Recent results on supergravity vacua

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Based on work in collaboration with

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- George Papadopoulos (King's College, London)
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  - ⋆ `math.AG/0211170`
- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
  - ⋆ `in preparation`

Introduction

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Our aim: to classify supergravity vacua.

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N = 2$			$N = 1$			
8	$N = 2$			$N = 1$			
7	$N = 4$			$N = 2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N = 8$	$N = 6$		$N = 4$		$N = 2$	
4	$N = 8$	$N = 6$	$N = 5$	$N = 4$	$N = 3$	$N = 2$	$N = 1$

[Van Proeyen, hep-th/0301005]

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- classify vacua of theories at the top of each column, and
- investigate their possible Kaluza–Klein reductions.

Strategy

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  defined by the supersymmetric variation of the gravitino:

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- the gravitino variation defines the connection

$$D_\mu = \nabla_\mu - \tfrac{1}{288} F_{\nu\rho\sigma\tau} \left(\Gamma^{\nu\rho\sigma\tau}{}_\mu + 8 \Gamma^{\nu\rho\sigma} \delta^\tau_\mu \right)$$

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where I is an index labeling the following elements

$$\Gamma_a \quad \Gamma_{ab} \quad \Gamma_{abc} \quad \Gamma_{abcd} \quad \Gamma_{abcde}$$

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Summarising the results:

- F is covariantly constant: $\nabla_\mu F_{\nu\rho\sigma\tau} = 0$
- F obeys the *Plücker relations*

$$F_{\alpha\beta\gamma[\mu}F_{\nu\rho\sigma\tau]} = 0$$

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- the Riemann curvature tensor is determined *algebraically* in terms of F and g:

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- F null: a one parameter $\mu \in \mathbb{R}$ family of *symmetric plane waves*:

$$g = 2dx^+dx^- - \tfrac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2 \right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

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[Blau–FO–Hull–Papadopoulos hep-th/0201081]

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KG

$\mathrm{AdS}_4 \times S^7 \nearrow$ KG $\nwarrow S^4 \times \mathrm{AdS}_7$

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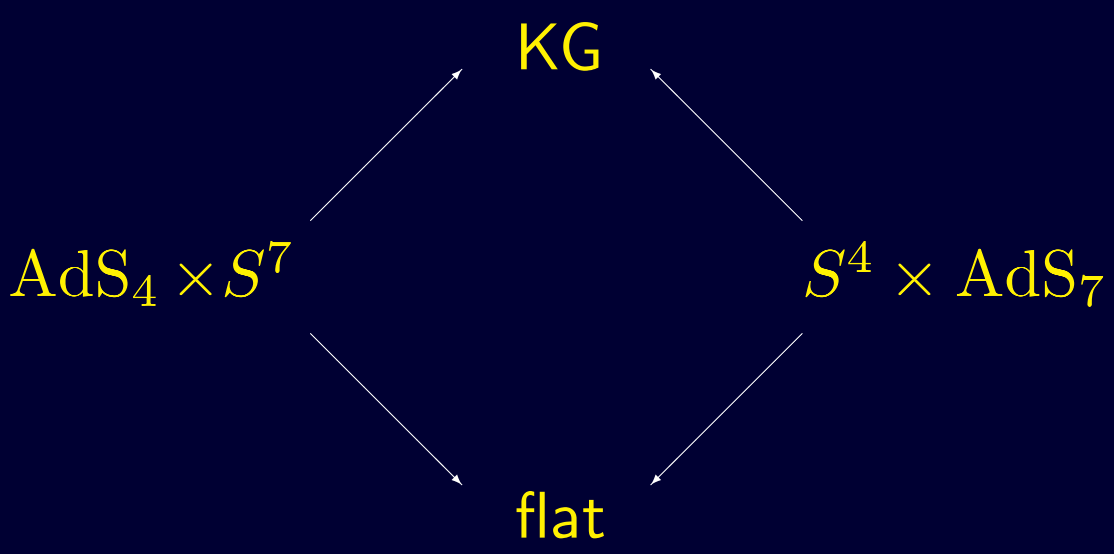

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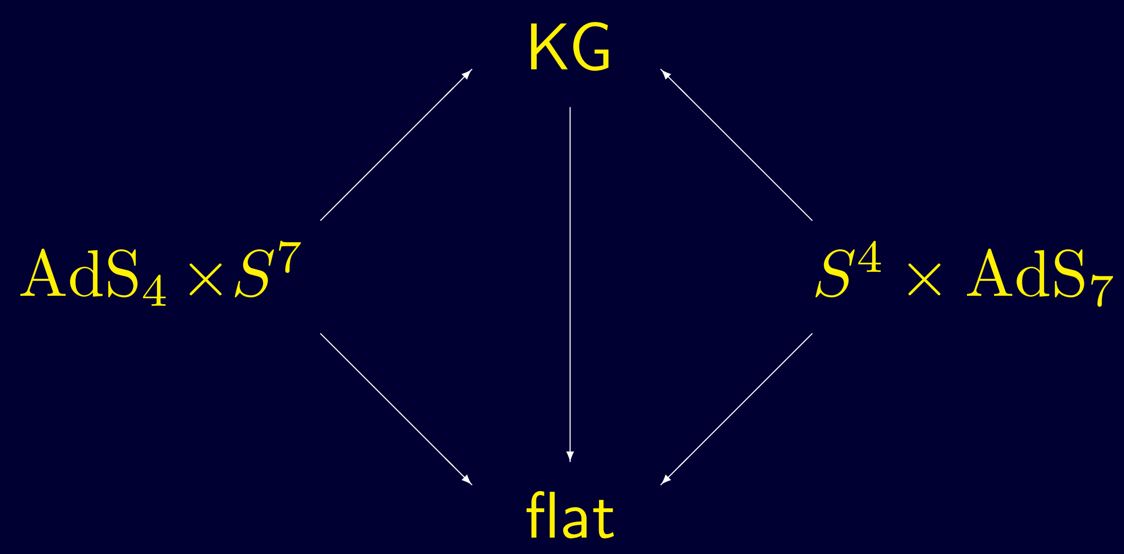

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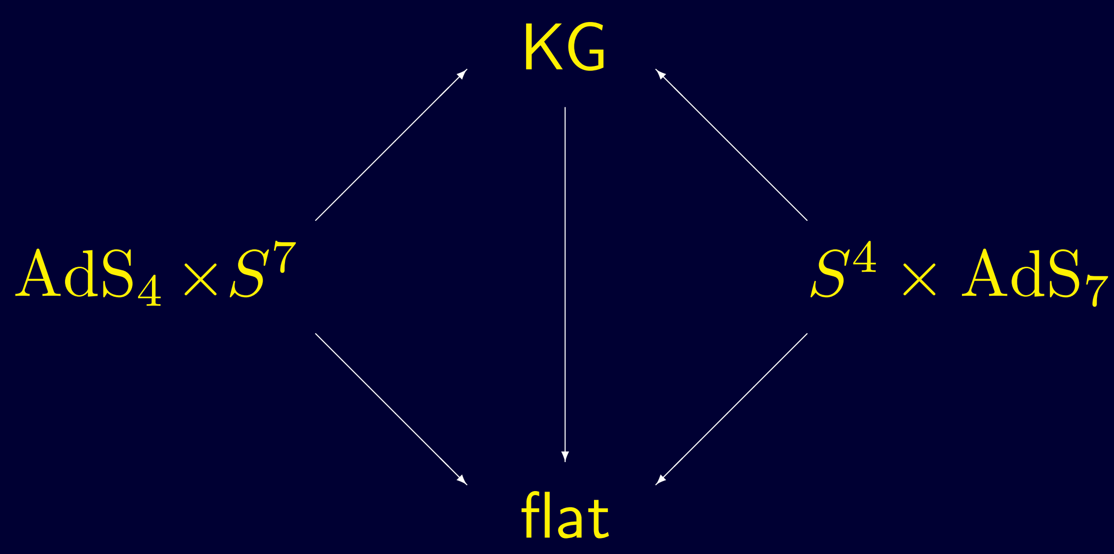

[Back]

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The solution to this problem is known.

Lorentzian Lie algebras

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But there is a more general construction.

The double extension

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- since $\mathfrak{h}$ preserves the metric on $\mathfrak{g}$, there is a linear map

$$\mathfrak{h} \longrightarrow \Lambda^2\mathfrak{g}$$

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relative to bases X_a, H_i and H^i for $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively.

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- $\mathfrak{h}$ acts on $\mathfrak{g} \times_\omega \mathfrak{h}^*$ preserving the Lie bracket, so we can form the *double extension*

$$\mathfrak{d}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{h} \ltimes (\mathfrak{g} \times_\omega \mathfrak{h}^*)$$

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This construction is due to Medina and Revoy who proved an important structure theorem.

The structure theorem of Medina and Revoy

A metric Lie algebra is *indecomposable* if it is not the direct sum of two orthogonal ideals.

Theorem (Medina–Revoy (1985)).

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Every metric Lie algebra is obtained as an orthogonal direct sum of indecomposables.

[See also FO–Stanciu `hep-th/9506152`]

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The third case is a six-dimensional version of the Nappi-Witten spacetime, found by Meessen. [Meessen hep-th/0111031]

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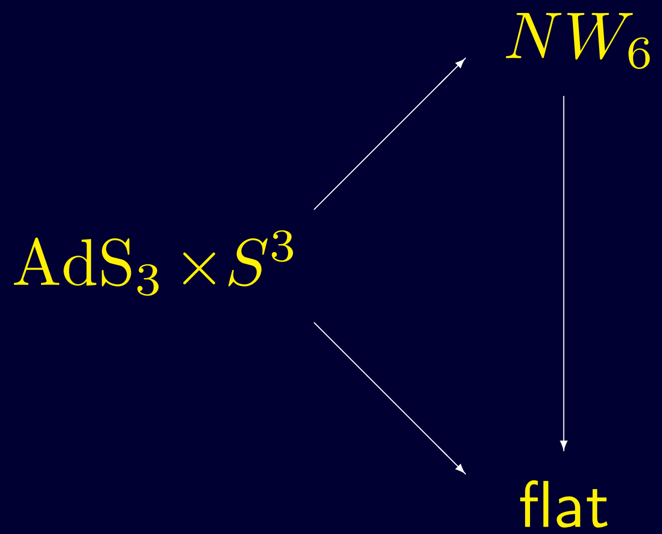

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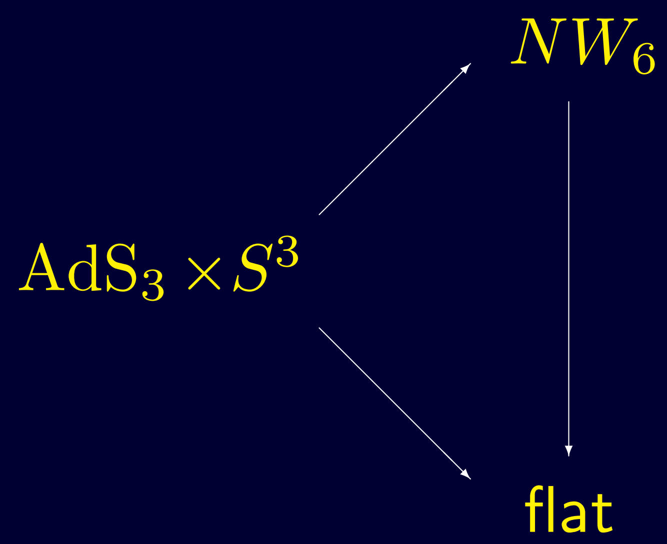

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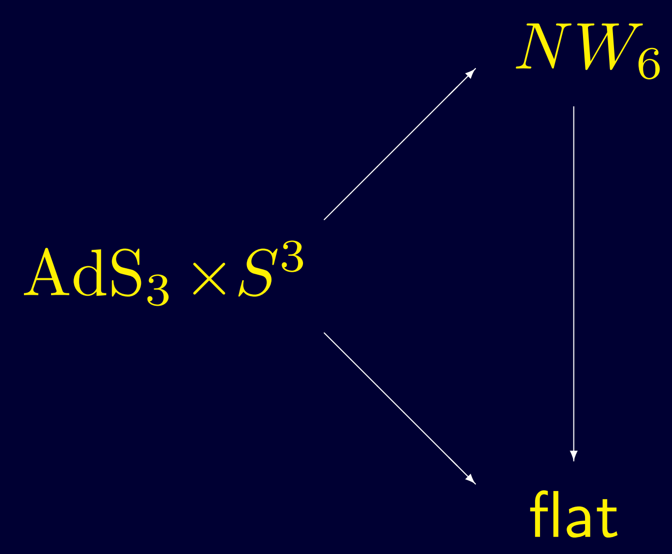

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[Back]

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Furthermore it is a 4-Lie algebra admitting an invariant metric.

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n-Lie algebras also appear naturally in the context of Nambu dynamics. [Nambu (1973)]

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[FO–Papadopoulos `math.AG/0211170`]

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$\mathrm{AdS}_5 \times S^5$ → BFOHP

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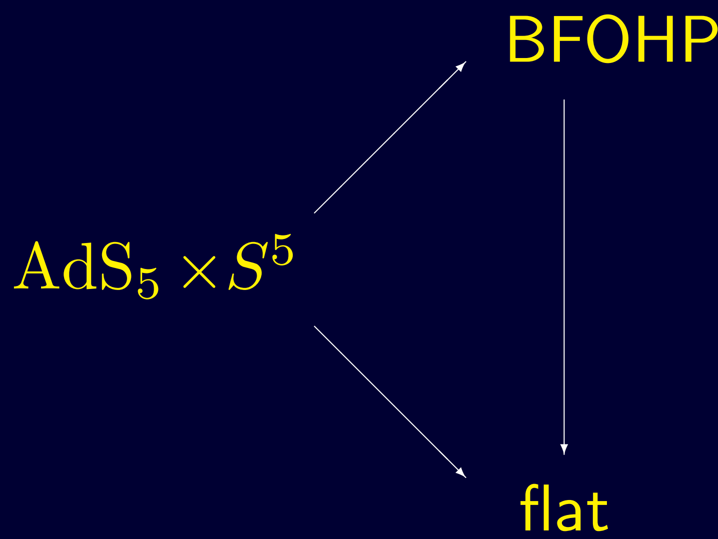

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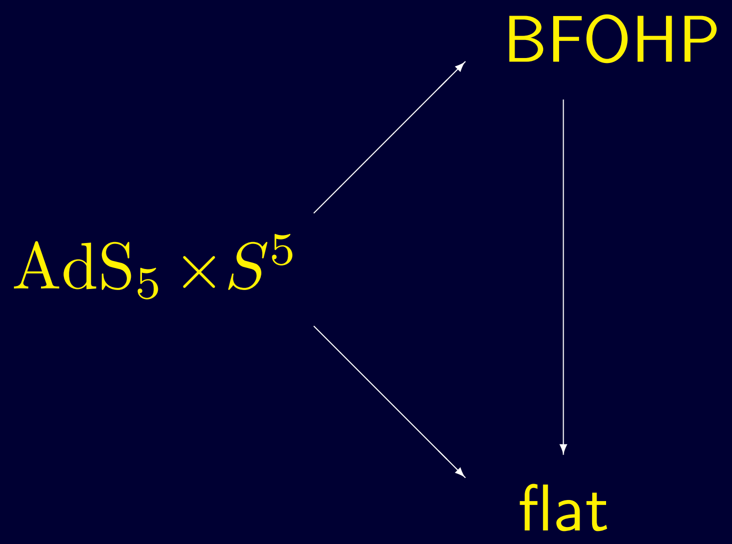

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[Gauntlett–Gutowsky–Hull–Pakis–Reall hep-th/0209114]

[Lozano-Tellechea–Meessen–Ortín hep-th/0206200]

Moltes gràcies.