Lie algebra deformations and d=11 supergravity backgrounds

(joint work with Andrea Santi)

(V,η) 11-dim'l Lorentzian vector space

$Cl(V) \supset \mathfrak{so}(V)$

S irred $Cl(V)$-mod : $\dim_{\mathbb{R}} S = 32$ $\omega : S \times S \longrightarrow \mathbb{R}$ symplectic structure

$\omega(v\cdot s_1, s_2) = -\omega(s_1, v\cdot s_2)$

$\kappa : S \times S \longrightarrow V$ $\eta(\kappa(s_1,s_2),v) = \omega(s_1, v\cdot s_2)$ $\kappa(s_1,s_2)^\mu \sim s_1^T C\Gamma^\mu s_2$

$\kappa(s_1,s_2) = \kappa(s_2,s_1)$

On $\underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V}$ $\mathbb{Z}$-graded LSA $\Rightarrow$ Poincaré superalgebra $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}$

Nahm ('78) Smallest massless unitary irrep $\Rightarrow$ induced from $\underset{\text{traceless}}{\odot^2_0 W} \oplus \underset{\text{spinor of } \mathfrak{so}(9)}{(\Sigma \otimes W)_0} \oplus \Lambda^3 W$

vector rep of $\mathfrak{so}(9)$

Γ-traceless

Conjecture : $\exists$ d=11 SUGRA $(\underset{\text{Lor. metric}}{g}, \underset{\text{gravitino}}{\Psi}, \underset{\text{3-form potential}}{A})$

Cremmer + Julia + Scherk ('78) constructed d=11 SUGRA theory

(M,g) Lorentzian spin mfd $\$ \rightarrow M$ spinor bundle $F \in \Omega^4(M)$, $dF = 0$ D connection on $\$$

$$D_X = \nabla_X - \tfrac{1}{24}\left(X^\flat\cdot F - 3F\cdot X^\flat\right) = \nabla_X + \tfrac{1}{12} X^\flat \wedge F + \tfrac{1}{6}\iota_X F$$

$\ker D = \{\text{Killing spinors}\}$

$\dim \ker D \in \{0,1,2,\ldots,32\}$

Γ-trace of $R^D = 0$ $\Leftrightarrow$ $dF = 0$
"Maxwell"
Einstein } equations

$\therefore$ D is the fundamental geometric object in $d=11$ SUGRA.

Hull ('02) $\mathrm{Hol}_0(D) \leqslant SL(32,\mathbb{R})$

Duff et al. ( ) studied $\mathrm{Hol}_0(D)$ for many backgrounds

Much remains to be understood.

$$\left.\begin{array}{l} \mathfrak{k}_{\bar 0} = \{ X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0 = \mathcal{L}_X F \} \\ \mathfrak{k}_{\bar 1} = \{ \varepsilon \in \Gamma(\$) \mid D\varepsilon = 0 \} \end{array}\right\} \quad \mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$$

Lie superalgebra   Killing superalgebra

Meessen ('04) conjectured $\dim\ker D > 16$
$\Downarrow$
(M,g,F) homogeneous

JMF + Hustler ('12) $\dim\ker D > 16 \Rightarrow (M,g,F)$ locally homogeneous
(sharp: M2, M5 $\frac{1}{2}$-BPS ~~not~~ loc. homog.)

$\left(\Leftarrow \quad \kappa : \odot^2 S' \twoheadrightarrow V \text{ for } \dim S' > 16 \right)$

State of the art:
$(>\frac{1}{2})$-BPS

dim ker D	32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17
	✓ FRs Mink KG	∄	∄	?	?	?	PP	?	PP FR	?	PP	?	PP Gö	?	PP	?

Freund+Rubin, Kowalski-Glikman,
Michelson, Pope, Gauntlett+Hull,
JMF+Papadopoulos, JMF+Gadhia
Gran+Gutowski+Papadopoulos, ...

Question: How big is the supersymmetry gap?

Problem: Classify $>\frac{1}{2}$-BPS backgrounds (up to local iso.)

Standard approaches:
- explicit construction via Ansätze
- G-structures
- spinorial geometry

have all hit a wall.

Novel approach : Filtered deformations of Lie superalgebras (Cheng + Kac '98)

Theorem The Killing superalgebra of a sugra background is filtered:

$$\mathfrak{k} = \mathfrak{k}^{-2} \supset \mathfrak{k}^{-1} \supset \mathfrak{k}^{0} \supset 0 \qquad [\mathfrak{k}^i, \mathfrak{k}^j] \subset \mathfrak{k}^{i+j}$$

associated graded

$$\mathrm{gr}(\mathfrak{k}) = \underset{\mathfrak{k}^{-2}/\mathfrak{k}^{-1}}{\underset{\parallel}{\mathrm{gr}_{-2}(\mathfrak{k})}} \oplus \underset{\mathfrak{k}^{-1}/\mathfrak{k}^{0}}{\underset{\parallel}{\mathrm{gr}_{-1}(\mathfrak{k})}} \oplus \underset{\mathfrak{k}^{0}}{\underset{\parallel}{\mathrm{gr}_{0}(\mathfrak{k})}} \cong V' \oplus S' \oplus \mathfrak{h}$$

$\mathbb{Z}$-graded subalgebra of $\mathfrak{p}$: $V' \subset V$, $S' \subset S$

Lie subalgebra $\longrightarrow$ $\mathfrak{h} < \mathfrak{so}(V)$

$\forall\, A, B \in \mathfrak{h}$, $v, w \in V'$, $s \in S'$

$$[A,B] = A \circ B - B \circ A \qquad [s,s] = \kappa(s,s) + \underset{2}{\gamma}(s,s)$$

$$[A,s] = \sigma(A)s \qquad [v,s] = \underset{2}{\beta_v}(s) \qquad \beta : V' \to \mathrm{End}(S')$$

$$[A,v] = Av + \underset{2}{\delta}(A,v) \qquad [v,w] = \underset{2}{\alpha}(v,w) + \underset{4}{\rho}(v,w) \qquad (\mathrm{fdeg})$$

$\dim S' > 16 \Rightarrow V' = V$ Filtered deformations governed by 'generalised Spencer cohomology'.

Some calculations: $\mathfrak{p} = \mathfrak{p}_0 \oplus \underbrace{\mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}}_{\mathfrak{p}_-}$ Poincaré superalgebra

Chevalley-Eilenberg $C^\bullet(\mathfrak{p}_-;\mathfrak{p})$. (Lie superalgebra; rep of $\mathfrak{p}_-$ restricting $\mathrm{ad}^{\mathfrak{p}}$ to $\mathfrak{p}_-$)

$\mathfrak{p}$ is $\mathbb{Z}$-graded $\rightarrow$ $\deg [\,,\,] = 0 \Rightarrow \deg \partial_{CE} = 0$

$$\Rightarrow C^\bullet(\mathfrak{p}_-;\mathfrak{p}) = \bigoplus_d \underset{\text{generalised Spencer complex}}{C^{d,\bullet}(\mathfrak{p}_-;\mathfrak{p})}$$

cf. inf. deformations of a LA $\mathfrak{g}$ classified by $H^2(\mathfrak{g};\mathfrak{g})$, but here we only deform in pos. degree

Example

$d = -1$

$$0 \to S \xrightarrow{\partial} \mathrm{Hom}(S,V) \to 0$$

$$(\partial s_1)(s_2) = \kappa(s_1,s_2)$$

$\Rightarrow$ ∂ is injective

$\Rightarrow$ $H^{-1,1}(\mathfrak{p}_-;\mathfrak{p}) \cong \dfrac{\mathrm{Hom}(S,V)}{\partial S}$

$$\begin{tikzcd}
0 \arrow[r] & (V\otimes S)_0 \arrow[r] & V\otimes S \arrow[r, "cl"] & S \arrow[r] & 0 \\
0 & H^{-1,1}(\mathfrak{p}_-;\mathfrak{p}) \arrow[l] & \mathrm{Hom}(S,V) \arrow[l] & S \arrow[l, "\partial"'] & 0 \arrow[l]
\arrow[from=1-2, to=2-2, leftrightarrow, "\text{dual}"]
\arrow[from=1-1, to=2-1, bend right, "\text{dualise}"']
\end{tikzcd}$$

"gravitino"

$d = 2$

$$\mathrm{Hom}(V,\mathfrak{so}(V)) \xrightarrow{\partial} \begin{matrix} \mathrm{Hom}(V,\operatorname{End} S) \\ \cong \\ \mathrm{Hom}(V\otimes S, S) \\ \oplus \\ \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V)) \\ \oplus \\ \mathrm{Hom}(\Lambda^2 V, V) \end{matrix} \xrightarrow{\partial} \begin{matrix} \mathrm{Hom}(\odot^3 S, S) \\ \oplus \\ \mathrm{Hom}(\odot^2 S\otimes V, V) \end{matrix} \xrightarrow{\partial} \mathrm{Hom}(\odot^4 S, V)$$

$$\mathrm{Hom}(V,\mathfrak{so}(V)) \xrightarrow{\cong} \mathrm{Hom}(\Lambda^2 V, V)$$

Every cohomology class has a unique cocycle representative whose $\mathrm{Hom}(\Lambda^2 V, V)$-component is 0.

(Adding coboundary = adding torsion to metric connection.)

$$H^{2,2}(\mathfrak{p}_-;\mathfrak{p}) \underset{\substack{\uparrow \\ \mathfrak{so}(V)\text{-rep}}}{\cong} \Lambda^4 V$$

Let $\varphi \in \Lambda^4 V$, then corresponding cocycle has components:

$$\beta^4 : V \longrightarrow \operatorname{End}(S)$$
$$v \longmapsto \boxed{\beta_v(s) = \tfrac{1}{24}(v\cdot\varphi - 3\varphi\cdot v)\cdot s} \quad \leftarrow \text{cf. } D_X = \nabla_X - \beta_X$$

$$\gamma^4 : \odot^2 S \longrightarrow \mathfrak{so}(V)$$
$$s^2 \longmapsto \gamma(s,s)\,v = -2\,\kappa(\beta_v(s), s)$$

Reconstruction

Let $\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$ be a filtered LSA $\mathfrak{k} = \mathfrak{k}^{-2} \supset \mathfrak{k}^{-1} \supset \mathfrak{k}^{0} \supset 0$ where $\mathrm{gr}(\mathfrak{k}) \cong \mathfrak{h} \oplus S' \oplus V' \underset{\uparrow}{<} \mathfrak{p}$ $\mathbb{Z}$-graded

If $\dim S' > 16$, then $V' = V$. If $\exists\ X : V \to \mathfrak{so}(V)$ s.t.

$$\alpha(v,w) = X_v w - X_w v$$
$$\beta(v,s) = \beta^{\varphi}_v(s) + \sigma(X_v)s$$
$$\gamma(s,s) = \gamma^{\varphi}(s,s) - X_{\kappa(s,s)}$$
$$\delta(A,v) = [A, X_v] - X_{Av}$$

$\exists\ \varphi \in (\Lambda^4 V)^{\mathfrak{h}}$ and $d\varphi = 0$.

then $\mathfrak{k}$ is the KSA of a $d=11$ SUGRA background (determined up to local isometry).

Question: What data need we specify in order to determine $\mathfrak{k}$?

Let $S' \subset S$ have $\dim > 16$. Then $\kappa|_{\odot^2 S'} : \odot^2 S' \to V$ is surjective. Let $\mathfrak{D} \subset \odot^2 S'$ denote its kernel:

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{D} & \to & \odot^2 S' & \xrightarrow{\kappa} & V & \to & 0 \\ & & \downarrow{\scriptstyle \gamma^{\varphi}|_{\mathfrak{D}}} & & \downarrow{\scriptstyle \gamma^{\varphi}|_{\odot^2 S'}} & \underset{\Sigma\ \text{section}}{\longleftarrow} & & & \\ & & \mathfrak{h}(S',\varphi) & \longrightarrow & \mathfrak{so}(V) & & & & \end{array}$$

We say that (S',φ) is a Lie pair if $\forall\ A \in \mathfrak{h}(S',\varphi)$, $A\cdot\varphi = 0$ & $\sigma(A) S' \subset S'$

Then $\mathfrak{h}(S',\varphi) < \mathfrak{so}(V)$ is a Lie subalgebra.

The Killing ideal of a $(>\frac{1}{2})$-BPS supergravity background $(\underline{\mathfrak{k}} : \mathfrak{k}_{\bar{0}} = [\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}])$ is a filtered deformation of $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$, where $\mathfrak{h} = \mathfrak{h}(S, \varphi')$ for a Lie pair (S', φ), $X = \gamma^{\varphi} \circ \Sigma$ for some section Σ.

Andrea's Refinement (Dec '19)

(S', φ) a Lie pair such that $\beta^{\varphi}_v(s) + \sigma(X_v)s \in S' \quad \forall s \in S', v \in V$

$R : \Lambda^2 V \longrightarrow \mathfrak{so}(V)$ given by

$$\tfrac{1}{2} R(v, \kappa(s,s)) w = \kappa\big((X_v \beta^{\varphi})_w(s), s\big) - \kappa\big(\beta^{\varphi}_v(s), \beta^{\varphi}_w(s)\big) - \kappa\big(\beta^{\varphi}_w \beta^{\varphi}_v(s), s\big)$$

and satisfying

$$\sigma(R(v,w))s = (X_v \beta^{\varphi})_w(s) - (X_w \beta^{\varphi})_v(s) + [\beta^{\varphi}_v, \beta^{\varphi}_w](s)$$

then we get a sugra by preserving $\frac{\dim S'}{\dim S}$ of susy.