BMS Working seminar

Carrollian geometry : "$c \to 0$" null hypersurfaces in Lorentzian mfds

cf. Galilean geometry : "$c \to \infty$" null reductions of Lorentzian mfds

Lorentzian

Carrollian $\quad$ Galilean

"duality"?

Examples

$\mathbb{M}^{D+1}$ (D+1)-dim'l Minkowski spacetime $\quad (x^+, x^-, x^a)$

$x^+ = \text{const}$ $\quad$ carrollian spacetime $\quad$ symmetric affine space of carroll gp. $\quad [B_a, P_b] = \delta_{ab} H$

$2x^+x^- = \sum_a (x^a)^2$ $\quad$ carrollian lightcone $\quad$ non-reductive homog space of $SO(1,D)$

$Q_{AdS} \subset \mathbb{R}^{D,2}$ $\quad -t_1^2 - t_2^2 + x_1^2 + \cdots + x_D^2 = -R^2$

$x_1 = t_1$ $\quad$ (Carrollian AdS)/$\sim$ $\quad$ symmetric homog space of $\boxdot$

$Q_{dS} \subset \mathbb{R}^{D+1,1}$ $\quad -t_1^2 + x_1^2 + \cdots + x_{D+1}^2 = R^2$

$x_1 = t_1$ $\quad$ (Carrollian dS)/$\sim$ $\quad$ symmetric homog space of euclidean LA

$i : N \hookrightarrow (M, g)$ $\quad$ $i^* g = h$ degenerate, co-rank 1

null hypers. $\quad$ Lorentzian

v nowhere-vanishing vector field $\quad h(v,-) = 0$

Def. $\xi \in \mathfrak{X}(M)$ is carrollian Killing if

$L_\xi h = 0 \qquad [\xi, v] = 0$

is conformal carrollian Killing (at level $N \in \mathbb{N}$) if

$L_\xi h = \lambda h \qquad [\xi, v] = -\frac{\lambda}{N} v \qquad \exists \lambda \in C^\infty(M)$.

cf. (M,g) Lorentzian $\xi \in \mathfrak{X}(M)$ is Killing if $L_\xi g = 0$

is conformal Killing if $L_\xi g = \lambda g$ $\exists \lambda \in C^\infty(M)$

$\underline{\mathfrak{carr}}(M,h,v) = \{ \xi \in \mathfrak{X}(M) \mid \xi \text{ carrollian Killing} \}$

$\underline{\mathfrak{ccarr}}_N(M,h,v) = \{ \xi \in \mathfrak{X}(M) \mid \xi \text{ conf. carr. Killing at level } N \}$

Both $\underline{\mathfrak{carr}}(M,h,v)$ & $\underline{\mathfrak{ccarr}}_N(M,h,v)$ are Lie subalgebras of $\mathfrak{X}(M)$.

Unlike the Killing & conf. Killing Lie algebras of (M,g), they need not be finite-dimensional.

Examples

Carrollian Killing vector fields

C^{D+1} $\qquad \xi = \underbrace{T(x)}_{\text{smooth fn on } \mathbb{E}^D} \frac{\partial}{\partial t} + \underbrace{\zeta(x)}_{\text{Killing vector of } \mathbb{E}^D}$

$$0 \to \overset{\text{abelian}}{C^\infty(\mathbb{E}^D)} \to \underline{\mathrm{carr}}(C^{D+1}) \rightleftarrows \mathfrak{e} \to 0$$

$1 \mapsto H$, $x^a \mapsto B_a$ $\qquad \underline{\mathrm{carr}}(C^{D+1}) \cong \mathfrak{e} \ltimes C^\infty(\mathbb{E}^D)$ $\qquad \mathfrak{e}$: J_{ab}, P_a

Conformal carrollian VFs

C^{D+1} $\qquad \xi = \Big(\underbrace{T(x)}_{C^\infty \text{ fn on } \mathbb{E}^D} + \frac{2t}{ND}\frac{\partial \zeta^a}{\partial x^a}\Big)\frac{\partial}{\partial t} + \underbrace{\zeta(x)}_{\text{CKV on } \mathbb{E}^D}$

$D \geq 3$

$$0 \to \cancel{C^\infty(\mathbb{E}^D)} \to \underline{\mathrm{ccarr}}(C^{D+1}) \rightleftarrows \underline{\mathfrak{so}}(D+1,1) \to 0$$

$\Gamma(\mathcal{L}^{2/N})$ $\qquad \zeta + \frac{2}{ND}\partial\cdot\zeta\, t\frac{\partial}{\partial t} \longleftarrow \zeta$

$h \in \Gamma(S^2(T^*M) \otimes \mathcal{L}^2)$

$D=2 \qquad \mathrm{CKV}(\mathbb{E}^2) \cong \mathcal{O}(\mathbb{C})$

$D=1 \qquad \mathrm{CKV}(\mathbb{E}^1) \cong C^\infty(\mathbb{R})$

For the carrollian light cone $\mathcal{C}_+^{D+1} \subset M^{D+2}$, $\underline{\mathrm{carr}}(\mathcal{C}_+^{D+1}) \cong \underline{\mathfrak{so}}(D+1,1)$ at least for $D \geq 2$

but $\underline{\mathrm{ccarr}}(\mathcal{C}_+^{D+1}) \cong \Gamma(\mathcal{L}^{2/N}) \rtimes \underline{\mathfrak{so}}(D+1,1)$

By contrast, the symmetries of the galilean structures are reminiscent of KM algebras:

$$0 \to C^\infty(\mathbb{R}_t, \mathfrak{e}^D) \to \underline{\mathrm{gal}}^{D+1} \to \mathbb{R} \to 0$$

$H = \frac{\partial}{\partial t} \longleftarrow 1$

and

$$0 \to C^\infty(\mathbb{R}_t, \mathfrak{e}^D) \to \underline{\mathrm{cgal}}^{D+1} \to C^\infty(\mathbb{R}_t) \to 0$$

"wronskian"

The other $\boxdot$-homogeneous spacetime: "AdSC"$^{D+1}$ $\quad (h, v) \quad$ coordinates $(t, \vec{x})$

$v = \frac{1}{\cosh(r)}\frac{\partial}{\partial t} \qquad h = \underbrace{dr^2 + \sinh^2(r)\, g_{S^{D-1}}}_{H^D}$

$\underline{\mathrm{carr}}(\mathrm{AdSC}_{D+1}) \cong \underline{\mathfrak{so}}(D,1) \ltimes C^\infty(\mathbb{H}^D)$ $\qquad \underline{\mathfrak{so}}(D,1)$: J_{ab}, P_a $\qquad 1 \mapsto H$, $x^a \mapsto B_a$

$\underline{\mathrm{ccarr}}_N(\mathrm{AdSC}_{D+1}) \cong \mathrm{CKV}(\mathbb{H}^D) \ltimes \Gamma(\mathcal{L}^{2/N})$

$\mathrm{CKV}(\mathbb{H}^{D\geq 3}) \cong \underline{\mathfrak{so}}(D+1,1)$ $\qquad$ (conf. flatness)

$\mathrm{CKV}(\mathbb{H}^2) \cong \mathcal{O}(\mathbb{H})$

$\hookrightarrow$ upper half $\mathbb{C}$-plane

$\mathrm{CKV}(\mathbb{H}^1) \cong C^\infty(\mathbb{R})$