed, V has no nonzero $(k-1)$-solvable ideal if $k \geq 3$. To show this we prove that each nonzero ideal of V contains I. Let J be an arbitrary nonzero ideal of V and $0 \neq v \in J$, $v = \sum_{j=1}^{n+1} \beta_j e_j$, $\beta_j \in K$. Then there is a $\beta_{j_0} \neq 0$. Let $i \leq k$, $i \neq j_0$ (such an i exists because $k \geq 3$), then

$$J \ni [e_1, \cdots, \widehat{e_i}, \cdots, e_{j_0-1}, v, e_{j_0+1}, \cdots, e_{n+1}]$$

$$= \alpha_i\beta_{j_0}\, e_i + (-1)^{j_0-i-1}\alpha_{j_0}\beta_i\, e_{j_0}.$$

Let $i_0 \le k$ such that $i_0 \ne i$, $i_0 \ne j_0$. Then

$$[e_1, \cdots, \widehat{e_{i_0}}, \cdots, e_{i-1}, \alpha_i\beta_{j_0}\, e_i + (-1)^{j_0-i-1}\alpha_{j_0}\beta_i\, e_{j_0}, e_{i+1}, \cdots, e_{n+1}] = \alpha_{i_0}\alpha_i\beta_{j_0}e_{i_0}.$$

Therefore $e_{i_0} \in J$. This forces $e_i \in J$ for all $i \le k$. Hence $I \subseteq J$.

Since I is not $(k-1)$-solvable ideal of V, V possesses no nonzero $(k-1)$-solvable ideal of V if $k \ge 3$.

In the following proposition we give some properties of k-solvable ideals.

Proposition 2.2: *Let V be an n-Lie algebra and $k \in \underline{n}$.*

1) *If V is k-solvable , then all subalgebras of V are k-solvable and all ideals of V are k-solvable ideals of V.*

2) *Let $\psi : V \longrightarrow V_1$ be a surjective n-Lie algebra homomorphism and I a k-solvable ideal of V, then ψI is a k-solvable ideal of V_1.*

3) *Let I, J be ideals of V, $J \subseteq I$. If J is a k-solvable ideal of V and I/J a k-solvable ideal of V/J, then I is a k-solvable ideal of V. In particular, if J is a k-solvable ideal of V such that V/J is a k-solvable n-Lie algebra, then V itself is k-solvable.*

4) *If I and J are k-solvable ideals of V, so is $I + J$.*

Proof: 1) If U is a subalgebra of V, then $U^{(s,k)} \subseteq V^{(s,k)}$. This implies the first assertion. If I is an ideal of V, then $I^{(s,k)} \subseteq V^{(s,k)}$. From it the second assertion follows.

2) It can be shown inductively that $\psi(I)^{(s,k)} = \psi(I^{(s,k)})$. This implies the assertion.

3) Denote the canonical homomorphism $V \longrightarrow V/J$ by π. Since I/J is a k-solvable ideal of V/J, we have $\pi(I^{(s,k)}) = \pi(I)^{(s,k)} = (I/J)^{(s,k)} = \{0\}$ for some s, hence $I^{(s,k)} \subseteq J$. But $J^{(s',k)} = \{0\}$ for some $s' \in \mathbb{N}$, so $I^{(s+s',k)} = (I^{(s,k)})^{(s',k)} \subseteq J^{(s',k)} = \{0\}$. Therefore I is a k-solvable ideal of V.

4) We consider the canonical homomorphism $\pi : V \longrightarrow V/J$. Since I is a k-solvable ideal of V, so $I/J = \pi(I)$ is a k-solvable ideal of V/J in view of 2). But $\pi^{-1}(I/J) = I + J$, so $I + J$ is a k-solvable ideal of V in view of 3). □

From now on we assume that V is finite dimensional over K. Let I be a maximal k-solvable ideal of V and J an arbitrary k-solvable ideal of V. According to Proposition 2.2 4) $I + J$ is also a k-solvable ideal of V which in turn implies

$I + J = I$ or $J \subseteq I$ due to the maximality of I. Hence we have proved the existence of a unique maximal k-solvable ideal of V.

Definition: *Let V be an n-Lie algebra, $k \in \underline{n}$.*

1) *The maximal k-solvable ideal of V is called the k-radical of V and will be denoted by $Rad_k(V)$.*

2) *If $Rad_k(V) = \{0\}$, then we call V is k-semisimple.*

Remark 2.1: It is proved by Kasymov [10] that $Rad_k(V)$ is invariant under all derivations of V, that is, $D(v) \in Rad_k(V)$ for any $D \in Der(V)$ and any $v \in Rad_k(V)$.

Recall that V is assumed to be finite dimensional. Thus the upper central series of V will be stationary, i.e. there exists an $s \in \mathbb{N}$ such that $C_s(V) = C_{s+1}(V)$. Assume that s is minimal. Since $C_s(V)$ is 1-solvable, $C_s(V) \subseteq Rad_1(V)$. On the other hand, $Rad_1(V)$ is included in some $C_{s'}(V)$ for $s' \leq s$. Thus $Rad_1(V) \subseteq C_s(V)$. Together with the foregoing inclusion we get $Rad_1(V) = C_s(V)$, in other words, the $Rad_1(V)$ is the greatest element in the upper central series. If V is 1-semisimple, then $C_s(V) = \{0\}$ forces $C(V) = \{0\}$; conversely, if $C(V) = \{0\}$, then $C_s(V) = \{0\}$ and V is 1-semisimple. Therefore an n-Lie algebra V is 1-semisimple if and only if $C(V) = \{0\}$.

Since a k-solvable ideal of an n-Lie algebra is also a k'-solvable ideal of the given n-Lie algebra for $k \leq k'$, we have

Proposition 2.3: *Let $k, k' \in \mathbb{N}$, $1 \leq k \leq k' \leq n$. If an n-Lie algebra V is k'-semisimple, then V is also k-semisimple. In particular, if V is n-semisimple, then V is k-semisimple for all $1 \leq k \leq n$.*

Example 2.2: For $k \geq 3$ the n-Lie algebra in Example 2.1 is $(k-1)$-semisimple but k-solvable. A simple n-Lie algebra is k-semisimple for all $k \in \underline{n}$.

Theorem 2.4: *Let V be an n-Lie algebra, then $V/Rad_k(V)$ is k-semisimple.*

Proof: Denote by $\pi : V \longrightarrow V/Rad_k(V)$ the canonical homomorphism. If I is the k-radical of $V/Rad_k(V)$, then we derive from $\pi^{-1}(I) \supseteq Rad_k(V)$ and 3) in Proposition 2.2 that $\pi^{-1}(I)$ is a k-solvable ideal of V, therefore $\pi^{-1}(I) \subseteq Rad_k(V)$, this implies $I = \{0\}$. □

In what follows we shall mean n-solvability by solvability and correspondingly n-semisimple by semisimple. Instead of $Rad_n(V)$ we shall write $Rad(V)$. In the following we introduce direct sums of n-Lie algebras before we give a characterization of semisimple n-Lie algebras over a field of characteristic 0.

Let V_i be n-Lie algebras, $i = 1, 2$. It can easily be proved that the vector space direct sum $V = V_1 + V_2$ is an n-Lie algebra with regard to the product

$$[u_1 + v_1, \cdots, u_n + v_n] := [u_1, \cdots, u_n] + [v_1, \cdots, v_n] \quad u_i \in V_1,\ v_i \in V_2,\ i \in \underline{n}.$$

Evidently V_i, $i = 1, 2$, are ideals of V. We call the n-Lie algebra V the direct sum of the n-Lie algebras V_1 and V_2 and write $V = V_1 \oplus V_2$. This definition can be generalized to direct sum of m n-Lie algebras: $V = \oplus_{i=1}^m V_i$. In this situation we have

Theorem 2.5: *If $V = \oplus_{i=1}^m V_i$ is a direct sum of n-Lie algebras V_i, $i \in \underline{m}$, then*

1) $Inder(V) \cong \oplus_{i=1}^m Inder(V_i)$,

2) For all $k \in \underline{n}$: $Rad_k(V) = \oplus_{i=1}^m Rad_k(V_i)$.

For the following proposition we delete the assumption that the n-Lie algebra V is finite dimensional over the field K. For the sake of convenience we write L' for $Der(V)$ and L for $Inder(V)$. If M is a subspace of L' (or L) and I is a subspace of V, we denote by $M^s(I)$ the subspace of V which is spanned by all elements of the form $D_1 D_2 \cdots D_s(v)$, $D_i \in M$, $v \in I$.

Proposition 2.6: *Let V be an arbitrary n-Lie algebra. If M is an ideal of L' (or L) and I an ideal of V, then $M(I)$ is an ideal of V and $(M(I))^{(s,n)} \subseteq M^{s+1}(I)$ for $s \geq 1$.*

Proof: 1) $[M(I), V, \cdots, V] = L(M(I)) \subseteq [L, M](I) + M(L(I)) \subseteq M(I)$. Thus $M(I)$ is an ideal of V.

2) We prove the inclusion inductively. Since $M(I)$ and $M^2(I)$ are ideals of V by 1), it follows that

$$\begin{aligned}
&(M(I))^{(1,n)} = [M(I), \cdots, M(I)] \\
&\subseteq M([I, M(I), \cdots, M(I)]) + [I, MM(I), M(I), \cdots, M(I)] \\
&\subseteq M^2(I),
\end{aligned}$$

that is, $(*)$ $\quad (M(I))^{(1,n)} \subseteq M^2(I)$. Suppose the inclusion is true for s. Replacing I in $(*)$ by $M^s(I)$ we obtain: $(M^{s+1}(I))^{(1,n)} \subseteq M^{s+2}(I)$. By induction hypothesis, $(M(I))^{(s+1,n)} \subseteq ((M(I))^{(s,n)})^{(1,n)} \subseteq M^{s+2}(I)$. □

Now we are ready to give a characterization of finite dimensional semisimple n-Lie algebras over K of characteristic 0. We remark that a simple ideal of an n-Lie algebra is an ideal of the given n-Lie algebra which is simple as an n-Lie algebra.

Theorem 2.7: *Let K be of characteristic 0. An n-Lie algebra V is semisimple if and only if V is a direct sum of simple ideals: $V = \oplus_{i=1}^m V_i$, $m \in \mathbb{N}$. Moreover, $Der(V)$ is semisimple and each derivation of V is inner.*

Proof: If V is a direct sum of simple ideals, then V is semisimple by Theorem 2.5.

Let now V be semisimple. We prove that V is a direct sum of simple ideals. Let R be the radical of L' (recall $L' = Der(V)$ and $L = Inder(V)$) and $M := [R, L']$. According to Theorem 12.38 in [13] M is included in the radical of the associative subalgebra of End(V) generated by L', hence $M^{s+1} = \{0\}$ for some $s \in \mathbb{N}$. By Proposition 2.6, $(M(V))^{(s,n)} \subseteq M^{s+1}(V) = \{0\}$. This says that $M(V)$ is a solvable ideal of V, hence by the assumption, $M(V) = \{0\}$ which implies $M = \{0\}$, that is, R coincides with the centre of L'. Therefore L' is reductive.

Let Z be the centre of L'. Since V is semisimple, $C(V) = \{0\}$. Applying Lemma 1.1.4 to elements of Z we obtain $Z = \{0\}$. This means that L' is a semisimple Lie algebra. Let L_1 be the ideal of L' with $L' = L \oplus L_1$. Once again by Lemma 1.1.4, $L_1 = \{0\}$ because of $[L_1, L] = \{0\}$. Therefore $L' = L$.

Now the L-module V is completely reducible. Suppose that $V = \oplus_{i=1}^m V_i$, where V_i are irreducible L-modules. This is at the same time a direct sum of ideals of V. If V_i is one dimensional, then $V_i \subseteq C(V)$. But $C(V) = \{0\}$ by the assumption. Therefore V_i is nontrivial. Since V_i is an irreducible L-module, V_i contains no ideal of V. But each ideal of V_i is also an ideal of V, hence V_i is simple as a subalgebra of V. This completes the proof of Theorem 2.7. □

Corollary 2.8: *Let K be of characteristic 0. If V is a semisimple n-Lie algebra, then every simple ideal of V coincides with one of the V_i in Theorem 2.7 and any ideal is a direct sum of certain simple ideals.*

Proof: Let $V = \oplus V_i$ be the decomposition of V into simple ideals. If I is a simple ideal of V, then $[I, V, \cdots, V]$ is an ideal of V, nonzero because the centre of V is zero. This forces $[I, V, \cdots, V] = I$, since I is simple. On the other hand, $[I, V, \cdots, V] = \oplus_{i=1}^m [I, V_i, \cdots, V_i]$, so all but one summand must be 0. Say $I = [I, V_i, \cdots, V_i]$. Then $I \subseteq V_i$, and $I = V_i$ because V_i is simple. Therefore V_i are all simple ideals of V, $i \in \underline{m}$.

Next we prove that each ideal of V is the sum of some V_i's. Let I be an arbitrary ideal of V. Then I is an invariant subspace of the L-module V. But L is semisimple, I is a direct sum of some irreducible L-submodule of V, in other words, I is a direct sum of simple ideals of V. □

In the following we are concerned with the connection between the radical of the Lie algebra of derivations of V and the radical of V. For this purpose we make the following consideration.

Let V_i, $i = 1, 2$, be two n-Lie algebras over an arbitrary field K. Let L_i denote the Lie algebra of deivations of V_i, $i = 1, 2$. Suppose that $\pi : V_1 \to V_2$ is a surjective homomorphism of n-Lie algebras such that the kernel of π is invariant under L_1. Define

$$\gamma : L_1 \to L_2 \quad \gamma(D)(\pi(v)) := \pi(D(v)), \quad D \in L_1,\, v \in V. \tag{2.1}$$

Evidently $\gamma(D)$ is an endomorphism of V_2. It is even an element of L_2 because we have for all $v_i \in V_1$, $i \in \underline{n}$:

$$\begin{aligned}
&\gamma(D)[\pi(v_1), \cdots, \pi(v_n)] \\
&= \gamma(D)(\pi[v_1, \cdots, v_n]) \\
&= \pi(D[v_1, \cdots, v_n]) \\
&= \pi(\sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, D(v_i), v_{i+1}, \cdots, v_n]) \\
&= \sum_{i=1}^{n} [\pi(v_1), \cdots, \pi(v_{i-1}), \pi(D(v_i)), \pi(v_{i+1}), \cdots, \pi(v_n)] \\
&= \sum_{i=1}^{n} [\pi(v_1), \cdots, \pi(v_{i-1}), \gamma(D)(\pi(v_i)), \pi(v_{i+1}), \cdots, \pi(v_n)].
\end{aligned}$$

Let $D_1, D_2 \in L_1$ and $v \in V_1$. Then

$$\begin{aligned}
\gamma([D_1, D_2])(\pi(v)) &= \pi([D_1, D_2]v) \\
&= \pi(D_1 D_2(v) - D_2 D_1(v)) \\
&= \gamma(D_1)(\pi(D_2(v))) - \gamma(D_2)(\pi(D_1(v))) \\
&= \gamma(D_1)\gamma(D_2)(\pi(v)) - \gamma(D_2)\gamma(D_1)(\pi(v))) \\
&= [\gamma(D_1), \gamma(D_2)](\pi(v)).
\end{aligned}$$

Hence γ is a Lie algebra homomorphism from L_1 to L_2. Moreover, we have for any $\mathrm{ad}_1(u_1, \cdots, u_{n-1}) \in L_1$:

$$\begin{aligned}
\gamma(\mathrm{ad}_1(u_1, \cdots, u_{n-1}))(\pi(v)) &= \pi([u_1, \cdots, u_{n-1}, v]) \\
&= [\pi(u_1), \cdots, \pi(u_{n-1}), \pi(v)] \\
&= \mathrm{ad}_2(\pi(u_1), \cdots, \pi(u_{n-1}))(\pi(v)),
\end{aligned}$$

that is, $\gamma(\mathrm{ad}_1(u_1, \cdots, u_{n-1})) = \mathrm{ad}_2(\pi(u_1), \cdots, \pi(u_{n-1}))$. Therefore

$$\gamma(Inder(V_1)) = Inder(V_2). \tag{2.2}$$

Now let V be an n-Lie algebra over a field K. Since the canonical homorphism $\pi : V \to V/Rad(V)$ is surjective and its kernel $Rad(V)$ is invariant under all derivations of V (see Remark 2.1), we can define (as (2.1)) a homomorphism $\gamma : Der(V) \to Der(V/Rad(V))$ with

$$Ker\,\gamma = \{D \in Der(V) \mid D(V) \subseteq Rad(V)\}. \tag{2.3}$$

By Theorem 2.4, $V/Rad(V)$ is semisimple. If char$K = 0$, then $Der(V/Rad(V))$ agrees with $Inder(V/Rad(V))$ and is semisimple by Theorem 2.7. But by (2.2), $\gamma(Inder(V)) = Inder(V/Rad(V))$, it follows that γ is a surjective Lie algebra homomorphism and Rad (Der(V)) $\subseteq Ker\gamma$. So we have proved

Theorem 2.9: *Let K be of characteristic 0. Let V be an n-Lie algebra over K. Let $\pi : V \to V/Rad(V)$ be the canonical homomorphism. Define a map $\gamma : Der(V) \to Inder(V/Rad(V))$ by $\gamma(D)(\pi(v)) := \pi(D(v))$. Then γ is a surjective Lie algebra homomorphism with the kernel given in (2.3). Moreover $Rad(Der(V)) \subseteq Ker\gamma$ and $Rad(Der(V))(V) \subseteq Rad(V)$.*

We now generalize the concept of reductive Lie algebras.

Definition: *An n-Lie algebra V is called reductive if $Rad(V) = C(V)$.*

Example 2.3: Let $n \geq 3$ and $V = (K^{n+1}, b, f)$ be the n-Lie algebra with the vector product. For given $x \in V$ we consider the $(n-1)$-Lie algebra $V(x)$ on K^{n+1} with the product $[v_1, \cdots, v_{n-1}]_x := [v_1, \cdots, v_{n-1}, x]$ (see Remark 1.1.1). Obviously Kx is an ideal of $V(x)$ included in the centre of $V(x)$. Set $V_0 := \{v \in V \mid b(v,x) = 0\}$. Now

$$\begin{aligned} & b([V(x), \cdots, V(x)]_x, x) \\ &= b([V, \cdots, V, x], x) \\ &= f(V, \cdots, V, x, x) = \{0\}. \end{aligned}$$

Thus $[V(x), \cdots, V(x)]_x \subseteq V_0$, which implies that V_0 is an ideal of $V(x)$.

Case 1: $b(x,x) = 0$.

By the assumption $x \in V_0$. Let W be a subspace of V_0 such that the sum $V_0 = Kx + W$ is direct. Then $V_0^{(1,n-1)} = [V_0, \cdots, V_0]_x = [W, \cdots, W]_x$ and has dimension 1 (notice dim$W = n-1$). Therefore V_0 is solvable. As a consequence $V(x)$ is solvable because $[V(x), \cdots, V(x)]_x \subseteq V_0$.

Case 2: $b(x,x) \neq 0$.

Since $V = Kx + V_0$ (direct sum of vector spaces), the $(n-1)$-Lie algebra $V(x)$ is the direct sum of its ideals Kx and V_0. We show that V_0 is a simple $(n-1)$-Lie algebra.

Let b_0 be the restriction of b to V_0 and f_0 be the nonzero determinant form on V_0 defined by $f_0(v_1, \cdots, v_n) := -f(v_1, \cdots, v_n, x)$. Then b_0 is a nondegenerate symmetric bilinear form on V_0 and for all $v_i \in V_0$, $i \in \underline{n}$ we have

$$\begin{aligned} b_0([v_1, \cdots, v_{n-1}]_x, v_n) &= b_0([v_1, \cdots, v_{n-1}, x], v_n) \\ &= f(v_1, \cdots, v_{n-1}, x, v_n) \\ &= -f(v_1, \cdots, v_{n-1}, v_n, x) \\ &= f_0(v_1, \cdots, v_{n-1}, v_n). \end{aligned}$$

This means that the $(n-1)$-Lie algebra V_0 can be realized as (K^n, b_0, f_0). Therefore V_0 is simple by Proposition 1.2.2. It follows that Kx is the centre of $V(x)$ and $V(x)$ is reductive.

Now let K be algebraically closed. Let $y \in V$, $y \neq x$, be such that $b(y,y) \neq 0$. We may ask whether the reductive $(n-1)$-Lie algebras $V(x)$ and $V(y)$ are isomorphic. Let $V(y) = Ky \oplus V_1$. Since V_0 and V_1 are simple $(n-1)$-Lie algebras, there exists an $(n-1)$-Lie algebra isomorphism T from V_0 onto V_1 (see Theorem 1.2.4). Extend T to $V(x)$ linearly via:

$$T_{x,y}(v + \alpha x) := T(v) + \alpha y, \quad v \in V_0, \ \alpha \in K.$$

One can easily check that $T_{x,y}$ is an $(n-1)$-Lie algebra isomorphism.

We can also construct an isomorphism from $V(x)$ onto $V(y)$ as follows. Since $b(x,x) \neq 0$ and $b(y,y) \neq 0$, there exists a scalar $\alpha \in K$, $\alpha \neq 0$ and an element $\tau \in O(K^{n+1}, b)$ such that $y = \alpha\, \tau(x)$. Let $\tau' := (\alpha \det(\tau))^{-\frac{1}{n-2}} \tau$. From

$$\begin{aligned} b([\tau' v_1, \cdots, \tau' v_{n-1}]_y, v_n) &= (\alpha \det(\tau))^{-\frac{n-1}{n-2}} b([\tau v_1, \cdots, \tau v_{n-1}]_y, v_n) \\ &= (\alpha \det(\tau))^{-\frac{n-1}{n-2}} f(\tau v_1, \cdots, \tau v_{n-1}, y, v_n) \\ &= (\alpha \det(\tau))^{-\frac{1}{n-2}} f(v_1, \cdots, v_{n-1}, x, \tau^{-1} v_n) \\ &= (\alpha \det(\tau))^{-\frac{1}{n-2}} b([v_1, \cdots, v_{n-1}]_x, \tau^{-1} v_n) \\ &= (\alpha \det(\tau))^{-\frac{1}{n-2}} b(\tau [v_1, \cdots, v_{n-1}]_x, v_n) \\ &= b(\tau' [v_1, \cdots, v_{n-1}]_x, v_n) \end{aligned}$$

it follows that $\tau'[v_1, \cdots, v_{n-1}]_x = [\tau' v_1, \cdots, \tau' v_{n-1}]$ for all $v_i \in V$, $i \in \underline{n-1}$, that is, τ' is an isomorphism from $V(x)$ onto $V(y)$.

In the following theorem we give two criteria for reductivity.

Theorem 2.10: *Let K be of characteristic 0 and V be an n-Lie algebra V over K. The following are equivalent.*

1) $Inder(V)$ is semisimple.

2) $V = C(V) \oplus V_0$, where $C(V)$ is the centre of V and V_0 is a semisimple n-Lie subalgebra of V.

3) V is reductive.

Proof: 1) $\Rightarrow$ 2) If L $(= Inder(V))$ is semisimple, then the L-module V is completely reducible. We decompose V into a direct sum of irreducible submodules: $V = \oplus V_i$. This is at the same time a direct sum of ideals of V. If V_i is one dimensional, that is, $L(V_i) = \{0\}$, then $V_i \subseteq C(V)$. If V_i is not trivial, then V_i is simple as a subalgebra of V (see the proof of Theorem 2.7). Hence $V = C(V) \oplus V_0$, where V_0 is the sum of the V_i with $\dim(V_i) > 1$ and thus semisimple.

2) $\Rightarrow$ 3) It is clear by Theorem 2.5.

3) $\Rightarrow$ 1) Suppose that V is reductive, that is, $Rad(V)$ is the centre $C(V)$ of V. Let R denote the radical of L. Since R is included in the radical of $Der(V)$ (note R is included in the intersectionof $Inder(V)$ with the radical of $Der(V)$, cf. [18] p. 204), we have $R(V) \subseteq C(V)$ by Theorem 2.9. Now

$$[R, \mathrm{ad}(V, \cdots, V)] \subseteq \mathrm{ad}(C(V), V, \cdots, V) = \{0\}$$

implies $R \subseteq Z$, the centre of L, thus L is reductive. Let L_0 be the semisimple subalgebra with $L = Z \oplus L_0$.

It remains to show that $Z = \{0\}$. Let us regard V as an L_0-module. Since L_0 is semisimple, there exists an L_0-invariant subspace V_0 of V such that $V = C(V) \oplus V_0$. Since $Z(C(V)) = \{0\}$, it suffices to prove that $Z(V_0) = \{0\}$. Let π denote the canonical homomorphism from V onto $V/Rad(V)$ and let γ be as in Theorem 2.9. Then $\gamma(L_0) = \gamma(L) = Inder(V/Rad(V))$ and $\pi(V_0) = \pi(V) = V/Rad(V)$. It follows that $\pi(L_0(V_0)) = \gamma(L_0)(\pi(V_0)) = V/Rad(V)$. Comparing dimension shows that $L_0(V_0) = V_0$. From $[Z, L_0] = \{0\}$ and $Z(V) \subseteq C(V)$ we get: $Z(V_0) = Z(L_0(V_0)) = L_0(Z(V_0)) \subseteq L_0(C(V)) = \{0\}$. □

One might ask whether every n-Lie algebra V possesses an ideal V_0 such that V is the direct sum of V_0 and its radical. The answer is no. In the following example we construct an n-Lie algebra $V = I + V_1$, where I is an abelian ideal and V a subalgebra (not an ideal) of V and $I \cap V_1 = \{0\}$. If V_1 is semisimple, then I is the radical of V. We shall see in chapter 5 that each n-Lie algebra over an algebraically closed field of characteristic 0 is a vector space direct sum of its radical and a semisimple subalgebra.

Example 2.4: Let V_1 be an n-Lie algebra with product $[v_1, \cdots, v_n]_1$. Let I be an $Inder(V_1)$-module such that

$$\mathrm{ad}_1(u_1, \cdots, u_{n-2}, [v_1, \cdots, v_n]_1).w$$

$$= \sum_{i=1}^{n} (-1)^{n-i} \mathrm{ad}_1(v_1, \cdots, \widehat{v_i}, \cdots, v_n).\mathrm{ad}_1(u_1, \cdots, u_{n-2}, v_i).w \tag{2.4}$$

(see (1.7)). Set $V := I + V_1$ (direct sum). For v_i from I or V ($i \in \underline{n}$) let

$$[v_1, \cdots, v_n] := \begin{cases} [v_1, \cdots, v_n]_1 & \text{if } v_i \in V_1 \text{ for all } i \\ (-1)^{n-i}\mathrm{ad}_1(v_1, \cdots, \widehat{v_i}, \cdots, v_n).v_i & \text{if only } v_i \in I \\ 0 & \text{if at least two of } v_i \in I \end{cases}$$

Extending it linearly to V, then we get an alternating n-ary operation $[v_1, \cdots, v_n]$ on V with respect to which V becomes an n-Lie algebra.

In fact, for the $2n-1$ elements u_i ($i \in \underline{n-1}$) and v_i ($i \in \underline{n}$) in V_1 or I let

$$\begin{aligned} a &:= [u_1, \cdots, u_{n-1}, [v_1, \cdots, v_n]] \\ b &:= \sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], v_{i+1}, \cdots, v_n]. \end{aligned}$$

If all u_i and all v_i are in V_1, then $a = b$, since V_1 is an n-Lie algebra; if only one of the u_i is in I and all v_i are in V_1 (since the product is alternating, we may assume that $u_{n-1} \in I$), then by (2.4)

$$\begin{aligned} a &= -[u_1, \cdots, u_{n-2}, [v_1, \cdots, v_n], u_{n-1}] \\ &= -\mathrm{ad}_1(u_1, \cdots, u_{n-2}, [v_1, \cdots, v_n]_1).u_{n-1} \\ &= -\sum_{i=1}^{n} (-1)^{n-i} \mathrm{ad}_1(v_1, \cdots, \widehat{v_i}, \cdots, v_n).\mathrm{ad}_1(u_1, \cdots, u_{n-2}, v_i).u_{n-1} \\ &= \sum_{i=1}^{n} (-1)^{n-i} [v_1, \cdots, \widehat{v_i}, \cdots, v_n, [u_1, \cdots, u_{n-1}, v_i]] \\ &= b; \end{aligned}$$

if all u_i are from V and only one of the v_i is from I, say v_n, then

$$\begin{aligned} a &= \mathrm{ad}_1(u_1, \cdots, u_{n-1}).\mathrm{ad}_1(v_1, \cdots, v_{n-1}).v_n \\ &= [\mathrm{ad}_1(u_1, \cdots, u_{n-1}), \mathrm{ad}_1(v_1, \cdots, v_{n-1})].v_n \\ &\qquad + \mathrm{ad}_1(v_1, \cdots, v_{n-1}).\mathrm{ad}_1(u_1, \cdots, u_{n-1}).v_n \\ &= \sum_{i=1}^{n-1} \mathrm{ad}_1(v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i]_1, v_{i+1}, \cdots, v_{n-1}).v_n \\ &\qquad + \mathrm{ad}(v_1, \cdots, v_{n-1}).\mathrm{ad}(u_1, \cdots, u_{n-1}).v_n \\ &= b; \end{aligned}$$

if at least two of the $2n-1$ vectors are from I, we have always $a=b=0$. In any case, $a=b$. Therefore V is an n-Lie algebra. Moreover, it is clear according to the definition that I is an ideal and V_1 a subalgebra of V. □

Ramark 2.2: A natural question is whether there exists an Inder(V_1)-module I with the property (2.4). The answer to this question is 'yes'. Indeed, let I be a space with $\dim(I)=\dim(V_1)=m$. Let $\{e_i\}_{i\in \underline{m}}$ be a basis of V_1 and $\{f_i\}_{i\in \underline{m}}$ of I. For $v_i\in V_1$, $i\in \underline{n-1}$, let $\rho(\mathrm{ad}(v_1,\cdots,v_{n-1}))$ be the endomorphism of I, for which the matrix relative to the basis $\{f_i\}_{i\in \underline{m}}$ equals the matrix for $\mathrm{ad}(v_1,\cdots,v_{n-1})$ relative to the basis $\{e_i\}_{i\in \underline{m}}$. Obviously $\rho(\mathrm{ad}(v_1,\cdots,v_{n-1}))$ is well defined and it gives rise to a representation of $Inder(V_1)$ in I. Moreover we have

$$\begin{aligned}&\rho(\mathrm{ad}(u_1,\cdots,u_{n-2},[v_1,\cdots,v_n]))\\&=\sum_{i=1}^{n}(-1)^{n-i}\rho(\mathrm{ad}(v_1,\cdots,\widehat{v_i},\cdots,v_n))\rho(\mathrm{ad}(u_1,\cdots,u_{n-2},v_i)).\end{aligned}$$

Chapter 3

Classification of simple n-Lie algebras

Let K be an algebraically closed field of characteristic 0. Theorem 2.7 shows that the finite dimensional semisimple n-Lie algebras over K can be decomposed into a direct sum of simple n-Lie algebras. Therefore in order to determine the finite dimensional semisimple n-Lie algebras we have to study the finite dimensional simple n-Lie algebras over K. By Theorem 1.1.3 the derivation algebra of such an n-Lie algebra is semisimple and acts irreducibly on the n-Lie algebra itself. This suggests the theory of representations of semisimple Lie algebras for our purpose. In this chapter we shall show that there is for every $n \geq 3$ only one finite dimensional simple n-Lie algebra over K up to isomorphism and this is just the n-Lie algebra with the vector product. With the help of this result we shall give all finite dimensional real simple n-Lie algebras up to isomorphism.

First let K be an arbitrary field. Let V be an n-Lie algebra over K. Notice that $\mathrm{ad} : \times^{n-1}V \to Inder(V)$ is a map such that for all $D \in Inder(V)$,

$$[D,\ \mathrm{ad}(v_1, \cdots, v_{n-1})] = \sum_{i=1}^{n-1} \mathrm{ad}(v_1, \cdots, v_{i-1}, Dv_i, v_{i+1}, \cdots, v_{n-1}),$$

and the the associated map $(v_1, \cdots, v_n) \to \mathrm{ad}(v_1, \cdots, v_{n-1})v_n$ from $\times^n V$ to V is alternating. If we regard V as an $Inder(V)$-module, then ad induces an $Inder(V)$-module morphism from $\wedge^{n-1}V$ to $Inder(V)$ (which we denote also by ad) such that the map $(v_1, \cdots, v_n) \to \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n$ is alternating. Conversely, if (L, V, ad) is a triple with L a Lie algebra, V an L-module and ad an L-module morphism from $\wedge^{n-1}V$ to L such that the map $(v_1, \cdots, v_n) \to \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n$ from $\times^n V$ to V is alternating, then V becomes an n-Lie algebra by

$$[v_1, \cdots, v_n] := \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1}).v_n.$$

Therefore we get a correspondence between the set of n-Lie algebras and the set of the triples (L, V, ad).

Let $\tau : V_1 \to V_2$ be an n-Lie algebra isomorphism. Let $L_i := Inder(V_i)$ and let ad_i be the map from $\wedge^{n-1}V_i \to L_i$, $i = 1, 2$. Then $\gamma : L_1 \to L_2$, defined by $\gamma(D)(\tau v) = \tau(Dv)$, is a Lie algebra isomorphism (see (2.1)) and

$$\gamma(\mathrm{ad}_1(v_1 \wedge \cdots \wedge v_{n-1})) = \mathrm{ad}_2(\tau v_1 \wedge \cdots \wedge \tau v_{n-1}) \tag{3.1}$$

for all $v_i \in V_1$, $i \in \underline{n-1}$. Conversely, if $(L_i, V_i, \mathrm{ad}_i)$, $i = 1, 2$, are two triples and if $\gamma : L_1 \to L_2$ is a Lie algebra isomorphism and $\tau : V_1 \to V_2$ is a vector space isomorphism such that identity (3.1) holds, then τ is an isomorphism of the associated n-Lie algebras. Therefore we shall view two triples with the above property as equivalent. It remains to determine the triples (L, V, ad) up to isomorphism of L and the equivalence of V.

From now on let K be an algebraically closed field of characteristic 0. We will assume that all vector spaces appearing in the following are finite dimensional over K.

If V is a simple n-Lie algebra over K, then Theorem 1.1.3 shows that the Lie algebra $Inder(V)$ ($= Der(V)$) is semisimple and that V as an $Inder(V)$-module is faithful and irreducible. Moreover the $Inder(V)$-module morphism ad: $\wedge^{n-1}V \to Inder(V)$ is surjective. If (L, V, ad) is a triple such that

G1) *L is a nonzero semisimple Lie algebra over K,*

G2) *V is a faithful irreducible L-module over K,*

G3) *ad is a surjective L-module morphism from $\wedge^{n-1}V$ onto the adjoint module L such that the map $\times^n V \to V$, $(v_1, \cdots, v_n) \to \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1}).v_n$ is alternating,*

then the corresponding n-Lie algebra is simple, since an ideal of it is also an L-submodule of V. Moreover the derivation algebra is isomorphic to L. In fact, if ρ is the representation of L in V, then ρ is an isomorphism from L to the derivation algebra. A triple with G1), G2) and G3) will be called a good triple. The problem of determining the simple n-Lie algebras over K can be translated into that of finding the good triples (L, V, ad).

For the following notations one compares Humphreys [7] or the appendix in this work. Let L be a semisimple Lie algebra, let H be a maximal toral subalgebra of L, Φ the root system of L relative to H and $L = H \oplus \oplus_{\alpha \in \Phi} L_\alpha$, the root space decomposition of L. Further let Δ be a base of Φ, $\prec$ the half ordering on H^* relative to Δ and Φ^+ (Φ^-) the subset of Φ of positive (negative) roots. Let $(\cdot, \cdot)$ be the nondegenerate symmetric bilinear form on H^* which comes from the Killing form of L and $\langle \mu, \nu \rangle := \frac{2(\mu,\nu)}{(\nu,\nu)}$, where $\mu, \nu \in H^*$ and $(\nu, \nu) \neq 0$. For an $\alpha \in \Phi^+$ choose $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$, $h_\alpha \in H$ such that $[x_\alpha, y_\alpha] = h_\alpha$, $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, y_\alpha] = -2y_\alpha$. Recall that for this choice we have $\mu(h_\alpha) = \langle \mu, \alpha \rangle$ for all $\mu \in H^*$.

Let V be a faithful irreducible L-module with maximal weight $\lambda \in \Lambda^+$. Denote by $\Pi(\lambda)$ the set of all its weights. Then $V = \oplus_{\mu \in \Pi(\lambda)} V_\mu$.

<u>**Lemma 3.1**</u>: *Let L and V be as above. Let τ be an L-module morphism from $\wedge^m V$ ($m \geq 1$) into L. If $v_{\mu_i} \in V_{\mu_i}$ is a vector of weight μ_i for all $i \in \underline{m}$, then $\tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m}) \in L_\gamma$, where $\gamma = \sum_{i=1}^m \mu_i$.*

Proof: Since τ is an L-module morphism, it follows that for all $h \in H$:

$$\begin{aligned}
&[h, \tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m})] \\
&= \sum_{i=1}^{m} \tau(v_{\mu_1} \wedge \cdots v_{\mu_{i-1}} \wedge h.v_{\mu_i} \wedge v_{\mu_{i+1}} \wedge \cdots \wedge v_{\mu_m}) \\
&= (\sum_{i=1}^{m} \mu_i)(h) \cdot \tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m}).
\end{aligned}$$

Thus $\tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m}) \in L_\gamma$ as asserted. □

Let (L, V, ad) be a good triple and the notations for L and V be as above. Let σ_0 denote the element in the Weyl group W of L such that $\sigma_0\Delta = -\Delta$ (see Proposition A5). Then $\sigma_0\lambda$ is the minimal weight of V. Further let v^+ (resp. v^-) be a maximal (resp. minimal) weight vector of V. We determine (L, V, ad) in three steps: We show

C1) *that H contains a nonzero element of the form $ad(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge v_{n-3})$, $v_i \in V$, $i \in \underline{n-3}$ if $n > 3$ and $ad(v^+ \wedge v^-)$ if $n = 3$ (see Corollary 3.4),*

C2) *that for each simple component of L there exists a simple root α of this component such that $\lambda - \sigma_0\lambda - \alpha \in \Phi$ (see Lemma 3.5), and*

C3) *that the number of the irreducible representations with the property in C2) is finite. Then all good triples (L, V, ad) can be found among them.*

We show the following lemma in advance.

Lemma 3.2: *If (L, V, ad) is a good triple, then $\lambda - \sigma_0\lambda \notin \Phi$.*

Proof: Suppose that $\lambda - \sigma_0\lambda$ is a root of L. We will first show that this implies L has to be simple. Suppose on the contrary that $L = L_1 \oplus L_2$, $L_i \neq \{0\}$, $i = 1, 2$. Let $H_i = H \cap L_i$. Then H_i is a maximal toral subalgebra of L_i (see Theorem A6). Let $\Phi_i \subseteq \Phi$ be the root system of L_i relative to H_i. Let $\Delta_i := \Delta \cap \Phi_i$. Then Δ_i is a base of Φ_i (see Remark A1). Since $\Phi = \Phi_1 \cup \Phi_2$, $\lambda - \sigma_0\lambda$ is an element in Φ_1 or in Φ_2. Assume that $\lambda - \sigma_0\lambda \in \Phi_1$ (the other case can be treated analogously). Then $\lambda - \sigma_0\lambda$ vanishes on H_2. Let $\lambda^{(2)}$ be the restriction of λ to H_2 and $\sigma_0 = \sigma_1 + \sigma_2$, where σ_i is the element in the Weyl group of L_i with $\sigma_i\Delta_i = -\Delta_i$. Then $\lambda - \sigma_0\lambda = (\lambda^{(1)} - \sigma_1\lambda^{(1)}) + (\lambda^{(2)} - \sigma_2\lambda^{(2)})$. So $(\lambda - \sigma_0\lambda)(H_2) = \{0\}$ can be translated into $\lambda^{(2)} - \sigma_2\lambda^{(2)} = \{0\}$. Since $\lambda^{(2)}$ and $\sigma_2\lambda^{(2)}$ are dominant weights of L_2, we get $\lambda^{(2)} = 0$, which implies that L does not operate faithfully on V (see Corollary A12). This contradicts our assumption that (L,V,ad) is a good triple and therefore L is simple.

Now, since the root $\lambda - \sigma_0\lambda$ of the simple Lie algebra L is a sum of two nonzero dominant weights and since the roots in Table 2 are the only roots which are dominant, it follows by checking Table 2 that $L \cong A_l$ respectively C_l and correspondingly $\lambda - \sigma_0\lambda = \lambda_1 + \lambda_l$ respectively $2\lambda_1$. In the first case we get $\lambda = \lambda_1$ respectively λ_l. In the second case we get $\lambda = \lambda_1$. Therefore V is either the natural L-module $V(\lambda_1)$ or its contragredient. Since the adjoint L-module is $V(\lambda_1 + \lambda_l)$ or $V(2\lambda_1)$ respectively (see Table 2), it follows from Table 3 that there is no nonzero L-module morphism from $\wedge^m V$ to L, $m \in \underline{d}$, where d denotes the dimension of V (notice that $(\wedge^m V)^* \cong \wedge^m V^*$). Therefore there is no good triple constructed from the pairs L and V. We obtain a contradiction to the assumption. □

We proceed to prove C1.

Since $v^+ \wedge v^-$ is a generator of the L-module $V \wedge V$ (see Lemma A10), it follows that $v^+ \wedge v^- \wedge \underbrace{V \wedge \cdots \wedge V}_{n-3}$ generates $\wedge^{n-1}V$. Since ad $\neq 0$ we can have $\mathrm{ad}(v^+ \wedge v^-) \neq 0$ if $n = 3$ and $\mathrm{ad}(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge v_{n-3}) \neq 0$ for some $v_i \in V$, $i \in \underline{n-3}$, if $n > 3$. Define

$$H_0 := \begin{cases} H \cap \mathrm{ad}(v^+ \wedge v^- \wedge \underbrace{V \wedge \cdots \wedge V}_{n-3}) & n > 3 \\ H \cap \{\mathrm{ad}(v^+ \wedge v^-)\} & n = 3. \end{cases}$$

__Lemma 3.3__: *Let $n > 3$. Let $z = ad(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \neq 0$, where $v_{\mu_i} \in V_{\mu_i}$, $\mu_i \in \Pi(\lambda)$, $i \in \underline{n-3}$. Let $\gamma = \lambda + \sigma_0\lambda + \sum_{i=1}^{n-3} \mu_i$. Then $z \in L_\gamma$. If $\gamma = 0$, then $z \in H_0$; if $\gamma \neq 0$, then $\{0\} \neq [z, L_{-\gamma}] \subseteq H_0$.*

Proof: It is clear by Lemma 3.1 that $z \in L_\gamma$. If $\gamma = 0$, then $z \in H_0$ because of $L_0 = H$. So let $\gamma \in \Phi^+$. For $\gamma \in \Phi^-$ we can proceed analogously. By multiplying v^+ by some scalar we can assume that $z = x_\gamma$. Then by the choice of x_γ, y_γ, h_γ and the minimality of v^-,

$$\begin{aligned} -h_\gamma &= [y_\gamma, x_\gamma] \\ &= \mathrm{ad}(y_\gamma.v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \\ &\quad + \sum_{i=1}^{n-3} \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{i-1}} \wedge y_\gamma v_{\mu_i} \wedge v_{\mu_{i+1}} \wedge \cdots \wedge v_{\mu_{n-3}}). \end{aligned}$$

If $y_\gamma.v^+ = 0$, then h_γ is already an element of H_0 and we are done with the proof.

Suppose now that $y_\gamma.v^+ \neq 0$ from which we shall deduce a contradiction. By Proposition A9, $\langle \lambda, \gamma \rangle \neq 0$, and this yields $x_\gamma.y_\gamma.v^+ = [x_\gamma, y_\gamma].v^+ = h_\gamma.v^+ =$

$\lambda(h_\gamma)v^+ = \langle\lambda,\gamma\rangle v^+ \neq 0$. On the other hand, as ad is alternating, we have

$$\begin{aligned} 0 \neq x_\gamma.y_\gamma.v^+ &= \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).y_\gamma.v^+ \\ &= \mathrm{ad}(y_\gamma.v^+ \wedge v^+ \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).v^-. \end{aligned}$$

In particular, $\mathrm{ad}(y_\gamma.v^+ \wedge v^+ \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \neq 0$. By Lemma 3.1 the weight of this element is $2\lambda + \sum_{i=1}^{n-3}\mu_i - \gamma$, which is equal to $\lambda - \sigma_0\lambda$ by the definition of γ. Therefore $\lambda - \sigma_0\lambda \in \Phi \cup \{0\}$. By Lemma 3.2, $\lambda - \sigma_0\lambda \notin \Phi$, hence $\lambda - \sigma_0\lambda = 0$. Since $-\sigma_0\lambda$ is also a dominant weight, $\lambda = 0$, which is a contradiction to the assumption that V is faithful. □

Corollary 3.4: $H_0 \neq \{0\}$

Proof: If $n > 3$, the assertion follows from Lemma 3.3. Let $n = 3$. Since the nonzero element $\mathrm{ad}(v^+ \wedge v^-)$ lies in $L_{\lambda+\sigma_0\lambda}$ by Lemma 3.1, $\lambda + \sigma_0\lambda \in \Phi \cup \{0\}$. But $\sigma_0(\lambda+\sigma_0\lambda) = \sigma_0\lambda + \sigma_0^2\lambda = \lambda + \sigma_0\lambda$, hence $\lambda + \sigma_0\lambda = 0$ because σ_0 maps the positive roots into the negative ones. This means that $\mathrm{ad}(v^+ \wedge v^-) \in H_0$. □

We continue to show C2.

Let $L = \sum_{i=1}^{m} L_i$, where L_i, $i \in \underline{m}$ are the simple ideals of L. Further $H_i := H \cap L_i$, $\Phi_i \subseteq \Phi$ the root system of L_i relative to H_i and $\Delta_i := \Delta \cap \Phi_i$. Δ_i is a base of Φ_i (see Theorem A6 and Remark A1)

Lemma 3.5: *Let (L, V, ad) be a good triple. Then for each $i \in \underline{m}$ there exists $\alpha \in \Delta_i$ with $\lambda - \sigma_0\lambda - \alpha \in \Phi$.*

Proof: Set $\Delta_{0,i} := \{\alpha \in \Delta_i \mid \alpha(H_0) \neq \{0\}\}$. Then for each i: $\Delta_{0,i} \neq \phi$. If $\Delta_{0,i} = \phi$ for some $i \in \underline{m}$, say $i = 1$, then $H_0 \subseteq \oplus_{i=2}^{m} H_i$. Set $M := \oplus_{i=2}^{m} L_i$. We show that $\mathrm{ad}(V \wedge \cdots \wedge V) \subseteq M$, which contradicts the surjectivity of ad. Since $v^+ \wedge v^-$ generates $V \wedge V$, it suffices to show that

$$\mathrm{ad}(v^+ \wedge v^- \wedge V \wedge \cdots \wedge V) \subseteq M. \tag{3.2}$$

Let $z = \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \neq 0$. If $z \in H_0$, then $z \in M$; if $z \notin H_0$, then $z \in L_\gamma$ for some root γ. By Lemma 3.3, $\{0\} \neq [z, L_{-\gamma}] \subseteq H_0$, this in turn gives $z \in M$. Hence (3.2) is true and $\Delta_{0,i} \neq \phi$.

We claim that for each i there exists an $\alpha \in \Delta_{0,i}$ such that $x_\alpha.v^- \neq 0$ or $y_\alpha.v^+ \neq 0$.

If $n = 3$, then $\mathrm{ad}(v^+ \wedge v^-)$ is the only nonzero element in H_0 up to scalar. By the definition of $\Delta_{0,i}$ we have for all its elements α:

$$\mathrm{ad}(v^+ \wedge x_\alpha.v^-) = [x_\alpha, \mathrm{ad}(v^+ \wedge v^-)]$$

$$\begin{aligned} &= -\alpha(\mathrm{ad}(v^+ \wedge v^-))x_\alpha \\ &\neq 0, \end{aligned}$$

which implies that $x_\alpha.v^- \neq 0$.

Now let $n > 3$. We proceed indirectly and suppose that for all $\alpha \in \Delta_{0,i}$: $x_\alpha.v^- = y_\alpha.v^+ = 0$. If $\Delta_{0,i} = \Delta_i$, then by the assumption $y_\alpha.v^+ = 0$ for all $\alpha \in \Delta_i$. Combining it with $x_\alpha.v^+ = 0$, we get $h_\alpha.v^+ = 0$, which implies that $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle = 0$. So the restriction $\lambda^{(i)}$ of λ on H_i is zero, and we obtain a contradiction to the assumption that V is faithful L-module (see Corollary A12). Therefore $\Delta_{0,i}$ is a nonempty proper subset of Δ_i. Let $\alpha \in \Delta_{0,i}$ and $h = \mathrm{ad}(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge v_{n-3}) \in H_0$ such that $\alpha(h) \neq 0$. Then since ad is a module morphism, we have

$$\alpha(h)y_\alpha = [y_\alpha, h] = \sum_{i=1}^{n-3} \mathrm{ad}(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge y_\alpha.v_i \wedge \cdots \wedge v_{n-3}),$$

which in turn implies that $y_\alpha \in \mathrm{ad}(v^+ \wedge v^- \wedge V \wedge \cdots \wedge V)$. Analogously we can show that x_α is an element in $\mathrm{ad}(v^+ \wedge v^- \wedge V \wedge \cdots \wedge V)$, so is $h_\alpha = [x_\alpha, y_\alpha]$. Then $h_\alpha \in H_0$. Now for any $\beta \in \Delta_i \setminus \Delta_{0,i}$ and any $\alpha \in \Delta_{0,i}$: $\langle \beta, \alpha \rangle = \beta(h_\alpha) = 0$, that is, $\Delta_{0,i} \perp \Delta_i \setminus \Delta_{0,i}$, which contradicts that L_i is a simple ideal of L. Therefore the assumption for $\Delta_{0,i}$ is false.

Now we can prove Lemma 3.5 as follows.

Let $\alpha \in \Delta_{0,i}$ such that $x_\alpha.v^- \neq 0$ or $y_\alpha.v^+ \neq 0$. Since we can proceed analogously if $y_\alpha.v^+ \neq 0$, we assume that $x_\alpha.v^- \neq 0$. Further let $h \in H_0$, $h = \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}})$ with $\alpha(h) \neq 0$. By pluging the expression for h in $[x_\alpha, h]$ we obtain:

$$\begin{aligned} &-\alpha(z)x_\alpha \\ &= \mathrm{ad}(v^+ \wedge x_\alpha.v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \\ &+ \sum_{j=1}^{n-3} \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{j-1}} \wedge x_\alpha.v_{\mu_j} \wedge v_{\mu_{j+1}} \wedge \cdots \wedge v_{\mu_{n-3}}). \end{aligned}$$

If the element $\mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge x_\alpha.v_{\mu_j} \wedge \cdots \wedge v_{\mu_{n-3}})$ is nonzero for some j, it is a weight vector of weight α and we might assume that it agrees with x_α by choosing v^+ appropriately. Then we get $x_\alpha.v^- = \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge x_\alpha.v_{\mu_i} \wedge \cdots \wedge v_{\mu_{n-3}}).v^- = 0$ which contradicts the assumption that $x_\alpha.v^- \neq 0$. Therefore all terms but the first one on the right side are 0, consequently $x_\alpha = \mathrm{ad}(v^+ \wedge x_\alpha.v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}})$ (where z is choosen such that $\alpha(z) = -1$). From

$$\begin{aligned} 0 \neq x_\alpha.v^- &= \mathrm{ad}(v^+ \wedge x_\alpha.v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).v^- \\ &= \mathrm{ad}(x_\alpha.v^- \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).v^+ \end{aligned}$$

we conclude that $\mathrm{ad}(x_\alpha.v^- \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}})$ is different from 0 and has $2\sigma_0\lambda + \alpha + \sum_{i=1}^{n-3}\mu_i$ as its weight. But $\lambda + \sigma_0\lambda + \sum_{i=1}^{n-3}\mu_i = 0$, hence $-\lambda + \sigma_0\lambda + \alpha \in \Phi \cup \{0\}$, or equivalently $\lambda - \sigma_0\lambda - \alpha \in \Phi \cup \{0\}$.

If $\lambda - \sigma_0\lambda - \alpha = 0$, then $\lambda - \sigma_0\lambda$ is a root of L which is impossible by Lemma 3.2. Therefore $\lambda - \sigma_0\lambda - \alpha \in \Phi$. □

Finally we proceed to C3).

To find all good triples we look for the pairs L and V where L is a finite dimensional semisimple Lie algebra and V is a faithful irreducible L-module with the property in Lemma 3.5. For this purpose we discuss two different cases, namely L is simple or not.

At first we assume that L is not simple, say $L = \oplus_{i=1}^{s} L_i$, where L_i are the simple ideals of L and $s \geq 2$. Further let $H_i := H \cap L_i$, $\Phi_i \subseteq \Phi$ the root system of L_i relative to H_i and $\Delta_i = \Delta \cap \Phi_i$ (see Theorem A6 and Remark A1). First, we can assume that $\alpha \in \Delta_1$. Then $\alpha(H_i) = \{0\}$ for all $2 \leq i \leq s$. Let $\beta = \lambda - \sigma_0\lambda - \alpha$. If $\beta \in \Phi_1$, then $\beta(H_i) = \{0\}$ for all $2 \leq i \leq s$. Therefore $(\lambda - \sigma_0\lambda)(H_i) = \{0\}$. Denote by σ_i the element in the Weyl group of L_i with the property: $\sigma_i\Delta_i = -\Delta_i$, and let $\lambda^{(i)}$ be the restriction of λ to H_i. Then $\lambda^{(i)} - \sigma_i\lambda^{(i)} = 0$, which in turn implies $\lambda^{(i)} = 0$ for all $i \geq 2$. But this is impossible because V is faithful (see Corollary A12). Hence $\beta \in \Phi_i$ for some $i \geq 2$, say $i = 2$. The same argument gives $s = 2$, that is, $L = L_1 \oplus L_2$. It follows that $\alpha = \lambda^{(1)} - \sigma_1\lambda^{(1)}$, that is, the simple root α is a sum of two nonzero dominant weights. This is only possible if the rank of L_1 is 1. Thus $L_1 \cong so(3, K)$ and $\lambda^{(1)}$ is the only fundamental dominant weight of L_1.

Assume now that $\alpha \in \Delta_2$. Repeating the above consideration, we can also conclude that $L_2 \cong so(3, K)$ and $\lambda^{(2)}$ is the only fundamental dominant weight L_2. Summarizing the foregoing results we obtain $L \cong so(4, K)$ and $V \cong K^4$.

Can we construct from $so(4, K)$ and K^4 a good triple? Since we have to deal with a 4-dimensional $so(4, K)$-module, this question is the same as to ask whether there exists an $so(4, K)$-module morphism of the form ad: $\wedge^2 K^4 \to so(4, K)$ with G3), or equivalently whether there exists a simple 3-Lie algebra structrue on K^4 such that the Lie algebra of its derivations is isomorphic to $so(4, K)$. Theorem 1.2.4 shows that such a 3-Lie algebra exists and it is unique up to isomorphism. Therefore we have proved

Theorem 3.6: *If V is a finite dimensional simple n-Lie algebra over an algebraically closed field K of characteristic 0 for which the Lie algebra of the derivations is not simple, then $n = 3$. Moreover V is isomorphic to the 3-Lie algebra (K^4, b, f) with the vector product.*

Let us now treat the case that L is simple. Denote by α_0 the maximal root of

L. We show

Lemma 3.7: *If L is a simple Lie algebra and (L, V, ad) is a good triple, then*

$$\lambda - \sigma_0\lambda = \alpha_0 + \alpha \tag{3.3}$$

for some simple root α of L.

Proof: To show the assertion we need the following representation for x_{α_0}:

$$x_{\alpha_0} = \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^+) \tag{3.4}$$

for some $v_{\mu_i} \in V_{\mu_i}$, $\mu_i \in \Pi(\lambda)$, $i \in \underline{n-2}$. In fact, there exist $v_{\mu_i} \in V_{\mu_i}$, $\mu_i \in \Pi(\lambda)$, $i \in \underline{n-1}$, such that

$$x_{\alpha_0} = \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-1}}), \tag{3.5}$$

since ad is surjective and L_{α_0} is one dimensional. If $\mu_{n-1} \neq \lambda$, then there exist $w_{\nu_j} \in V_{\nu_j}$, $\nu_j \in \Pi(\lambda)$, and $\beta_j \in \Phi^+$, $j \in \underline{s}$ such that $v_{\mu_{n-1}} = \sum_{j=1}^{s} y_{\beta_j}.w_{\nu_j}$ (cf. Theorem A8). Inserting this expression for $v_{\mu_{n-1}}$ in (3.5), we get

$$\begin{aligned} x_{\alpha_0} &= \sum_{j=1}^{s} [y_{\beta_j}, \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge w_{\nu_j})] \\ &\quad - \sum_{j=1}^{s} \sum_{i=1}^{n-2} \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{i-1}} \wedge y_{\beta_j}.v_{\mu_i} \wedge v_{\mu_{i+1}} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge w_{\nu_j}). \end{aligned}$$

The first sum on the right side is zero because $\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge w_{\nu_j}) \in L_{\alpha_0+\beta_j} = \{0\}$. Hence x_{α_0} is represented in the form as in (3.5), but the weight of the last component is greater. Repeating this process (we can do it only finitely many times) until the last position is occupied by a maximal vector, we have reached the expression (3.4).

We can show (3.3) by using (3.4) as follows. Since $\langle \sigma_0\lambda, \alpha_0 \rangle = -\langle \lambda, \alpha_0 \rangle \neq 0$ (otherwise $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Delta$, which implies that $\lambda = 0$ and V is one dimensional), we have $x_{\alpha_0}.v^- \neq 0$ by Proposition A9. On the other hand, we have by (3.4),

$$\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^+).v^- = -\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^-).v^+. \tag{3.6}$$

Therefore $\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^-).v^+ \neq 0$, in particular, $\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^-) \neq 0$. If its weight is $-\beta$, then $\beta \in \Phi^+ \cup \{0\}$ due to the maximality of v^+. By (3.6), $\alpha_0 + \sigma_0\lambda = \lambda - \beta$ or

$$\lambda - \sigma_0\lambda = \alpha_0 + \beta \tag{3.7}$$

If $\beta = 0$, then $\lambda - \sigma_0\lambda$ is the maximal root of L which is not true by Lemma 3.2. Therefore $\beta \in \Phi^+$. From Lemma 3.4 and (3.7) we get $\alpha_0 + \beta - \alpha \in \Phi$ for some simple root α of L. Since α_0 is maximal, it follows that $\alpha - \beta = \alpha_0 - (\alpha_0 + \beta - \alpha) \succ 0$. But α is simple, hence $\beta = \alpha$. Finally we get $\lambda - \sigma_0\lambda = \alpha_0 + \alpha$. □

In Proposition A13 all pairs (L, V) with (3.3) are determined. We list them here again for convenience.

L	A_1	A_3	$B_l, l \geq 2$	B_3	$D_l, l \geq 4$	D_4	G_2
λ	$2\lambda_1$	λ_2	λ_1	λ_3	λ_1	λ_3, λ_4	λ_1

We shall go through the list and discuss the cases respectively.

Case 1: $L \cong B_l$, $l \geq 2$ *or* D_l, $l \geq 4$, $\lambda = \lambda_1$.

In this case the L-module V is the natural module K^d $(= V(\lambda_1))$ of the orthogonal Lie algebra $so(d, K)$, where $d \geq 5$, $d \neq 6$. By Table 2 the adjoint module of $so(d, K)$ has the maximal weight λ_2 if $d \geq 6$ respectively $2\lambda_2$ if $d = 5$. By Table 3, if ad: $\wedge^n K^d \to so(d, K)$ is a nonzero $so(d, K)$-module morphism for some n, then $n = 2$ or $n = d - 2$. In the following we exclude the case $n = 2$.

There is up to scalar only one nonzero $so(d, K)$-module morphism τ from $\wedge^2 K^d$ to $so(d, K)$ (see Table 3) and it can be constructed as follows. Let b be the nondegenerate symmetric bilinear form on K^d which defines $so(d, K)$. Let $u, v \in V(\lambda_3)$ be fixed. For the linear form $L \to K : x \to b(x.u, v)$ there exists a unique $\tau(u, v) \in so(d, K)$ such that $\mathrm{Kill}(x, \tau(u, v)) = b(x.u, v)$. Obviously $\tau \neq 0$ and for all $x \in so(d, K)$ we have

$$\begin{aligned} \mathrm{Kill}(x, \tau(v, u)) &= b(x.v, u) \\ &= -b(v, x.u) \\ &= -b(x.u, v) \\ &= -\mathrm{Kill}(x, \tau(u, v)), \end{aligned}$$

which implies $\tau(v, u) = -\tau(u, v)$. Therefore τ induces a linear map from $\wedge^2 K^d$ to $so(d, K)$ and we write $\tau(u \wedge v)$ instead of $\tau(u, v)$. Moreover for all $x, y \in so(d, K)$,

$$\begin{aligned} &\mathrm{Kill}(x, [y, \tau(u \wedge v)]) \\ &= b([x, y], \tau(u \wedge v)) \\ &= b([x, y].u, v) \\ &= b(x.y.u, v) - b(y.x.u, v) \\ &= \mathrm{Kill}(x, \tau(y.u \wedge v)) + b(x.u, y.v) \\ &= \mathrm{Kill}(x, \tau(y.u \wedge v)) + \mathrm{Kill}(x, \tau(u \wedge y.v)). \end{aligned}$$

Thus $[y, \tau(u \wedge v)] = \tau(y.u \wedge v) + \tau(u \wedge y.v)$, i.e., τ is an $so(d, K)$-module morphism from $\wedge^2 K^d$ to $so(d, K)$. Since $so(d, K)$ $(d \geq 5)$ is irreducible as a module of itself, τ is surjective.

If $\tau(u, v).u = 0$ for all $u, v \in K^d$, then $\mathrm{Kill}(\tau(u \wedge v), \tau(u \wedge v)) = b(\tau(u \wedge v).u, v) = 0$. This means that the Killing form of L is skew-symmetric. Therefore the map $(u, v, w) \rightarrow \tau(u \wedge v).w$ is not alternating and $(so(d, K), K^d, \tau)$ is not a good triple.

It remains to check whether $(so(d, K), K^d, \mathrm{ad})$ is a good triple where ad is the (up to scalar) unique $so(d, K)$-module morphism from $\wedge^{d-2} K^d$ to $so(d, K)$. But we need not to do this because Theorem 1.2.4 shows that the vector product (K^d, b, f) is a simple $(d-1)$-Lie algebra with $so(d, K)$ as its derivation algebra and it is the only simple $(d-1)$-Lie algebra of dimension d up to isomorphism.

Case 2: $L \cong A_1$, $\lambda = 2\lambda_1$.

Since the $sl(2, K)$-module V with the maximal weight $2\lambda_1$ is 3-dimensional (notice that V is just the adjoint module of $sl(2, K)$), L and V give us no good triple.

Case 3: $L \cong A_3$, $\lambda = \lambda_2$.

The 6-dimensional $sl(4, K)$-module $V(\lambda_2)$ is self-contragredient, that is, there is an $sl(4, K)$-invariant nondegenerate bilinear form b on $V(\lambda_2)$ (b is unique up to scalar), since $\sigma_0 \lambda = -\lambda_2$ (see proof of Theorem A13). Then the representation of $sl(4, K)$ in $V(\lambda_2)$ gives a Lie algebra isomorphism $\rho : sl(4, K) \rightarrow L_1$, where L_1 is the set of the endomorphisms τ of $V(\lambda_2)$ with $b(\tau u, v) + b(u, \tau v) = 0$ for all $u, v \in V(\lambda_2)$, since $\dim sl(4, K) = \dim L_1 = 15$ and ρ is faithful. We know that b is either symmetric or skew-symmetric (cf. [14]). If the second case is true, then $sl(4, K)$ is isomorphic to $sp(6, K)$ which is impossible because $\dim sp(6, K) = 21$. Therefore b is symmetric. It follows that $sl(4, K) \cong so(6, K)$ and the $sl(4, K)$-module $V(\lambda_2)$ can be regarded as the natural $so(6, K)$-module. By equivalence of the good triples it suffices to consider the good triples based on $so(6, K)$ and K^6. As in Case 1 we can conclude that there is only one n-Lie algebra structure on K^6 up to isomorphism and its derivation algebra is isomorphic to $so(6, K)$.

Case 4: $L \cong B_3$, $\lambda = \lambda_3$.

V is the 8-dimensional spin representation of $so(7, K)$. By the computer program " Lie " by Arjeh M. Cohen, Bert Lisser, Bart de Smit and Ron Sommeling we have the following decompositions (all decompositions which appear later are also based on this programm) where $\wedge^m V$ ($\vee^m V$) denotes the alternating (symmetric) part of the tensor product.

$$\wedge^2 V(\lambda_3) \cong V(\lambda_1) \oplus V(\lambda_2) \tag{3.8}$$

$$\wedge^3 V(\lambda_3) \cong V(\lambda_1+\lambda_3) \oplus V(\lambda_3) \quad (3.9)$$
$$\wedge^4 V(\lambda_3) \cong V(\lambda_1) \oplus V(2\lambda_1) \oplus V(2\lambda_3) \oplus V(0) \quad (3.10)$$
$$\wedge^5 V(\lambda_3) \cong V(\lambda_1+\lambda_3) \oplus V(\lambda_3) \quad (3.11)$$
$$\wedge^6 V(\lambda_3) \cong V(\lambda_1) \oplus V(\lambda_2) \quad (3.12)$$
$$\wedge^7 V(\lambda_3) \cong V(\lambda_3) \quad (3.13)$$
$$\vee^2 V(\lambda_3) \cong V(2\lambda_3) \oplus V(0) \quad (3.14)$$

Notice that $V(\lambda_2)$ is the adjoint module of $so(7,K)$. According to these decompositions we could have at most two good triples from $so(7,K)$ and $V(\lambda_3)$. (see (3.8) and (3.12)).

At first we check wether $(so(7,K), V(\lambda_3), \tau)$, where τ is an $so(7,K)$-module morphism from $\wedge^2 V(\lambda_3)$ onto $so(7,K)$ (τ is unique up to scalar), is a good triple. Since there is an L-invariant nondegenerate symmetric bilinear form on $V(\lambda_3)$ (see (3.14)), we can show that there is no 3-Lie algebra on $V(\lambda_3)$ such that the Lie algebra of its derivations is isomorphic to $so(7,K)$ as in Case 1.

We now study whether there exists an 7-Lie algebra on $V(\lambda_3)$ with its derivation algebra isomorphic to $so(7,K)$. Theorem 1.2.4 shows each 7-Lie algebra on $V(\lambda_3)$ is isomorphic to the vector product (K^8, b, f). But the derivation algebra of the vector product is $so(8,K)$. Hence there is no good triple of the form $(so(7,K), V(\lambda_3), \tau)$.

Case 5: $L \cong D_4$, $\lambda = \lambda_3$ *or* λ_4.

$V(\lambda_3)$ and $V(\lambda_4)$ are the two 8-dimensional spin modules of $so(8,K)$. Since there is an automorphism τ of $so(8,K)$ such that the natural module K^8 becomes $V(\lambda_3)$ or $V(\lambda_3)$ via $x.v := (\tau x)(v)$ or $x.v := (\tau^2 x)(v)$ for $x \in so(8,K)$ and $v \in K^8$. In view of the equivalence of good triples we are led to Case 1.

Case 6: $L \cong G_2$, $\lambda = \lambda_1$.

$V(\lambda_1)$ is the natural 7-dimensional G_2-module. $V(\lambda_2)$ is the adjoint module.

$$\wedge^2 V(\lambda_1) \cong V(\lambda_1) \oplus V(\lambda_2) \quad (3.15)$$
$$\wedge^3 V(\lambda_1) \cong V(2\lambda_1) \oplus V(\lambda_1) \oplus V(0) \quad (3.16)$$
$$\wedge^4 V(\lambda_1) \cong V(2\lambda_1) \oplus V(\lambda_1) \oplus V(0) \quad (3.17)$$
$$\wedge^5 V(\lambda_1) \cong V(\lambda_1) \oplus V(\lambda_2) \quad (3.18)$$
$$\wedge^6 V(\lambda_1) \cong V(\lambda_1) \quad (3.19)$$
$$\vee^2 V(\lambda_1) \cong V(2\lambda_1) \oplus V(0) \quad (3.20)$$

Since $V(\lambda_1)$ is a self-contragredient G_2-module and there is up to scalar only one nonzero G_2-module morphism from $\wedge^2 V(\lambda_1)$ onto G_2 (see (3.15) and (3.20)), we can show as in Case 1 that there is no 3-Lie algebra on $V(\lambda_1)$ with its derivation algebra isomorphic to G_2. Therefore such an n-Lie algebra is necessarily a 6-Lie algebra (see (3.18)). But the derivation algebra of a simple 6-Lie algebra on a 7-dimensional vector space is isomorphic to $so(7, K)$ and therefore is 21-dimensional. Now $\dim(G_2) = 14 < 21 = \dim(so(7, K))$. Therefore we obtain no good triple in this case.

Summarizing the results from Case 1 to Case 6 we get

Theorem 3.8: *If V is a simple finite dimensional n-Lie algebra over an algebraically closed field of characteristic 0 such that the Lie algebra of its derivations is simple, then $n \geq 4$. Moreover V is isomorphic to the vector product (K^{n+1}, b, f).*

Combining Theorem 3.6 and 3.8, we obtain the following

Theorem 3.9: *For every $n \geq 3$ all finite dimensional simple n-Lie algebras over an algebraically closed field K of characteristic 0 are isomorphic to the vector product on K^{n+1}.*

As an application of Theorem 3.9 we will classify the finite dimensional real simple n-Lie algebras. The idea is analogous as in the case of Lie algebras, i.e. we investigte the complexified n-Lie algebras.

Let V be an arbitrary real n-Lie algebra. We form the tensor product $\widetilde{V} := \mathbb{C} \otimes_{\mathbb{R}} V$ and regard it as a vector space over $\mathbb{C}$: $z(z' \otimes v) := zz' \otimes v$. Obviously $\widetilde{V}$ is an n-Lie algebra with

$$[z_1 \otimes v_1, \cdots, z_n \otimes v_n] = z_1 \cdots z_n \otimes [v_1, \cdots, v_n].$$

This complex n-Lie algebra is called the complexification of V. We can formally think of $\widetilde{V}$ as

$$\widetilde{V} = \{u + iv \mid u, v \in V \text{ and } i^2 = -1\}.$$

Note that $V \subseteq \widetilde{V}$ by identifying V with $V + i\{0\}$. Let C be the map from $\widetilde{V}$ into itself with $C(u + iv) = u - iv$. Then V is just the set of fixed points of C.

Conversely, given a complex n-Lie algebra $\widetilde{V}$, then by restricting the ground field to the real numbers we obtain a real n-Lie algebra $\widetilde{V}_{\mathbb{R}}$, which will be called the realification of $\widetilde{V}$.

A real n-Lie algebra V is called a real form of $\widetilde{V}$ if its complexification is isomorphic to $\widetilde{V}$.

Proposition 3.10: *Let $\widetilde{V}$ be an arbitrary complex simple n-Lie algebra. Then the realification $\widetilde{V}_{\mathbb{R}}$ of $\widetilde{V}$ is simple.*

Proof: $\widetilde{V}_{\mathbb{R}}$ is semisimple, since if I is a solvable ideal of $\widetilde{V}_{\mathbb{R}}$, then $I + i\,I$ is a solvable ideal of $\widetilde{V}$. Let I be an arbitrary ideal of $\widetilde{V}_{\mathbb{R}}$. We show that I is also an ideal of $\widetilde{V}$. Now for all $v_j \in I$, $j \in \underline{n}$, we have by definition of scalar multiplication in $\widetilde{V}$ that

$$i[v_1, \cdots, v_n] = [iv_1, v_2, \cdots, v_n] \in [\widetilde{V}, v_2, \cdots, v_n] \subseteq I.$$

Thus $iI \subseteq I$, since $[I, \cdots, I] = I$. This means that I is closed relative to the scalar multiplication by complex numbers. Thus I is an ideal of $\widetilde{V}$. But $\widetilde{V}$ is simple, hence $I = \widetilde{V} = \widetilde{V}_{\mathbb{R}}$ or $\{0\}$. □

Proposition 3.11: *The complexification $\widetilde{V}$ of a real semisimple n-Lie algebra V is semisimple.*

Proof: By Theorem 1.1.3 the Lie algebra $L := Inder(V)$ of the derivations of V is semisimple. Thus the complexification $\widetilde{L}$ of L is semisimple in view of Proposition A3 . It turns out that $Inder(\widetilde{V})$ agrees with $\widetilde{L}$, thus $\widetilde{V}$ is reductive by Theorem 2.10.

Let $u + iv$ be an element of the centre of $\widetilde{V}$. From $[u + iv, V, \cdots, V] = \{0\}$ we get $[u, V, \cdots, V] = \{0\}$ and $[v, V, \cdots, V] = \{0\}$, that is, u and v are in the centre of V, hence $u = v = 0$ because V is semisimple. This means that the centre of $\widetilde{V}$ is zero. □

Theorem 3.12: *A real simple n-Lie algebra V is isomorphic to the realification of a simple complex n-Lie algebra or to a real form of a simple complex n-Lie algebra.*

Proof: We consider the complexification $\widetilde{V}$ of V. By Proposition 3.10, $\widetilde{V}$ is semisimple. If $\widetilde{V}$ is simple, then V is a real form for some simple complex n-Lie algebra. Assume that $\widetilde{V}$ is not simple. Let C denote the map on $\widetilde{V}$ with $C(u + i\,v) = u - i\,v$. We show that an ideal I of $\widetilde{V}$ with $C(I) = I$ is either $\{0\}$ or $\widetilde{V}$. The fixed points of C in I form an ideal of V: $I_0 := \{x + C(x) \mid x \in I\}$, thus $I_0 = V$ or $\{0\}$ because V is simple. In the second case we have that $I \subseteq iV$ which is impossible since I is an ideal of $\widetilde{V}$. Thus $V = I_0 \subseteq I$ and $I = \widetilde{V}$.

If I is a simple ideal of $\widetilde{V}$, so is $C(I)$. Therefore $I \cap C(I)$ is either $\{0\}$ or I. If $I \cap C(I) = I$, that is $C(I) = I$, then $I = \widetilde{V}$ as shown above, contradiction. If $I \cap C(I) = \{0\}$, then the sum $J := I \oplus C(I)$ is direct and an ideal of $\widetilde{V}$ satisfying $C(J) = J$, it follows again from above that $\widetilde{V} = I + C(I)$.

As a result of it, $V = \{x + C(x) \mid x \in I\}$. Then the correspondence $x \to x + C(x)$ gives a real isomorphism from I onto V, so V is $\mathbb{R}$-isomorphic to the realification of the simple ideal I of $\widetilde{V}$. □

According to Theorem 3.12, in order to find all real simple n-Lie algebras ($n \geq 3$)

we have to calculate the real forms and the realification of all complex simple n-Lie algebras. As Theorem 3.8 shows, there is up to isomorphism only one finite dimensional complex simple n-Lie algebra $\widetilde{V}$ and this can be realized by means of a nondegenerate symmetric form b and a determinant form f ($\neq 0$) on a complex vector space $\widetilde{V}$ of dimension $n+1$ as in Example 1.1.1. Therefore as one simple real n-Lie algebra we have the realification of the vector product on $\mathbb{C}^{n+1}$ and it is of dimension $2(n+1)$ over $\mathbb{R}$.

Now we consider the real forms of $\widetilde{V}$. Since $\widetilde{V}$ is $(n+1)$-dimensional, each real form of $\widetilde{V}$ has also dimension $n+1$. By Lemma 1.2.1 any $(n+1)$-dimensional simple real n-Lie algebra V can be given as (K^{n+1}, b, f) as in Example 1.1.1. By Proposition 1.2.3 two such n-Lie algebras $(\mathbb{R}^{n+1}, b_1, f)$ and $(\mathbb{R}^{n+1}, b_2, f)$ are isomorphic if and only if there exists an automorphism τ of the space $\mathbb{R}^{n+1}$ with property (1.12). But each nondegenerate symmetric bilinear form b on $\mathbb{R}^{n+1}$ is congruent to one of the bilinear forms b_s, $0 \leq s \leq n+1$, whose associated matrix relative to the canonical base is $\text{diag}(1, \cdots, 1, \underbrace{-1, \cdots -1}_{s})$, that is, there is an automorphism σ of $\mathbb{R}^{n+1}$ such that $b(\sigma u, \sigma v) = b_s(u, v)$. If $\det\sigma < 0$, we choose an element $\sigma' \in O(\mathbb{R}^{n+1}, b)$ with $\det\sigma' = -1$. Then $\det(\sigma'\sigma) > 0$ and $b(\sigma'\sigma u, \sigma'\sigma v) = b(\sigma u, \sigma v) = b_s(u, v)$. So we may assume that $\det\sigma > 0$. Set $\tau = (\det\sigma)^{-\frac{1}{n-1}}\sigma$. Then one can show that τ satisfies the identity (1.12) and is an isomorphism from $(\mathbb{R}^{n+1}, b, f)$ onto $(\mathbb{R}^{n+1}, b_s, f)$ (see Proposition 1.2.3). This means that every $(n+1)$-dimensional real simple n-Lie algebra is isomorphic to one of the n-Lie algebras $(\mathbb{R}^{n+1}, b_s, f)$, $0 \leq s \leq n+1$, where f is a fixed nonzero determinant form on $\mathbb{R}^{n+1}$ and b_s as above.

We claim that $(\mathbb{R}^{n+1}, b_s, f)$ is isomorphic to $(\mathbb{R}^{n+1}, b_{n+1-s}, f)$. In fact, let τ_1 be element in $O(\mathbb{R}^{n+1}, b_s)$ with $\det\tau_1 = -1$. Further let τ_2 be an isomorphism of $\mathbb{R}^{n+1}$ such that $-b_s(\tau_2 u, \tau_2 v) = b_{n+1-s}(u, v)$. As above we may assume that $\det\tau_2 > 0$. Set $\tau := (\det\tau_2)^{-\frac{1}{n-1}}\tau_1\tau_2$. Then we have $\det\tau = -(\det\tau_2)^{-\frac{2}{n-1}}$ and

$$\begin{aligned} b_s(\tau u, \tau v) &= (\det\tau_2)^{-\frac{2}{n-1}} b_s(\tau_1\tau_2 u, \tau_1\tau_2 v) \\ &= (\det\tau_2)^{-\frac{2}{n-1}} b_s(\tau_2 u, \tau_2 v) \\ &= -(\det\tau_2)^{-\frac{2}{n-1}} b_{n+1-s}(u, v) \\ &= \det\tau \; b_{n+1-s}(u, v). \end{aligned}$$

By Proposition 1.2.3, τ is an isomorphism from the n-Lie algebra $(\mathbb{R}^{n+1}, b_s, f)$ onto $(\mathbb{R}^{n+1}, b_{n+1-s}, f)$. Therefore we have proved that each real simple $n+1$-dimensional n-Lie algebra is isomorphic to one of the n-Lie algebras $(\mathbb{R}^{n+1}, b_s, f)$, $0 \leq s \leq [\frac{n+1}{2}]$.

Chapter 4

Levi decomposition

Throughout this chapter K will be an algebraically closed field of characteristic 0 and all vector spaces over K will be finite dimensional.

We know that any Lie algebra L over K admits a Levi decomposition, that is, there exists a (Levi) subalgebra L_0 of L such that $L = Rad(L) + L_0$ and $L_0 \cap Rad(L) = \{0\}$ (cf. [18] p. 225). In this chapter we shall show using this results for Lie algebras that n-Lie algebras ($n \geq 3$) over K have an analogous decomposition. Given an n-Lie algebra V, we call a subalgebra V_0 a Levi subalgebra of V if $V = Rad(V) + V_0$ and $Rad(V) \cap V_0 = \{0\}$. The corresponding decomposition of V is called a Levi decomposition. We remark that a Levi subalgebra is semisimple if it exists.

Theorem 4.1: *(Levi decomposition)*
Let V be an n-Lie algebra over K and $n \geq 3$. Then V admits a Levi subalgebra.

Let us describe the idea of the proof of Theorem 4.1. By induction on the dimension of the radical of V we will reduce the proof to the case that $Rad(V)$ is a minimal ideal of V. Then we consider the Lie algebra $Der(V)$. First, we make the Levi decomposition: $Der(V) = Rad(Der(V)) + L_0$, where L_0 is a Levi subalgebra of $Der(V)$. With the help of the Lie algebra homomorphism γ defined in Theorem 2.9 we decompose L_0 into a sum of two ideals L_1 and L_2: $L_0 = L_1 + L_2$, where L_1 is a Levi subalgebra of $Ker\gamma$. Then we decompose $Rad(Der(V))$ into the sum of N and A where N is the ideal of $Der(V)$ consisting of all nilpotent endomorphisms in $Rad(Der(V))$ and A is an abelian subalgebra consisting of semisimple endomorphisms of V and $[A, L_0] = \{0\}$. Since V is a completely reducible $(A + L_0)$-module and I is a submodule of V, there exists a submodule V_0 such that $V = I + V_0$ (vector space direct sum). In case $A + L_1 \neq \{0\}$, we show that V_0 is the subspace of elements in V killed by all elements in $A + L_1$ which shows that V_0 a Levi subalgebra. In caṣe $A + L_1 = \{0\}$, we discuss the possibilities of the appearence of I in the isotypic components of $[V_0, \cdots, V_0]$. By using the decomposition of the n-fold wedge product of the natural $so(n, K)$-module K^{n+1}, we can show that either V_0 is a Levi subalgebra or there exists a nonzero n-Lie algebra homomorphism from $V/Rad(V)$ to V whose image gives a Levi subalgebra.

Proof of Theorem 4.1:

We will set $I := Rad(V)$ in the following. If $I = \{0\}$, then V is semisimple and we are done. So let $I \neq \{0\}$ and suppose that the assertion is true in case I is a minimal ideal of V. Under this hypothesis we prove the existence of a Levi subalgebra for any n-Lie algebra by induction on the dimension of I. Assume that the assertion is true for all n-Lie algebras whose radical has dimension less than $\dim I$. If I is a minimal ideal of V, then we are done. Now assume that I is not minimal. Let $J \neq \{0\}$ be an ideal of V which is properly included in I and $\pi : V \to V/J$ be the canonical map. Then $\pi(I)$ is a solvable ideal of V/J by Proposition 2.2, 2). Since $\pi^{-1}(Rad(V/J))$ is a solvable ideal of V by Proposition 2.2 3), it is included in I. Hence $Rad(V/J) \subseteq \pi(I)$ and $\pi(I)$ coincides with the radical of V/J. Since $\dim\pi(I) < \dim I$, there exists a Levi subalgebra V' of V/J with $V/J = \pi(I) + V'$ and $\pi(I) \cap V' = \{0\}$ by the induction hypothesis. Set $V_1 := \pi^{-1}(V')$. Then V_1 is a subalgebra of V with $V = I + V_1$. Moreover, we have $I \cap V_1 = J$. In fact, from $\pi(I \cap V_1) \subseteq \pi(I) \cap \pi(V_1) = \pi(I) \cap V' = \{0\}$, we obtain that $I \cap V_1 \subseteq J$. The other inclusion is evident. We show that J is the radical of V_1. Since J is a solvable ideal of V, it is a solvable ideal of V_1, and thus $J \subseteq Rad(V_1)$. On the other hand, it follows from $\pi(Rad(V_1)) = \{0\}$ that $Rad(V_1) \subseteq J$, hence $J = Rad(V_1)$. Since $\dim J < \dim I$, we can find a Levi subalgebra V_0 of V_1 by the induction hypothesis. Then V_0 is also a Levi subalgebra of V because $V = I + (J + V_0) = I + V_0$ and $I \cap V_0 = \{0\}$.

It remains to show Theorem 4.1 in case I is a minimal ideal of V.

We set $\overline{V} := V/I$ and let $\pi : V \to \overline{V}$ be the canonical homomorphism. Let L (resp. $\overline{L}$) be the derivation algebra of V (resp. $\overline{V}$). If $\overline{V} = \{0\}$, then V is solvable and we are done. Assume that $\overline{V} \neq \{0\}$. Since $\overline{V}$ is a semisimple n-Lie algebra, it is the direct sum of its simple ideals, say $\overline{V} = \oplus_{i=1}^{m} \overline{V}_i$ for some $m \in \mathbb{N}$ (see Theorem 2.7). Moreover $\overline{L} \cong \oplus_{i=1}^{m} \overline{L}_i$, $\overline{L}_i = Inder(\overline{V}_i) \cong so(n+1, K)$ (see Theorem 1.2.4, Theorem 2.5 and Theorem 3.9). Let $\gamma : L \to \overline{L}$ be the Lie algebra homomorphism defined as in Theorem 2.9. Recall that γ is surjective with $Ker\gamma = \{D \in L \,|\, D(V) \subseteq I\}$ and $R \subseteq Ker\gamma$, where R denotes the radical of L. Because for any Lie algebra L and any ideal M of L we have $Rad(M) = Rad(L) \cap M$ (cf. [18] p. 204), R is the radical of $Ker\gamma$. Let L_1 be a Levi subalgebra of $Ker\gamma$. Since L_1 is semisimple, there exists a Levi subalgebra L_0 of L such that $L_1 \subseteq L_0$ (cf. [18] p. 226 and 228). In fact, L_1 is an ideal of L_0. To see this we first show that $L_0 \cap Ker\gamma = L_1$. Let $x \in L_0 \cap Ker\gamma$ and $x = y + z$, $y \in R$, $z \in L_1$. Because of $yx - z$ and $x - z \in L_0$ we must have $y \in L_0$, hence $y \in R \cap L_0 = \{0\}$ which implies that $x \in L_1$, that is $L_0 \cap Ker\gamma \subseteq L_1$. The other inclusion is trivial. Now, since L_0 is a subalgebra and $Ker\gamma$ is an ideal of L, it follows that $[L_0, L_1] \subseteq L_0 \cap Ker\gamma = L_1$, which implies that L_1 is an ideal of L_0. Let L_2 be the ideal of L_0 with $L_0 = L_1 \oplus L_2$. Let N be the set of nilpotent endomorphisms of V belonging to R. Then N is an ideal of L (cf. [7] p. 45 or [13] p. 257). Since L is an algebraic Lie algebra (recall that an algebraic Lie algebra is the Lie algebra of an algebraic group, cf. [2] and [8]), R can be represented as the direct sum of N and an abelian algebra A whose elements are semisimple endomorphisms and commute with elements of L_0 (cf. [2]). In one word, we have the following decompositions:

$$L = R + L_0, \quad R = N + A, \quad L_0 = L_1 + L_2$$

with $[A, L_0] = \{0\}$. Set $L' := A + L_0$. Then V is a completely reducible L'-module. Since I is invariant under all derivations (see Remark 2.1), it is an L'-submodule of V. Let V_0 be a complement of I in V: $V = I \oplus V_0$. Set $L'' := A + L_1$. Then $L'' \subseteq L'$ and $L'' \subseteq Ker\gamma$.

In the following we discuss two cases.

Case 1: $L'' \neq \{0\}$

Let V_1 denote the null space of L''. Then $V_1 = \{v \in V \mid D(v) = 0, \forall D \in L''\}$. We see easily that V_1 is a subalgebra of V. If V_1 agrees with V_0, then it is a Levi subalgebra of V because the sum $V = I + V_0$ is direct.

We are going to show that $V_1 = V_0$:

The inclusion $V_0 \subseteq V_1$ results from the fact that $L''(V_0) \subseteq L'(V_0) \subseteq V_0$ and $L''(V_0) \subseteq Ker\gamma(V_0) \subseteq I$, i.e. $L''(V_0) \subseteq V_0 \cap I = \{0\}$. To get the equality of V_1 and V_0 we put $I_0 := I \cap V_1$. Then V_1 is the vector space direct sum of I_0 and V_0. In fact, let $x \in V_1$ and $x = y + z$, $y \in I$, $z \in V_0$. Since $y = x - z \in V_1$, we get $y \in I_0$. It remains to show $I_0 = \{0\}$. For this purpose we prove that I_0 is invariant under all derivations. This is equivalent to the following two inclusions: $L_2(I_0) \subseteq I_0$ and $N(I_0) \subseteq I_0$ because $L'(I_0) = \{0\} \subseteq I_0$. We first show $L_2(I_0) \subseteq I_0$: Since $[L'', L_2] = \{0\}$, we get $L_2(V_1) \subseteq V_1$, hence $L_2(I_0) \subseteq L_2(V_1) \subseteq V_1$. We also have that $L_2(I_0) \subseteq I$ because I is invariant under L, in particular, under L_2. Therefore we get $L_2(I_0) \subseteq I_0$. We now show $N(I_0) \subseteq I_0$: Since $N(I)$ is an ideal of I (see Proposition 2.6), $N(I) = I$ or $\{0\}$. Since the elements of N are nilpotent endomorphisms of V, $N(I)$ is properly included in I by Engel's Theorem. By assumption, I is minimal, $N(I) = \{0\}$ follows. In particular, $N(I_0) = \{0\}$. Consequently I_0 is invariant under all elements of L which implies that I_0 is an ideal of V. Since I is a minimal ideal of V, $I_0 = \{0\}$ or I If $I_0 = I$, we get together with the first inclusion that $V_1 = V_0$ which implies that $L''(V) = \{0\}$, which is a contradiction to the assumption that $L'' \neq \{0\}$. Therefore $I_0 = \{0\}$ and $V_1 = V_0$ follows.

Case 2: $L'' = \{0\}$

Now we have $L = R + L_0$, $R = Ker\gamma = N$ and $L_0 = L' = L_2$. Clearly π induces a Lie algebra isomorphism from L_2 onto $\overline{L}$. Its inverse map will be denoted by σ. Then we can regard V as an $\overline{L}$-module as follows: $X.v = \sigma(X)(v)$.

1) $N(I) = \{0\}$. This has been shown above.

2) $[I, I, V, \cdots, V] = \{0\}$. Since $\mathrm{ad}(I, V, \cdots, V) \subseteq Ker\gamma$, it follows from the assumption that $\mathrm{ad}(I, V, \cdots, V) \subseteq N$. By 1), $[I, I, V, \cdots, V] = \{0\}$.

3) *I is an irreducible $\overline{L}$-module.* If J is a proper L'-submodule of I, then because of $N(J) \subseteq N(I) = \{0\}$, J must be an ideal of V contradicting the minimality of I. Therefore I is an irreducible L'-module which implies the assertion.

4) *The complementary $\overline{L}$-submodule V_0 is equivalent to the $\overline{L}$-module $\overline{V}$.* In fact, let π_1 denotes the restriction of π on V_0. Then we have for all $X \in \overline{L}$ and all $v \in V_0$:

$$\pi_1(X.v) = \pi(\sigma(X)(v)) = \gamma(\sigma(X))(\pi(v)) = X(\pi(v)).$$

That is, $\pi_1 : V_0 \to \overline{V}$ is an $\overline{L}$-module isomorphism. Let μ be the inverse of π_1.

5) *Set $V_i := \mu(\overline{V}_i)$, $i \in \underline{m}$. Then V_i is an $\overline{L}$-submodule of V_0 that is equivalent to $\overline{V}_i$ and $V_0 = \oplus_{i=1}^m V_i$. Moreover, $\overline{L}_i(V_j) = \{0\}$ for $i \neq j$ and V_i is the natural $\overline{L}_i$-module (or the natural $so(n+1, K)$-module $V(\lambda_1)$).*

6) *The $\overline{L}$-module $[V_0, \cdots, V_0]$ contains a submodule that is equivalent to V_0, hence it contains an $\overline{L}$-submodule equivalent to V_i.* This is evident because we have

$$\pi[V_0, \cdots, V_0] = [\pi(V_0), \cdots, \pi(V_0)] = [\overline{V}, \cdots, \overline{V}] = \overline{V},$$

7) *We have the following decomposition:*

$$[V_0, \cdots, V_0] = \sum_{\substack{0 \leq n_1, \cdots, n_m \leq n \\ n_1 + \cdots + n_m = n}} V_{(n_1, \cdots, n_m)},$$

where

$$V_{(n_1, \cdots, n_m)} := [\underbrace{V_1, \cdots, V_1}_{n_1}, \cdots, \underbrace{V_m, \cdots, V_m}_{n_m}].$$

Clearly, each $V_{(n_1, \cdots, n_m)}$ is an $\overline{L}$-submodule of V and is equivalent to a submodule of $V^{(n_1, \cdots, n_m)} := \wedge^{n_1} V_1 \otimes \wedge^{n_2} V_2 \otimes \cdots \otimes \wedge^{n_m} V_m$. Furthermore, any two distinct $\overline{L}$-modules $V^{(n_1, \cdots, n_m)}$ and $V^{(n'_1, \cdots, n'_m)}$ contain no nonzero equivalent submodules. We show the last assertion by showing that if U resp. U' is a nonzero submdule of $V^{(n_1, \cdots, n_m)}$ resp. $V^{(n'_1, \cdots, n'_m)}$ with $U \cong U'$, then $(n_1, \cdots, n_m) = (n'_1, \cdots, n'_m)$. Let

$$U \cong U_1 \otimes \cdots \otimes U_m \text{ and } U' \cong U'_1 \otimes \cdots \otimes U'_m,$$

where U_i and U_i' are $\overline{L}_i$-modules. Then U_i is equivalent to U_i', in other words, $\wedge^{n_i}V_i$ and $\wedge^{n_i'}V_i$ contain equivalent submodules for all $i \in \underline{m}$. By Table 3, this is only possible when $n_i = n_i'$ or $n_i + n_i' = n + 1$ for all i. If $n_i = n_i'$ for all $i \in \underline{m}$, we are done. Assume that $n_i + n_i' = n + 1$ for some i, say $i = 1$. Then we have $\sum_{i=2}^{m}(n_i + n_i') = \sum_{i=2}^{m} n_i + \sum_{i=2}^{m} n_i' = 2n - (n_1 + n_1') = n - 1$, which forces $m \geq 2$ and $n_i + n_i' \leq n - 1 < n + 1$ for all $i \geq 2$. Note that for all $i \in \underline{m}$, one of the two possibilities $n_i = n_i'$, $n_i + n_i' = n + 1$ must hold. Therefore we get $n_i = n_i'$ for all $i \in \underline{m}$, $i \geq 2$ which in turn implies that $n_1 = n_1'$ because $\sum_{i=1}^{m} n_i = \sum_{i=1}^{m} n_i' = n$. The assertion follows.

8) $[V_i, \cdots, V_i] \cong V_i$ *as* $\overline{L}$*-modules for all* $i \in \underline{m}$. The $\overline{L}$-module $V_{(0,\cdots,n,\cdots,0)}$ is equivalent to a submodule of $V^{(0,\cdots,n,\cdots,0)}$ by 7) while $V^{(0,\cdots,n,\cdots,0)} = \wedge^n V_i \cong V_i$ because V_i is $(n+1)$-dimensional. Therefore $V_{(0,\cdots,n,\cdots,0)}$ is either $\{0\}$ or equivalent to V_i because V_i is irreducible. Once again by 7), the $\overline{L}$-module $V^{(n_1,\cdots,n_m)}$, $(n_1, \cdots, n_m) \neq (0, \cdots, n, \cdots, 0)$, contains no submodule equivalent to $V^{(0,\cdots,n,\cdots,0)}$, so does its submodule $V_{(n_1,\cdots,n_m)}$. Therefore the relation $V_{(0,\cdots,n,\cdots,0)} = \{0\}$ will imply that $[V_0, \cdots, V_0]$ contains no submodule equivalent to V_i which contradicts 6). Therefore $V_{(0,\cdots,n,\cdots,0)} = [V_i, \cdots, V_i] \cong V_i$.

9) $V_{(n_1,\cdots,n_m)}$ *is nonzero for at most one tuple* $(n_1, \cdots, n_m)$ *with* $0 \leq n_1, \cdots, n_m < n$ *and* $n_1 + \cdots + n_m = n$, *say* $(n_1^0, \cdots, n_m^0)$. Since I is an irreducible $\overline{L}$-module by 3), $V = I + \sum_{i=1}^{m} V_i$ being a direct sum of $m+1$ irreducible submodules. As a submodule of V, $[V_0, \cdots, V_0]$ will be a sum of at most $m+1$ irreducible submodules. By 8), $[V_i, \cdots, V_i] \cong V_i$ for all $i \in \underline{m}$, then the decomposition of $[V_0, \cdots, V_0]$ in 7) implies the assertion.

Now we can write

$$[V_0, \cdots, V_0] = \sum_{i=1}^{m}[V_i, \cdots, V_i] + V_{(n_1^0,\cdots,n_m^0)}.$$

In the following we discuss three cases.

Case 2.1: I is equivalent to no submodule of $[V_0, \cdots, V_0]$.

Because the decomposition of a module into its isotypic components is unique, I and V_i, $i \in \underline{m}$ are the only irreducible submodules of V (which are pairwise inequivalent). As a submodule of V, $[V_0, \cdots, V_0]$ must be a sum of some $V_i's$. Therefore V_0 is a (Levi) subalgebra of V.

Case 2.2: I is equivalent to a submodule of $V^{(n_1^0,\cdots,n_m^0)}$.

Once again I and V_i, $i \in \underline{m}$ are the only irreducible submodules of V and they are pairwise inequivalent. By 8), $[V_i, \cdots, V_i] = V_i$, that is, V_i is a subalgebra of V

for all $i \in \underline{m}$. If $m = 1$, then we done because $V_0 = V_1$. Now let $m \geq 2$. We will show that $V_{(n_1^0,\cdots,n_m^0)} = \{0\}$ which implies that V_0 is a subalgebra of V and therefore a Levi subalgebra.

By renumbering we can assume that $0 < n_1^0 < n-1$. Then because of 9) we have for all $2 \leq i \leq m$,

$$[\underbrace{V_1, \cdots, V_1}_{n-1}, V_i] = \{0\}, \tag{4.1}$$

since $(n_1^0, \cdots, n_m^0) \neq (n-1, 0, \cdots, 1, \cdots, 0)$. Since $\pi_1(V_1) = \overline{V}_1$, V_1 is $(n+1)$-dimensional simple n-Lie algebra. Let $\{e_1, \cdots, e_{n+1}\}$ be a basis of V_1 with

$$[e_1, \cdots, \widehat{e_i}, \cdots, e_{n+1}] = e_i, \quad i \in \underline{n+1} \tag{4.2}$$

(see [5]). Let $v_{i,j} \in V_i$, $2 \leq i \leq m$, $j \in \underline{n_i^0}$. Let v_j, $j \in \underline{n_1^0}$ be arbitrary distinct elements in $\{e_1, \cdots, e_{n+1}\}$. By an appropriate numbering we can assume that $v_j = e_j$, $j \in \underline{n_1^0}$. Then we have by (4.1), (4.2) and the G.J.I.,

$$\begin{aligned}
&[e_1, \cdots, e_{n_1^0}, v_{2,1}, \cdots, v_{2,n_2^0}, \cdots, v_{m,1}, \cdots, v_{m,n_m^0}] \\
&= [e_1, \cdots, e_{n_1^0-1}, [e_1, \cdots, \widehat{e_{n_1^0}}, \cdots, e_{n+1}], v_{2,1}, \cdots, v_{m,n_m^0}] \\
&= \sum_{s=n_1^0+1}^{n+1} [e_1, \cdots, \widehat{e_{n_1^0}}, \cdots, [e_1, \cdots, e_{n_1^0-1}, e_s, v_{2,1}, \cdots, v_{m,n_m^0}], \cdots, e_{n+1}] \\
&= \sum_{s=n_1^0+1}^{n+1} [e_1, \cdots, e_{n_1^0-1}, [e_1, \cdots, \widehat{e_{n_1^0}}, \cdots, e_{n+1}], v_{2,1}, \cdots, v_{m,n_m^0}] \\
&= \sum_{s=n_1^0+1}^{n+1} [e_1, \cdots, e_{n_1^0}, v_{2,1}, \cdots, v_{m,n_m^0}] \\
&= (n+1-n_1^0)[e_1, \cdots, e_{n_1^0}, v_{2,1}, \cdots, v_{m,n_m^0}].
\end{aligned}$$

Because $n_1^0 < n$, we get $[e_1, \cdots, e_{n_1^0}, v_{2,1}, \cdots, v_{m,n_m}] = 0$, which implies the relation $V_{(n_1,\cdots,n_m)} = \{0\}$.

Case 2.3: I is equivalent to some V_i, say V_1.

By the assumption, I is equivalent to $\overline{V}_1$. Notice that $\overline{V}_1$ is the only $\overline{L}$-submodule of $\overline{V}$ that is equivalent to I by the assumption. Therefore all nonzero $\overline{L}$-module morphisms from $\overline{V}$ to I are proportional to each other by Schur's Lemma. The same is true for $\overline{L}$-module morphisms from $\wedge^n \overline{V}$ to I in view of the following isomorphism:

$$\wedge^n \overline{V} \cong \sum_{\substack{0 \leq n_1, \cdots, n_m \leq n \\ n_1 + \cdots + n_m = n}} \wedge^{n_1} \overline{V}_1 \otimes \cdots \otimes \wedge^{n_m} \overline{V}_m,$$

because the $\overline{L}$-module $\wedge^n \overline{V}_1$ is the only summand on the right side which is equivalent to $\overline{V}_1$ by 7) (note that $\overline{V}_i \cong V_i$ as $\overline{L}$-modules for every i). We consider the map $\tau(v_1, \cdots, v_n) := [\mu(v_1), \cdots, \mu(v_n)] - \mu[v_1, \cdots, v_n]$. Since $\pi\tau(v_1, \cdots, v_n)) = 0$ for all $v_i \in \overline{V}$, $i \in \underline{n}$, τ is n-linear map from $\times^n \overline{V}$ to I. It is clear that τ is an alternating map. Moreover, since μ is an $\overline{L}$-module morphism, we have for all $X \in \overline{L}$ and all $v_i \in \overline{V}$, $i \in \underline{n}$:

$$\begin{aligned}
& X.(\tau(v_1, \cdots, v_n)) \\
&= X.([\mu(v_1), \cdots, \mu(v_n)]) - X.(\mu([v_1, \cdots, v_n])) \\
&= \sum_{i=1}^{n} [\mu(v_1), \cdots, \mu(v_{i-1}), \mu(X.v_i), \mu(v_{i+1}), \cdots, \mu(v_n)] \\
&\quad - \sum_{i=1}^{n} \mu[v_1, \cdots, v_{i-1}, X.v_i, v_{i+1}, \cdots, v_n] \\
&= \sum_{i=1}^{n} \tau(v_1, \cdots, v_{i-1}, X.v_i, v_{i+1}, \cdots, v_n).
\end{aligned}$$

Therefore τ induces an $\overline{L}$-module morphism $\tilde{\tau} : \wedge^n \overline{V} \to I$.

Now let $\nu : \overline{V} \to I$ be a nonzero $\overline{L}$-module morphism. Consider the map $\nu_1 : \times^n \overline{V} \to I$ defined by

$$\begin{aligned}
\nu_1(v_1, \cdots, v_n) \quad &:= \quad \nu[v_1, \cdots, v_n] \\
&\quad - \sum_{i=1}^{n} [\mu(v_1), \cdots, \mu(v_{i-1}), \nu(v_i), \mu(v_{i+1}), \cdots, \mu(v_n)].
\end{aligned}$$

One can prove that ν_1 is alternating and

$$X.(\nu_1(v_1, \cdots, v_n)) = \sum_{i=1}^{n} \nu_1(v_1, \cdots, v_{i-1}, X.v_i, v_{i+1}, \cdots, v_n).$$

Therefore ν_1 induces an $\overline{L}$-module morphism $\tilde{\nu}_1 : \wedge^n \overline{V} \to I$. We assume that $\nu_1 \neq 0$ (in case $\nu_1 = 0$ we take $\beta\mu$ instead of μ where $\beta^{n-1} \neq 1$). Let $\alpha \in K$ such that $\tilde{\tau} = \alpha \tilde{\nu}_1$, i.e. for all $v_i \in \overline{V}$, $i \in \underline{n}$,

$$\begin{aligned}
& [\mu(v_1), \cdots, \mu(v_n)] - \mu[v_1, \cdots, v_n] \\
&= \alpha\nu[v_1, \cdots, v_n] - \alpha \sum_{i=1}^{n} [\mu(v_1), \cdots, \mu(v_{i-1}), \nu(v_i), \mu(v_{i+1}), \cdots, \mu(v_n)].
\end{aligned}$$

By 2) this identity can be reformulated as

$$(\mu + \alpha\,\nu)[v_1, \cdots, v_n] = [(\mu + \alpha\,\nu)v_1, \cdots, (\mu + \alpha\,\nu)v_n],$$

this means that $\mu + \alpha\,\nu$ is an n-Lie algebra homomorphism from $\overline{V}$ to V. Since $\pi \circ (\mu + \alpha\,\nu) = id_{\overline{V}}$, the image of $\mu + \alpha\nu$ gives a Levi subalgebra of V. □

__Theorem 4.2__: *Let V be an n-Lie algebra over K. If V_0 is a Levi subalgebra of V, then V_0 is also a Levi subalgebra of $V^{(1,n)}$ $(= [V, \cdots, V])$ and $V^{(1,n)} = [V, \cdots, V, Rad(V)] + V_0$ is a Levi decomposition of $V^{(1,n)}$.*

Proof: Since $V = Rad(V) + V_0$ and $[V_0, \cdots, V_0] = V_0$, we have

$$\begin{aligned} V^{(1,n)} &= [Rad(V) + V_0, \cdots, Rad(V) + V_0] \\ &= [V, \cdots, V, Rad(V)] + [V_0, \cdots, V_0] \\ &= [V, \cdots, V, Rad(V)] + V_0 \end{aligned}$$

From $[V, \cdots, V, Rad(V)] \cap V_0 \subseteq Rad(V) \cap V_0 = \{0\}$, we obtain $Rad\ V^{(1,n)} = [V, \cdots, V, Rad(V)]$ and the assertion. □

Appendix A

Lie algebras and their representations

We shall make some reviews and preparations in the theory of Lie algebras and their representations. We assume that all vector spaces, which will be considered, are finite dimensional over a field K of characteristic 0, unless otherwise is noted. For our notations and terminologies see Humphreys [7].

Recall that a Lie algebra is a vector space together with a bilinear operation $(x,y) \to [x,y]$ with the following properties:

1) $[x,y] = -[y,x]$,

2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

for all $x,y,z \in L$. The identity in 2) is the so-called Jacobi identity. A subspace I of L is called an ideal if $[I,L] \subseteq L$. If $[L,L] \neq \{0\}$ and if L and $\{0\}$ are the only ideals of L, then we say L is simple. L is called solvable if $L^{(s)} = \{0\}$ for some $s \in \mathbb{N}$, where $L^{(s)}$ is defined recursively via: $L^{(0)} := L$, $L^{(s+1)} := [L^{(s)}, L^{(s)}]$. The radical $Rad(L)$ is the unique maximal solvable ideal of L. By definition L is semisimple if $Rad(L) = \{0\}$; reductive if $Rad(L) = C(L)$, where $C(L)$ denotes the centre of L: $C(L) := \{x \in L \mid [x,L] = \{0\}\}$. It is obvious that a simple Lie algebra is semisimple, and therefore reductive.

Let V be a vector space. The set $gl(V)$ of all endomorphisms of V forms a Lie algebra relative to the Lie bracket $[f,g] := fg - gf$. A homomorphism ρ of a Lie algebra L into $gl(V)$ is said to be a representation of L in V. Define $x.v := \rho(x)v$, then we get a bilinear map $L \times V \longrightarrow V$ with $[x,y].v = x.y.v - y.x.v$ for all $x,y \in L$ and all $v \in V$. V together with such a map is called an L-module. Conversely if V is an L-module, then V defines a representation ρ of L by $\rho(x)(v) := x.v$. Hence we obtain two equivalent formulations. In this work we shall use both terminologies.

Given an L-module. V is called a faithful L-module if for all $x \in L$ $x.V = \{0\}$ implies $x = 0$. A submodule of V is a subspace U satisfying $L.U \subseteq U$. V is irreducible if $V \neq \{0\}$, and $\{0\}$ and V are the only submodules of V and completely reducible if each submodule possesses a complementary submodule. Further a linear map τ of V into an another L-module W is called an L-module morphism if $\tau(x.v) = x.\tau(v)$ for all $x \in L$ and $v \in V$.

In representation theory we have the well-known

Schur's Lemma: *If K is algebraically closed and if V and W are irreducible L-modules for an arbitrary Lie algebra L, then the vector space of all L-module morphism from V to W is one dimensional if V and W are equivalent; and is $\{0\}$ if V an W are inequivalent.*

Let V be an arbitrary L-module. The dual space V^* of V becomes an L-module by $(x.f)v := -f(x.v)$ $(x \in L,\ f \in V^*,\ v \in V)$, to which we refer as the dual module V^* of V. If V is equivalent to its adjoint module V^*, then we say that V is self-contragredient. We can define an L-invariant bilinear form b on a self-contragredient L-module V as follows: $b(u,v) := (\tau u)(v)$, where τ is an L-module isomorphism from V onto V^*. If V is irreducible and self-contragredient, then there exists up to a scalar only one such form on V (cf. [14]). Moreover it is symmetric or skew-symmetric.

Given another Lie algebra L' and an L'-module W. Let $L_1 = L \oplus L'$. Then the tensor product $V \otimes W$ over K of the underlying vector spaces is an L_1-module with respect to $(x+y).(v \otimes w) := x.v \otimes w + v \otimes y.w$. This module is called the tensor product of L-modules V and and L'-module W. If $L' = L$, then we obtain an L-module by defining $x.(v \otimes w) := x.v \otimes w + v \otimes x.w$. This L-module is said to be the tensor product of L-modules V and W.

Let V be an L-module and $n \in \mathbb{N}$. Then the symmetric group S_n acts on $\otimes^n V$ via: $\sigma.(v_1 \otimes \cdots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for $\sigma \in S_n$, $v_i \in V$, $i \in \underline{n}$. The elements w of $\otimes^n V$ satisfying $\sigma.w = w$ (resp. $\sigma.w = sign(\sigma)\, w$) for all $\sigma \in S_n$ form a subspace $\vee^n V$ $(\wedge^n V)$ of $\otimes^n V$. One can prove that they are L-invariant subspaces of $\otimes^n V$.

Given a representation ρ of L in V, we call the symmetric bilinear form $\kappa : L \times L \to K$, defined by $\kappa(x,y) := \mathrm{tr}(\rho(x)\rho(y))$, the associated form of the representation (or the module) of L. If ρ is the adjoint representation, then the associated form is called the Killing form of L and denoted by $\mathrm{Kill}(\cdot,\cdot)$. In the following we give two theorems and some criteria for semisimplicity and reductivity.

Theorem A1: *(cf. [1] p. 50, p. 52 and p.53) Let L be a Lie algebra. Then the following conditions are equivalent:*

1) L is semisimple.

2) L is a direct sum of its simple ideals.

3) Each L-module is completely reducible.

4) The Killing form of L is nondegenerate.

Theorem A2: *(cf. [1] p. 56) The following are equivalent for a Lie algebra L:*

1) L is reductive.

2) $L = C \oplus [L, L]$, where C is the centre of L and $[L, L]$ is semisimple.

3) There exists a faithful completely reducible L-module.

4) There exists a representation ρ of L whose associated form is nondegenerate.

5) L is completely reducible as an L-module.

Let P be an algebraic closure of K. Then $P \otimes_K L$, regarded as a vector space over P, is a Lie algebra relative to the product

$$[\alpha \otimes x, \beta \otimes y] := \alpha\beta \otimes [x, y].$$

If $K = \mathbb{R}$ and $P = \mathbb{C}$, then we refer to the Lie algebra $\mathbb{C} \otimes_{\mathbb{R}} L$ as the complexification of L.

Theorem A3: *([13] p. 261) L is semisimple over K if and only if $P \otimes_K L$ is semisimple over P.*

If L is a subalgebra of $gl(V)$, where V is a vector space over K, then $P \otimes_K L$ can be regarded as a subalgebra of $gl(P \otimes_K V)$.

Thereom A4: *([13] p. 251) L is completely reducible in V over K if and only if $P \otimes_K V$ is completely reducible in $P \otimes_K V$ over P.*

With the help of the last two theorems many problems can be reduced to the case that the ground field K is algebraically closed. In the following we assume that K is such a field.

A very powerful means to describe semisimple Lie algebras is the theory of root systems. Let E be a Euclidean space with scalarproduct $(\cdot,\cdot) : E \times E \longrightarrow R$. For an $\alpha \in E$, $\alpha \neq 0$ we define a linear map $\sigma_\alpha : E \longrightarrow E$ via

$$\sigma_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \qquad \forall \beta \in E.$$

For brevity we set $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. The map σ_α clearly satisfies the following properties: $\sigma_\alpha(\alpha) = -\alpha$, $\sigma_\alpha(\beta) = \beta$ for all $\beta \in P_\alpha := \{\gamma \in E \mid (\gamma, \alpha) = 0\}$ and $(\sigma_\alpha(\beta), \sigma_\alpha(\gamma)) = (\beta, \gamma)$ for all $\beta, \gamma \in E$. We call σ_α the reflection with respect to the hyperplane P_α. A root system in E is a set Φ with the following properties:

1) $|\Phi| < \infty$, Φ spans E and $0 \notin \Phi$.

2) If $\alpha \in E$, then $k\alpha \in \Phi$ implies $k = \pm 1$.

3) If $\alpha \in \Phi$, then $\sigma_\alpha(\Phi) = \Phi$.

4) $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The elements of Φ are called roots. A base of Φ is a linearly independent subset Δ of Φ such that each β can be written as $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, where k_α are all nonnegative or nonpositive. In the first case β is called a positive root ($\alpha \succ 0$) and in the second case a negative root ($\alpha \prec 0$) with respect to the base Δ. Let Φ^+ (Φ^-) denote the set of positive (resp. negative) roots, then $\Phi = \Phi^+ \cup \Phi^-$. For given $\lambda, \mu \in E$, we write $\lambda \prec \mu$ if $\mu - \lambda$ is a linear combination of elements in Φ^+ with nonnegative coefficients. This gives a half ordering on E.

The subgroup W of $\mathrm{Aut}(E)$ generated by the reflections σ_α, $\alpha \in \Phi$ is the so-called Weyl group of Φ. W acts simply transitively on the bases of Φ. If Δ is a base of Φ, then so is $-\Delta$ and therefore there is a unique element $\sigma_0 \in W$ satisfying $\sigma_0 \Delta = -\Delta$ and $\sigma_0^2 = 1$. For later use we write this fact as

Proposition A5: *For each base Δ of Φ there exists a $\sigma_0 \in W$ with $\sigma_0^2 = 1$, $\sigma_0 \Delta = -\Delta$.*

Define $\Lambda := \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Delta\}$. A $\lambda \in \Lambda$ is called a weight of Φ. It is dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$. We denote by Λ^+ the set of dominant weights. The elements $\lambda_i \in \Lambda^+$ ($i \in \underline{l}$) with $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $\alpha_j \in \Delta = \{\alpha_1, \cdots, \alpha_l\}$, are called the fundamental dominant weights. Obviously $\{\lambda_1, \cdots, \lambda_l\}$ is a base of E, and every element $\mu \in E$ possesses the representation $\mu = \sum_{i=1}^{l} \langle \mu, \alpha_i \rangle \lambda_i$.

Assume that L is a semisimple Lie algebra and H a maximal toral subalgebra of L. A toral subalgebra is by definition a subalgebra consisting of semisimple elements. Relative to H we can decompose L into root spaces: $L = H \oplus \oplus_{\alpha \in \Phi} L_\alpha$, where $L_\alpha := \{x \in L \mid [h, x] = \alpha(h)x,\ \forall h \in H\}$ and Φ is the set of $\alpha \in H^*$ for which $L_\alpha \neq \{0\}$. Since all maximal toral subalgebras are conjugate to each other, the above decomposition (Cartan decomposition) is unique up to isomorphim (cf. [7] p. 82).

View H^* as a vector space over $\mathbb{Q}$ and let $E_{\mathbb{Q}}$ be the collection of all $\mu \in H^*$ which are a linear combination of elements in Φ with rational coefficients. Extending the field $\mathbb{Q}$ to $\mathbb{R}$, we obtain a real vector space E. Since the restriction of the Killing form to H is nondegerate, the identity $(\lambda, \mu) := \mathrm{Kill}(t_\lambda, t_\mu)$ gives rise to a nondegenerate symmetric bilinear form on H^*, where t_λ is the unique element in H satisfying $\mathrm{Kill}(t_\lambda, h) = \lambda(h)$ for all $\lambda \in H^*$. The bilinear form $(\cdot, \cdot)$ on H^* induces in turn a positive definite symmetric bilinear form on E. Therewith E becomes a Euclidean space and Φ a root system in E. Φ is called the root system of L relative to H. We know that there is only one semisimple Lie algebra to a given root system up to isomorphism.

Theorem A6: *Let L be a semisimple Lie algebra and $L = \oplus_{i=1}^{s} L_i$, where L_i are ideals of L, $i \in \underline{s}$.*

1) *If H is a maximal toral subalgebra of L, then $H_i := H \cap L_i$ is a maximal toral subalgebra of L_i. If Φ is the root system of L relative to H and $\Phi^{(i)}$ is the set of the elements $\alpha \in \Phi$ with $\alpha(H_i) \neq \{0\}$, then*

$$\Phi^{(i)}|_{H_i} := \{\alpha|_{H_i} \mid \alpha \in \Phi^{(1)}\}$$

is the root system of L_i relative to H_i. Moreover $\Phi = \cup_{i=1}^{s} \Phi^{(i)}$ and $H = \oplus H_i$.

2) *If H_i is a maximal toral subalgebra of L_i and Φ_i the root system of L_i relative to H_i, then $H := \oplus H_i$ is a maximal toral subalgebra of L and $\Phi := \cup_{i=1}^{s} \Phi^{(i)}$ is the root system of L relative to H, where*

$$\Phi^{(i)} := \{\alpha \in H^* \mid \alpha|_{H_i} \in \Phi_i, \alpha|_{H_j} = 0, \forall j \neq i\}.$$

Remark A1: In view of Theorem A6 1) we shall identify the root system Φ_i of L_i relative to H_i with the subset $\Phi^{(i)}$ of Φ for a given maximal toral subalgebra H of L. Further we shall take the set $\Delta_i := \Phi^{(i)} \cap \Delta$ as a base of Φ_i.

Theorem A7: *A root system Φ of a simple Lie algebra is irreducible, that is, if $\Phi = \Phi_1 \cup \Phi_2$ with $(\Phi_1, \Phi_2) = \{0\}$, then $\Phi_1 = \Phi$ or $\Phi_2 = \Phi$.*

At most two lengths of roots occur in a root system of a simple Lie algebra. Among the roots of the same length there is a highest element relative to "$\prec$". In Table 1 we give the simple Lie algebras by their Dynkin diagram; in Table 2 the highest long and short roots (the numbering of the simple roots is as in Humphreys [7]). They are the only roots which are dominant. The highest long root will also be called the maximal root of L.

In the following let L be a semisimple Lie algebra over K, H a maximal toral subalgebra of L, Φ the root system of L relative to H, Δ a base of Φ, W the Weyl group of Φ. For later use we choose for each $\alpha \in \Phi$ three elements $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ and $h_\alpha \in H$ such that

$$[x_\alpha, y_\alpha] = h_\alpha,\ [h_\alpha, x_\alpha] = 2x_\alpha,\ [h_\alpha, y_\alpha] = -2y_\alpha.$$

For this choice we have $h_\alpha = \frac{2t_\alpha}{\mathrm{Kill}(t_\alpha, t_\alpha)}$. Therefore $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle$ for any $\lambda \in \Lambda$.

Now suppose V is a (not necessarily finite dimensional) L-module. For a $\mu \in H^*$ let $V_\mu := \{v \in V \mid h.v = \mu(h)v,\ \forall h \in H\}$. If $V_\mu \neq \{0\}$, then V_μ is called a weight space of V and μ a weight of V. A maximal vector in V with respect to H is a nonzero vector $v^+ \in V_\lambda$ (of weight λ) killed by all x_α ($\alpha \in \Phi^+$). If $V = U(L).v^+$

($U(L)$ denotes the universal enveloping algebra of L), then we say briefly that V is standard cyclic and call λ the maximal weight of V.

Theorem A8: *Let V be a standard cyclic L-module with maximal vector $v^+ \in V_\lambda$. Let $\Phi^+ = \{\beta_1, \cdots, \beta_m\}$. Then:*

1) *V is spanned by the vectors $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} v^+$ ($i_j \in \mathbb{Z}^+$); in particular, V is the direct sum of its weight spaces.*

2) *The weights of V are of the form $\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i$ ($k_i \in \mathbb{Z}^+$), i.e., all weights of V satisfy $\mu \prec \lambda$.*

3) *For each weight μ of V, $\dim(V_\mu) < \infty$, and $\dim(V_\lambda) = 1$.*

4) *If V is an irreducible L-module, then v^+ is the unique maximal vector in V up to nonzero scalar multiples.*

For each $\lambda \in H^*$ there exists one and (up to isomorphism) only one irreducible standard cyclic L-module $V(\lambda)$ of maximal weight λ. $V(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$. Let $\lambda \in \Lambda^+$ in the following. We write $\Pi(\lambda)$ for the set of weights of $V(\lambda)$. It holds that

$$\Pi(\lambda) = \{\sigma\mu \mid \sigma \in W,\ \mu \in \Lambda^+,\ \mu \prec \lambda\} = \{\mu \in \Lambda \mid \sigma\mu \prec \lambda, \forall \sigma \in W\},$$

and $\dim V_{\sigma\mu} = \dim V_\mu$ for all $\sigma \in W$ and all $\mu \in \Pi(\lambda)$. Let $\mu \in \Pi(\Lambda)$ and $\alpha \in \Phi$. If $r, q \in \mathbb{N}_0$ are maximal such that $\mu - r\alpha \in \Pi(\lambda)$ respectvely $\mu + q\alpha \in \Pi(\lambda)$, then

$$r - q = \langle \mu, \alpha \rangle. \tag{A.1}$$

Let $\sigma_0 \in W$ be as in Proposition 5. For any $\mu \in \Pi(\lambda)$, $\sigma_0\mu \in \Pi(\lambda)$. From $\lambda - \sigma_0\mu \succ 0$ it follows that $0 \succ \sigma_0(\lambda - \sigma_0\mu) = \sigma_0\lambda - \mu$. Hence $\mu - \sigma_0\lambda \succ 0$ and we call $\sigma_0\lambda$ the minimal weight of $V(\lambda)$. A weight vector $v^- \in V_{\sigma_0\lambda}$ which is not 0 is said to be minimal. A minimal vector v^- is characterized by the relation: $y_\alpha.v^- = 0$ for all $\alpha \succ 0$. Because of $\dim V_{\sigma_0\lambda} = 1$ any two minimal vectors are proportional.

By definition it is clear that the weights of the dual module $V^* := V(\lambda)^*$ (it is also irreducible) are $-\mu$, where $\mu \in \Pi(\lambda)$. In view of the above discussion $-\sigma_0\lambda$ must be the maximal weight of V^*. Therefore $V^* = V(-\sigma_0\lambda)$. So $V(\lambda)$ is self-contragredient if and only if $\sigma_0\lambda = -\lambda$.

Proposition A9: *Let L be a semisimple Lie algebra and α a positive root of L. Further let $V(\lambda)$ be an irreducible L-module with a maximal (minimal) vector v^+ (v^-). Then $y_\alpha.v^+ \neq 0$ if and only if $\langle \lambda, \alpha \rangle \neq 0$. Similarly $x_\alpha.v^- \neq 0$ if and only if $\langle \sigma_0\lambda, \alpha \rangle \neq 0$.*

Proof: We show the first assertion only because the second one can be shown analogously.

If $y_\alpha.v^+ = 0$, then $h_\alpha.v^+ = [x_\alpha, y_\alpha].v^+ = 0$. But $h_\alpha.v^+ = \lambda(h_\alpha)v^+ = \langle\lambda,\alpha\rangle v^+$, hence $\langle\lambda,\alpha\rangle = 0$. If $y_\alpha v^+ \neq 0$. Then $\lambda - \alpha \in \Pi(\lambda)$. On the other hand, $\lambda + \alpha \notin \Pi(\lambda)$ due to the maximality of λ, hence $\langle\lambda,\alpha\rangle \geq 1 > 0$ by (A.1). □

Theorem A10: *Let V and W be two irreducible L-modules. Let v^+ be a maximal vector of V and w^- a minimal of W. Then $v^+ \otimes w^-$ ($v^+ \wedge v^-$ resp. $v^+ \vee v^-$) generates the L-module $V \otimes W$ ($V \wedge V$ resp. $V \vee V$).*

Proof: Let U be the L-submodule of $V \otimes W$ generated by $v^+ \otimes w^-$. Apply y_{β_i} to $v^+ \otimes w^-$ repeatedly (notice that $y_\alpha.w^- = 0$), we get

$$y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.(v^+ \otimes w^-) = y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.v^+ \otimes w^-.$$

From Theorem A8 one concludes $V \otimes w^- \subseteq U$ because V is spanned by the elements $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.v^+$.

Let W_0 be the collection of the elements $w \in W$ such that $V \otimes w \subseteq U$. W_0 is a subspace of W and $w^- \in W_0$. For any $x \in L$ and any $w \in W_0$, $V \otimes x.w \subseteq x.(V \otimes w) + x.V \otimes w \subseteq U$. This means W_0 is an invariant subspace of W. But W is irreducible, hence $W_0 = W$. □

Theorem A11: *(cf. [14] p. 109) Let L be a semisimple Lie algebra and V an irreducible L-module with maximal weight λ. If $L = L_1 \oplus L_2$ is the direct sum of semisimple Lie algebras L_i, then V can be identified with the tensor product $V_1 \otimes V_2$ of L_i-modules V_i with maximal weight $\lambda^{(i)} = \lambda|_{H_i}$, where $H_i := H \cap L_i$.*

From Theorem A11 we can conclude

Corollary A12: *The hypothesis is as in Theorem A11. If V is faithful in addition, then so are V_i ($i = 1,2$); in particular, $\lambda^{(i)} \neq 0$.*

Proof: Let $x \in L_1$ and $x.V_1 = \{0\}$. Then $x.V = x.(V_1 \otimes V_2) = x.V_1 \otimes V_2 = \{0\}$. Because V is a faithful L-module, $x = 0$. □

For the element σ_0 for each type of simple Lie algebras one is referred to [14] p. 147-151. Recall that $\sigma \in W$ and $\sigma\Delta = -\Delta$.

Proposition A13: *Let L be a simple Lie algebra with the maximal root α_0. If $V(\lambda)$ is a faithful irreducible L-module with the property*

$$\alpha_0 + \alpha = \lambda - \sigma_0\lambda \tag{A.2}$$

for some simple root α of L, then the pair (L, λ) is one of following:

L	λ
A_1	$2\lambda_1$
A_3	λ_2
$B_l, l \geq 2$	λ_1
B_3	λ_3
$D_l, l \geq 4$	λ_1
D_4	λ_3, λ_4
G_2	λ_1

where l denotes the rank of L.

Proof. From (A.2) we get

$$\sigma_0(\alpha_0 + \alpha) + (\alpha_0 + \alpha) = \sigma_0(\lambda - \sigma_0\lambda) + (\lambda - \sigma_0\lambda) = 0.$$

But $\sigma_0\alpha_0 = -\alpha_0$, hence the simple root α satisfies

$$\sigma_0\alpha + \alpha = 0 \tag{A.3}$$

If $l = 1$, then L possesses only one positive root α_0. Since each L-module is self-contragredient, that is, $\sigma_0\lambda = -\lambda$ for all $\lambda \in \Lambda^+$, we get from $2\lambda = 2\alpha_0$ that $\lambda = \alpha_0$, in other words, V is the adjoint A_1-module. If $l = 2$, then L is of type A_2, C_2 or G_2. Let $\Delta = \{\alpha_1, \alpha_2\}$. If $L \cong A_2$, then $\sigma_0\alpha_1 = -\alpha_2$, $\sigma_0\alpha_2 = -\alpha_1$. Thus there is no simple root of A_2 satisfying (A.3). If $L \cong G_2$, then $\alpha_0 = \lambda_2$ and $\sigma_0 = -1$. Now (A.2) becomes $2\lambda = \lambda_2 + \alpha$. The simple roots of G_2 are $2\lambda_1 - \lambda_2$ and $-3\lambda_1 + 2\lambda_2$. Thus $2\lambda = 2\lambda_1$ or $-3\lambda_1 + 3\lambda_2$. Since λ is dominant, $\lambda = \lambda_1$. The maximal root of C_2 is $2\lambda_1$ and $\sigma_0 = -1$. If $\alpha = \alpha_1$, then $\alpha_0 + \alpha$ is not dominant since $\langle \alpha_0 + \alpha, \alpha_2 \rangle = \langle 2\lambda_1 + \alpha_1, \alpha_2 \rangle = -1 < 0$. This contradicts to the condiction that $\alpha_0 + \alpha$ is dominant. If $\alpha = \alpha_2$. $2\lambda = 2\lambda_1 + \alpha_2 = 2\lambda_1 + (-2\lambda_1 + 2\lambda_2) = 2\lambda_2$, thus $\lambda = \lambda_2$. It is well-known that the C_2-module $V(\lambda_2)$ is isomorphic to the B_2-module $V(\lambda_1)$. Now let $l \geq 3$. If α' is a neighbour of α in the Dynkindiagram, then

$$\langle \alpha_0, \alpha' \rangle = \langle \lambda - \sigma_0\lambda - \alpha, \alpha' \rangle > 0. \tag{A.4}$$

Hence if the simple root α possesses two neighbours in the Dynkindiagram, then there are at least two positive coefficients in the expression of α_0 relative to the fundamental weights. By Table 2, L is of type A_l. Since $\sigma_0\alpha_i = -\alpha_{l+1-i}$ for all $i \in \underline{l}$, $l > 3$ must be odd by (A.3) and $\alpha = \alpha_{\frac{l+1}{2}}$. If $l > 3$, then $\alpha_0 + \alpha_{\frac{l+1}{2}}$ is not dominant, hence $l = 3$ and $\alpha = \alpha_2$. This gives rise to $L \cong A_3$ and $\lambda = \lambda_2$.

Now assume that α is an end point and α_i is the unique neighbour of α in the Dynkindiagram (notice that α_i is no end point). Then $\langle \alpha_0, \alpha_i \rangle > 0$, i.e. the coefficient of λ_i in the representation of α_0 relative to the fundamental weights is greater than 0. By Table 1 and Table 2 the simple root α is one of the following: B_l, $l \geq 3$, $\alpha_1 = 2\lambda_1 - \lambda_2$; B_3, $\alpha_3 = -\lambda_2 + 2\lambda_3$; D_l, $l \geq 4$, $\alpha_1 = 2\lambda_1 - \lambda_2$; D_4, $\alpha_3 = -\lambda_2 + 2\lambda_3$ or $\alpha_4 = -\lambda_2 + 2\lambda_4$. From this we obtain the following possibilities:

$B_l, l \geq 3$	λ_1
B_3	λ_3
$D_l, l \geq 4$	λ_1
D_4	λ_3, λ_4

□

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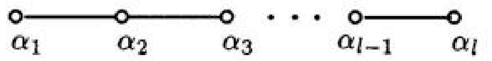
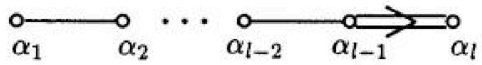
The computer program 'Lie' by Arjeh M. Cohn, Bert Lisser, Bart de Smit and Ron Sommeling

Table 1

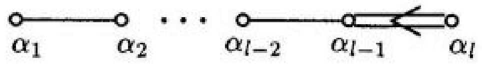
A_l, $l \geq 1$



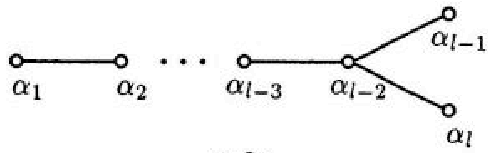
B_l, $l \geq 3$



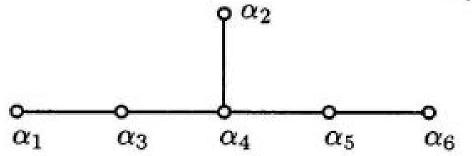
C_l, $l \geq 2$



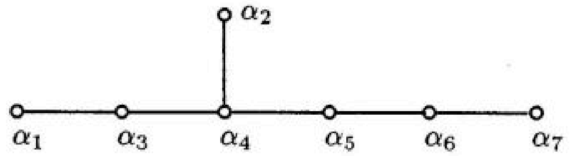
D_l, $l \geq 4$



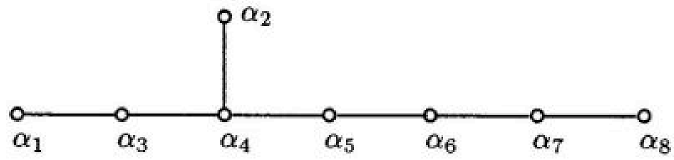
E_6



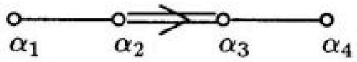
E_7



E_8



F_4



G_2

Table 2

Algebra	the highest long root	the highest short root
$A_l, l \geq 1$	$\lambda_1 + \lambda_l$	
$B_l, l \geq 3$	λ_2	λ_1
$C_l, l \geq 2$	$2\lambda_1$	λ_2
$D_l, l \geq 4$	λ_2	
E_6	λ_2	
E_7	λ_1	
E_8	λ_8	
F_4	λ_1	λ_4
G_2	λ_2	λ_1

Table 3

$A_l, l \geq 1$	$\wedge^m V(\lambda_1) \cong V(\lambda_m), 1 \leq m \leq l$
$B_l, l \geq 3$	$\wedge^m V(\lambda_1) \cong V(\lambda_m), 1 \leq m \leq l-1$ $\wedge^l V(\lambda_1) \cong V(2\lambda_l)$
$C_l, l \geq 2$	$\wedge^m V(\lambda_1) \cong V(\lambda_m) \oplus \wedge^{m-2} V(\lambda_1), 2 \leq m \leq l$
$D_l, l \geq 4$	$\wedge^m V(\lambda_1) \cong V(\lambda_m), 1 \leq m \leq l-2$ $\wedge^{l-1} V(\lambda_1) \cong V(\lambda_{l-1} \oplus \lambda_l)$ $\wedge^l V(\lambda_1) \cong V(2\lambda_{l-1}) \oplus V(2\lambda_l)$

Remark: In all tables above l denotes the rank of the Lie algebra. Table 1 and Table 3 can be found in [7] and [17] respectively. The highest long root and the highest short root in Table 2 are given in [7] by the simple roots. In [7] one can also find the type B_2. But it turns out that B_2 is the same thing as C_2.