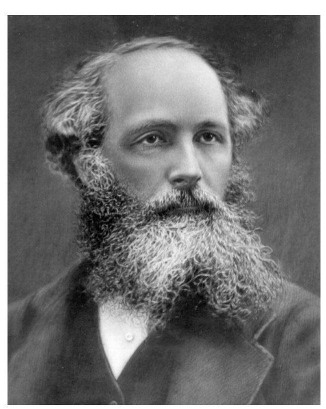
ARNOLD REGGE CENTER

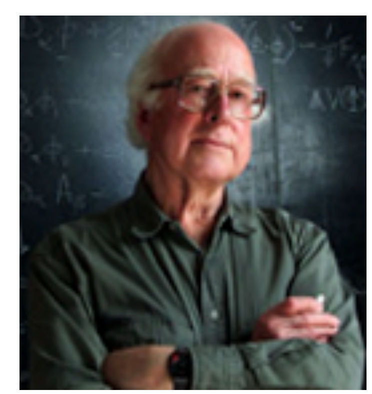
An Erlangen Programme for Supergravity

José Miguel Figueroa-O’Farrill

1 March 2017



THE UNIVERSITY OF EDINBURGH



References

- arXiv:1511.08737 [hep-th], with **Andrea Santi**
- arXiv:1511.09264 [hep-th], with **Andrea Santi**
- arXiv:1605.00881 [hep-th], with **Paul de Medeiros + Andrea Santi**
- arXiv:1608.05915 [hep-th], with **Andrea Santi**
- work in progress with **Paul de Medeiros + Andrea Santi**

Klein's Erlangen Programme

Felix Klein (1849-1925)

"geometry via symmetry"

Physics via symmetry

Physics via the relativity transformation group:

classical ↔ **galilean**

special relativistic ↔ **Poincaré**

general relativistic ↔ **Diffeomorphisms**

d=11 SUGRA backgrounds

(M, g) lorentzian, spin, 11-dimensional

$\$ \to M$ spinor bundle, real, rank 32, symplectic

$F \in \Omega^4(M) \quad dF = 0$

$$\left.\begin{aligned} d \star F &= -\tfrac{1}{2} F \wedge F \\ \mathrm{Ric} - \tfrac{1}{2} R g &= T(g, F) \end{aligned}\right\} \text{field equations}$$

Natural Problem (v1)

Classify supergravity backgrounds.

Hopelessly hard!

Supersymmetric backgrounds

A connection on spinors

$$D_X = \nabla_X - \tfrac{1}{24}(X \cdot F - 3F \cdot X)$$

A **supersymmetric** background admits **Killing spinors**

$$D\varepsilon = 0$$

$$\nu := \frac{\dim\{\varepsilon \in \$ \mid D\varepsilon = 0\}}{\operatorname{rank}\ \$} \in \{0, \tfrac{1}{32}, \tfrac{1}{16}, \tfrac{3}{32}, \dots, \tfrac{31}{32}, 1\}$$

Natural Problem (v2)

Classify *supersymmetric* backgrounds.

$$\nu > 0$$

Still too hard!

Kaluza-Klein supergravity

Lots of work in the late 1970s and early-to-mid 1980s, much of it in Torino, on backgrounds which are metrically a product (e.g., Freund-Rubin, Englert,…).

The main tools are homogeneous geometry and riemannian holonomy.

Closely related to the classification of homogeneous Einstein manifolds.

More recent approaches

- **holonomy** (of D) ⇒ classification for $\nu = 1$
- **spinorial geometry** ⇒ rules out $\nu = \frac{31}{32}, \frac{30}{32}$
- **G-structures** ⇒ general Ansätze for $\nu = \frac{1}{32}$
- **generalised geometry** ⇒ warped products $(\nu \leq \frac{1}{2})$

Natural Problem (v2.5)

Classify supersymmetric backgrounds which are **>½-BPS**

$$\nu > \frac{1}{2}$$

Still hard, but perhaps doable.

Why **>½-BPS**?

Conjecture (Meessen, ’04)

>½-BPS supergravity backgrounds are homogeneous:

$G \curvearrowright M$ transitively, preserving g, F

(e.g., M2 shows the conjecture is sharp.)

Theorem (JMF + Hustler, ‘12)

>½-BPS supergravity backgrounds are *locally* homogeneous

Key ingredient: the **Killing superalgebra**

Killing superalgebra

A Lie superalgebra generated by the Killing spinors

$$\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$$

$$\mathfrak{k}_{\bar{0}} = \{\xi \in TM \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\}$$

$$\mathfrak{k}_{\bar{1}} = \{\varepsilon \in \$ \mid D\varepsilon = 0\}$$

Killing superalgebra

$[\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{0}}] \subset \mathfrak{k}_{\bar{0}}$ Lie bracket of vector fields

$[\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{1}}$ Lichnerowicz Lie derivative

$[\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{0}}$ Dirac current

$\kappa^a = \bar{\varepsilon}\gamma^a\varepsilon$ is a causal vector field

$$D\varepsilon = 0 \implies \mathcal{L}_\kappa g = \mathcal{L}_\kappa F = 0$$

Local homogeneity

$$\dim \mathfrak{k}_{\bar{1}} > 16$$

Every tangent space is spanned by the Dirac currents of Killing spinors.

Killing Lie algebras

Some (2-graded) Lie algebras

$$\mathfrak{so}(9) \qquad \mathfrak{f}_4 \qquad \mathfrak{e}_8$$

are also generated by (geometric) Killing spinors* on

$$S^7 \qquad S^8 \qquad S^{15}$$

* $\nabla_X \varepsilon = \frac{1}{2} X \cdot \varepsilon$

KSA as a Lie superalgebra

Theorem (JMF + Santi, ’16)
The KSA is a filtered Lie superalgebra. It is a *filtered deformation* of a graded subalgebra of the Poincaré superalgebra.

Poincaré superalgebra is graded

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}$$

Lorentz

$\mathfrak{so}(V)$

supercharges

S

translations

V

$$[\mathfrak{p}_i, \mathfrak{p}_j] = \mathfrak{p}_{i+j}$$

$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2}$ graded subalgebra of $\mathfrak{p}$

$\mathfrak{a}_0 \subset \mathfrak{so}(V) \qquad \mathfrak{a}_{-1} \subset S \qquad \mathfrak{a}_{-2} \subset V$

Modify the Lie brackets by terms of positive degree:

$$
\begin{aligned}
[\mathfrak{a}_0, \mathfrak{a}_0] &\subset \mathfrak{a}_0 \\
[\mathfrak{a}_0, \mathfrak{a}_{-1}] &\subset \mathfrak{a}_{-1} \\
[\mathfrak{a}_0, \mathfrak{a}_{-2}] &\subset \mathfrak{a}_{-2} \oplus \mathfrak{a}_0 \\
[\mathfrak{a}_{-1}, \mathfrak{a}_{-1}] &\subset \mathfrak{a}_{-2} \oplus \mathfrak{a}_0 \\
[\mathfrak{a}_{-1}, \mathfrak{a}_{-2}] &\subset \mathfrak{a}_{-1} \\
[\mathfrak{a}_{-2}, \mathfrak{a}_{-2}] &\subset \mathfrak{a}_0 \oplus \mathfrak{a}_{-2}
\end{aligned}
$$

This is the structure of the Killing superalgebra

Filtered deformations

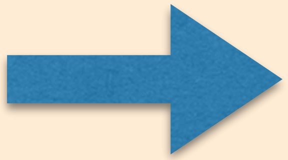
The passage from a *graded* to a *filtered* algebra is very familiar when passing to the very small:

classical	→	**quantum**
$\Lambda^\bullet V$	→	$C\ell(V)$
symbols	→	diff. ops.

The deformation parameter is $\hbar$

Filtered deformations

But also when passing to the very large:

flat  **curved**

The isometry Lie algebra of flat space is graded, but that of a curved manifold is filtered.

The deformation “parameter” is the curvature.

Spencer Cohomology

Deformations of algebraic structures are typically governed by a cohomology theory.

e.g., Lie algebra deformations are governed by Chevalley-Eilenberg cohomology.

Filtered deformations of graded Lie superalgebras are governed by generalised Spencer cohomology.

This is a bigraded refinement of Chevalley-Eilenberg cohomology.

Spencer complex

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0$$

graded Lie superalgebra

$$C^{d,p}(\mathfrak{g}_-, \mathfrak{g}) = \left(\Lambda^p \mathfrak{g}_-^* \otimes \mathfrak{g}\right)_{\text{degree } d}$$

$$C^{2,1}(\mathfrak{g}_-, \mathfrak{g}) = \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0$$

$$C^{2,2}(\mathfrak{g}_-, \mathfrak{g}) = \left(\Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}\right) \oplus \left(\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}\right) \oplus \left(\odot^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0\right)$$

$$\partial : C^{d,p}(\mathfrak{g}_-, \mathfrak{g}) \to C^{d,p+1}(\mathfrak{g}_-, \mathfrak{g})$$

Chevalley-Eilenberg

Infinitesimal deformations

Infinitesimal filtered deformations are controlled by

$H^{2,2}(\mathfrak{g}_-, \mathfrak{g})$

whereas $H^2(\mathfrak{g}, \mathfrak{g})$ controls *all* deformations

cocycles $\alpha + \beta + \gamma$

$$\alpha : \Lambda^2 \mathfrak{g}_{-2} \to \mathfrak{g}_{-2}$$

$$\beta : \mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$$

$$\gamma : \odot^2 \mathfrak{g}_{-1} \to \mathfrak{g}_0$$

Some calculations

$\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ d=11 Poincaré superalgebra

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^4 V$$

The cocycle component $\beta : V \otimes S \to S$

of a class $F \in \Lambda^4 V$ is given

$$\beta(v,s) = \tfrac{1}{24}(v \cdot F - 3F \cdot v) \cdot s$$

cf. the connection in the gravitino variation!

$\mathfrak{p}$ d=4 N=1 Poincaré superalgebra

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$$

bosonic field content of "old" minimal off-shell formulation of supergravity

and we again recover the gravitino variation from the cocycle.

Reconstruction

Theorem (JMF + Santi, ’16)
Every “admissible” filtered deformation of a graded subalgebra of the Poincaré superalgebra is (contained in) the KSA of a >½-BPS supergravity background.

In particular, >½-BPS implies the field equations.

⇒ Classify >½-BPS supergravity backgrounds via their KSAs.

Supergravity backgrounds via their (super)symmetries.

Some further results

- Recovered classification of maximally supersymmetric backgrounds of d=11supergravity.
- Recovered the Festuccia-Seiberg classification of four-dimensional geometries admitting rigid N=1 supersymmetry.
- N=(1,0) d=6 Poincaré: more than supergravity
- N>1 d=4 Poincaré
- Superconformal algebras in various low dimensions

Conclusions

- We have reformulated the classification problem of **>½-BPS backgrounds** of d=11 supergravity as the classification problem of certain Lie superalgebras ("admissible" **filtered deformations** of graded subalgebras of the Poincaré superalgebra)

- Generalised Spencer cohomology **knows** about supergravity and, more generally, about the **Killing spinor equations** which allow us to define **rigidly supersymmetric field theories**.