Homological approach to hamiltonian reduction (JMF, 3/4/2019)

Recall the setting: (M,ω) connected symplectic manifold
G connected Lie group, $G \curvearrowright M$ a hamiltonian group action with (equivariant) moment map $\mu : M \longrightarrow \mathfrak{g}^*$ and co-moment map $\mu^* : \mathfrak{g} \to C^\infty(M)$

Theorem (Marsden–Weinstein '74)

Under these conditions and if $0 \in \mathfrak{g}^*$ is a regular value of μ and the G action M_0 is free and proper, $\tilde{M} := M_0/G$ is a smooth manifold with symplectic form $\tilde{\omega}$ satisfying $\pi^*\tilde{\omega} = \iota^*\omega$ for

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & M \\ \tilde{M} & \end{tikzcd}$$

For M symplectic (more generally Poisson) $C^\infty(M)$ is a Poisson algebra. Indeed, if $f \in C^\infty(M)$, then $\xi_f = \omega^\#(-df)$ and $[f,g] := \xi_f g$.
$\Leftrightarrow \iota_{\xi_f}\omega = -df$

Symplectic reduction can be understood as a gadget which starts from a Poisson algebra $C^\infty(M)$ and produces a Poisson algebra $C^\infty(\tilde{M})$. This gadget has a homological description first discovered in perturbative Yang–Mills theory (playing a crucial rôle in the proof of renormalisability) and goes by the name of **BRST cohomology** (Becchi, Rouet, Stora & Tyutin).

If 0 is a regular value of μ, then $M_0 := \mu^{-1}(0)$ is a submanifold of M and $C^\infty(M_0) \cong C^\infty(M)/I$, where $I = C^\infty(M)\cdot\mu^*(\mathfrak{g})$ is the ideal of $C^\infty(M)$ consisting of functions vanishing at M_0, equivalently the ideal generated by the image of the comoment map. Equivariance of the co-moment map says that I is a coisotropic ideal: $[I,I] \subset I$.

Everything I say extends more or less straightforwardly to coisotropic Poisson reduction, but this is a working seminar on moment maps and this version is perhaps the simplest to describe.

Equivariance also implies that $G \curvearrowright M_0$ and hence on $C^\infty(M_0)$. Since $\tilde{M} = M_0/G$ we have that $C^\infty(\tilde{M}) \cong C^\infty(M_0)^G \cong \left(C^\infty(M_0)/I\right)^G$ and if G is connected then $C^\infty(\tilde{M}) = \left(C^\infty(M)/I\right)^{\mathfrak{g}}$. This is an isomorphism of associative, commutative algebras, but $C^\infty(M)$ and $C^\infty(\tilde{M})$ are Poisson.
A different description of $C^\infty(\tilde{M})$ makes this manifest.

Proposition (Śniatycki–Weinstein '83)

$$C^\infty(\tilde{M}) \cong N(I)/I \quad \text{where} \quad N(I) = \{f \in C^\infty(M) \mid [f,I] \subset I\}$$

Poisson alg.

(More generally, for $I \subset P$ a coisotropic ideal of a Poisson algebra, the Poisson reduction of P by I is the Poisson algebra $N(I)/I$.)

In Sue's lecture (but for the C^∞ category) she discussed the case of $M = T^*N$ with the hamiltonian G action induced from a G action on N. In that case $\tilde{M} \cong T^*(N/G)$. Sue's talk was about the quantisation of hamiltonian reduction. The basic ingredients are $T^*N \rightsquigarrow \mathrm{Diff}(N)$ acting on $C^\infty(N)$, and the comoment map quantises to $\mu^*: \mathcal{U}\mathfrak{g} \to \mathrm{Diff}(N)$ and Sue wrote the quantisation of $\tilde{M} = M/\!\!/G$ as $\left(\mathrm{Diff}(N)/\mathrm{Diff}(N)\cdot\mu^*(I_0)\right)^G$ where I_0 is the augmentation ideal of $\mathcal{U}\mathfrak{g}$.

In this seminar I wish to describe another algebraic description of hamiltonian (more generally, Poisson) reduction suggesting a different (?) way to quantise.

There are two steps to coisotropic reduction, neither of which is "Poisson"

① $C^\infty(M) \longrightarrow C^\infty(M_0) \cong C^\infty(M)/I$
② $C^\infty(\tilde{M}) \cong C^\infty(M_0)^{\mathfrak{g}}$

} although the initial and final points are Poisson algebras.

① admits a homological description: the Koszul resolution

$$C^\infty(M) \xleftarrow{\delta} C^\infty(M)\otimes\mathfrak{g} \xleftarrow{\delta} C^\infty(M)\otimes\Lambda^2\mathfrak{g} \xleftarrow{\delta} \cdots$$

where $\delta f = 0$ and $\delta X = \phi_X$ for $f \in C^\infty(M)$, $X \in \mathfrak{g}$.

Lemma For $0 \in \mathfrak{g}^*$ a regular value of μ, $H^b_\delta \cong \begin{cases} C^\infty(M_0), & b = 0 \\ 0, & b > 0 \end{cases}$

$C^\infty(M)$-module

② The passage from $C^\infty(M_0)$ to $C^\infty(\tilde{M})$ is passing to invariants

$$H^0(\mathfrak{g}; C^\infty(M_0)) \cong C^\infty(M_0)^{\mathfrak{g}}$$

where $H^0(\mathfrak{g}; C^\infty(M_0))$ is the 0^{th} Chevalley–Eilenberg cohomology:

$$C^\infty(M_0) \xrightarrow{d} C^\infty(M_0)\otimes\mathfrak{g}^* \xrightarrow{d} C^\infty(M_0)\otimes\Lambda^2\mathfrak{g}^* \xrightarrow{d} \cdots$$

where $(df)(X) = [\phi_X, f]$ and $(d\alpha)(X,Y) = -\alpha([X,Y])$ for $f \in C^\infty(M_0)$ and $\alpha \in \mathfrak{g}^*$.

The BRST complex puts these two together, first into a double complex and then into the total complex.

Notice that $C^\infty(M)$ is also a $\mathfrak{g}$-module and of course so is $\Lambda^\cdot \mathfrak{g}$, and that the Koszul complex is actually a complex of $\mathfrak{g}$-modules with δ equivariant:

AUGMENTED COMPLEX

$$\begin{tikzcd}
C^\infty(M_0) \arrow[d, "d"'] & C^\infty(M) \arrow[l, "\varepsilon"'] \arrow[d, "d"'] & C^\infty(M)\otimes\mathfrak{g} \arrow[l, "\delta"'] \arrow[d, "d"'] & C^\infty(M)\otimes\Lambda^2\mathfrak{g} \arrow[l, "\delta"'] \arrow[d, "d"'] & \cdots \arrow[l, "\delta"'] \\
C^\infty(M_0)\otimes\mathfrak{g}^* \arrow[d, "d"'] & C^\infty(M)\otimes\mathfrak{g}^* \arrow[l, "\varepsilon"'] \arrow[d, "d"'] & C^\infty(M)\otimes\mathfrak{g}\otimes\mathfrak{g}^* \arrow[l, "\delta"'] \arrow[d, "d"'] & C^\infty(M)\otimes\Lambda^2\mathfrak{g}\otimes\mathfrak{g}^* \arrow[l, "\delta"'] \arrow[d, "d"'] & \cdots \arrow[l, "\delta"'] \\
C^\infty(M_0)\otimes\Lambda^2\mathfrak{g}^* \arrow[d, "d"'] & C^\infty(M)\otimes\Lambda^2\mathfrak{g}^* \arrow[l, "\varepsilon"'] \arrow[d, "d"'] & C^\infty(M)\otimes\mathfrak{g}\otimes\Lambda^2\mathfrak{g}^* \arrow[l, "\delta"'] \arrow[d, "d"'] & C^\infty(M)\otimes\Lambda^2\mathfrak{g}\otimes\Lambda^2\mathfrak{g}^* \arrow[l, "\delta"'] \arrow[d, "d"'] & \cdots \arrow[l, "\delta"'] \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where $\delta\alpha = 0$ for $\alpha\in\mathfrak{g}^*$ and $(dY)(X) = [X,Y]$ for $Y\in\mathfrak{g}$

So we have a double complex : $C^{b,c} := C^\infty(M)\otimes\Lambda^b\mathfrak{g}\otimes\Lambda^c\mathfrak{g}^*$

$\delta : C^{b,c} \longrightarrow C^{b-1,c}$ and $d : C^{b,c} \longrightarrow C^{b,c+1}$.

Define the total degree of $C^{b,c}$ to be $c-b$ ("ghost number")

and define $K^n := \bigoplus_{c-b=n} C^{b,c}$ $\qquad D = D' + D''$, $D' = d$ & $D'' = (-1)^c\delta$

$\therefore D : K^n \longrightarrow K^{n+1}$, $D^2 = 0$

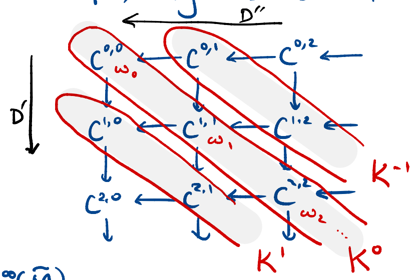
Theorem

$$H^n(K^\cdot) \cong H^n(\mathfrak{g}; C^\infty(M_0))$$

The proof follows from the acyclicity of the Koszul complex by 'tic-tac-toe'.

eg: Let $\omega\in K^n$, $D\omega = 0$.

Then $\omega = \omega_0 + \omega_1 + \omega_2 + \cdots + \omega_{top}$



$D\omega = 0$ becomes

$D''\omega_0 = 0$
$D'\omega_0 = -D''\omega_1$
$D'\omega_1 = -D''\omega_2$
$\vdots$
$D'\omega_{top} = 0$

and the isomorphism in the Theorem sends $[\omega]\in H^n(K^\cdot)$ to $[\omega_0]\in H^n(\mathfrak{g}; C^\infty(M^0))$.

Theorem $H^n(K^\cdot) \cong H^n(\mathfrak{g})\otimes C^\infty(\tilde{M})$

In particular, $H^0(K^\cdot) \cong C^\infty(\tilde{M})$ not yet as Poisson algebras !

To show that $H^0(K^\bullet) \cong C^\infty(\tilde{M})$ as Poisson algebras, we make the following observation (this is the crucial aspect of BRST!).

$C^\infty(M)$ is a Poisson algebra

$\Lambda^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*)$ is a Poisson superalgebra with $[\alpha, X] = \alpha(X) = [X, \alpha]$ for $\alpha \in \mathfrak{g}^*$, $X \in \mathfrak{g}$

actually a graded Poisson superalgebra

by ghost number

$\therefore$ $C^\infty(M) \otimes \Lambda^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*)$ is a graded Poisson superalgebra.

Proposition $\exists\, Q \in K^1$ such that $[Q, -] = D$

Therefore $H^\bullet(K)$ is a graded Poisson superalgebra and $H^0(K) \cong C^\infty(\tilde{M})$ is a Poisson isomorphism.

Proof The construction of Q is via "homological perturbation theory" in the general coisotropic case (due to Stasheff) but for the case of moment map reduction is very explicit: pick a basis X_i for $\mathfrak{g}$ and canonical dual basis θ^i for $\mathfrak{g}^*$. Let $[X_i, X_j] = \sum_k C_{ij}{}^k X_k$. Then

$$Q = \sum_i \theta^i \phi_{X_i} - \frac{1}{2} \sum_{i,j,k} C_{ij}{}^k \theta^i \theta^j X_k$$

← BRST quantisation consists in quantising Q, ...

Natural questions

1. Is there a similar homological description for other "reductions" beyond coisotropic Poisson reduction? eg: quasi-hamiltonian, contact, hyperkähler?

2. Take $M = T^*N$ and the hamiltonian G-action induced from a G action on N. Then $\phi_X = \sum_a \xi_X^a(q)\, p_a$ and can quantise Q:

rel to a Darboux chart (q^i, p_i)

$C^\infty(T^*N) \rightsquigarrow \mathrm{Diff}(N)$ acting on $C^\infty(N)$

$\Lambda^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*) \rightsquigarrow Cl(\mathfrak{g} \oplus \mathfrak{g}^*)$ acting on $\Lambda^\bullet \mathfrak{g}^*$, say.

So Q is promoted to an operator acting on $C^\infty(N) \otimes \Lambda^\bullet \mathfrak{g}^*$

and if $Q^2 = 0$ again, we define the reduced Hilbert space as the cohomology and the observables as Q-invariant, ...

How does this compare with Sue's quantum hamiltonian reduction?