Supersymmetric space forms

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Edinburgh Mathematical Physics Group

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Bath, 30 January 2004

Based on work in collaboration with

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- George Papadopoulos (King's College, London)
  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170` (*J Geom Phys*, in print)

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- Joan Simón (University of Pennsylvania)
  - ⋆ `hep-th/0401206`

A cosmological motivation

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$\Longrightarrow$ spatial universe has the geometry of a 'space form'

Space forms

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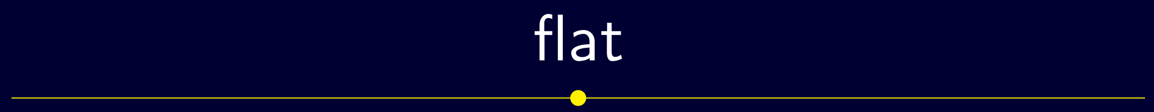

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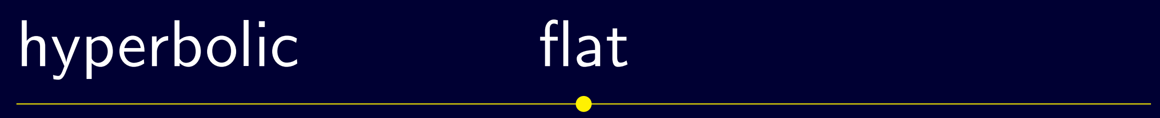

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- ‘maximally symmetric’

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- $\dim\left(T_pM \oplus \mathfrak{so}(T_pM)\right) = n$

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  - ⋆ hyperbolic: still open despite many partial results

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- quadric is not simply-connected; its universal cover is AdS_n

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The natural context is supergravity theory.

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- ⋆ $g^{(3)}$ a three-dimensional space form

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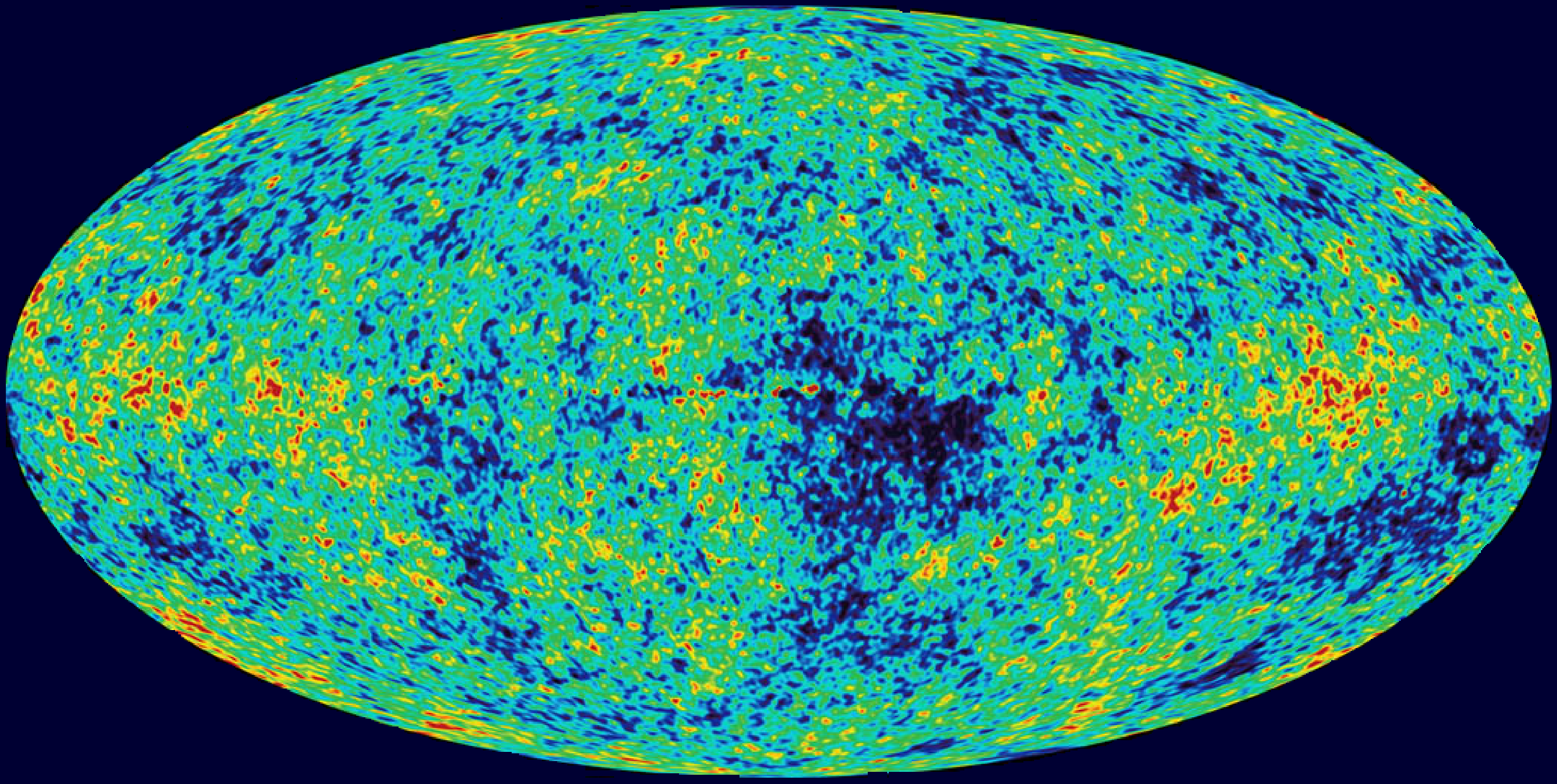

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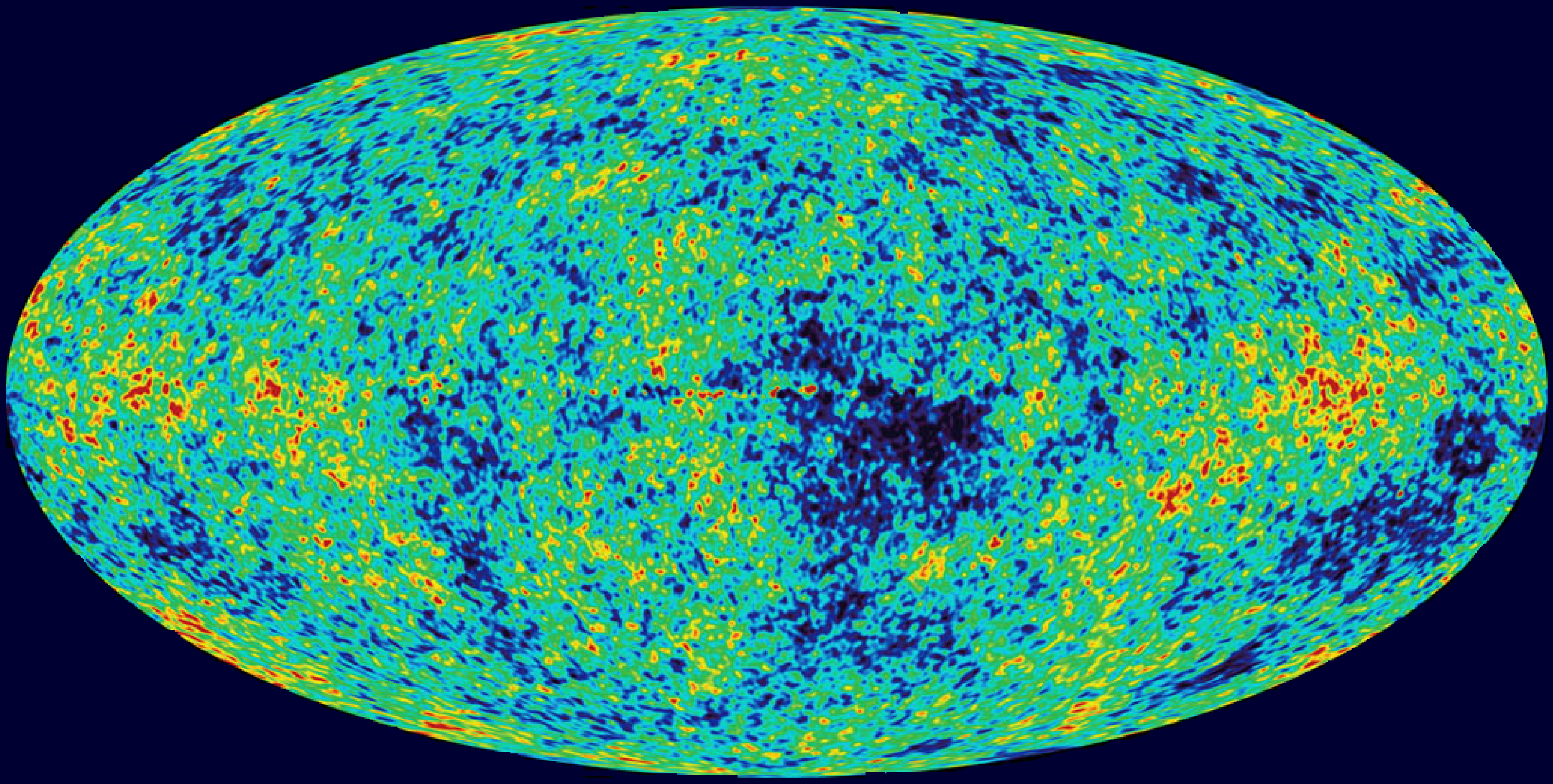

suggest that $g^{(3)}$ is the Poincaré dodecahedral space S^3/E_8

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  all in dimension ≤ 11

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N=2$			$N=1$			
8	$N=2$			$N=1$			
7	$N=4$			$N=2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
4	$N=8$	$N=6$	$N=5$	$N=4$	$N=3$	$N=2$	$N=1$

[Van Proeyen, hep-th/0301005]

A natural question

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Which are the maximally supersymmetric backgrounds of supergravity theories?

Eleven-dimensional supergravity

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* E is degenerate: a one parameter $\mu\in\mathbb{R}$ family of plane waves

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This equation defines a 4-Lie algebra (with an invariant metric). [Filippov (1985)]

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$$\mathrm{ad}_X[Y, Z] = [\mathrm{ad}_X Y, Z] + [Y, \mathrm{ad}_X Z]$$

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- $\langle -, - \rangle$ is an <u>invariant metric</u> $\iff$ F totally skew-symmetric

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[FO–Papadopoulos `math.AG/0211170`]

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$$\mathrm{AdS}_5(-R) \times S^5(R) \qquad F = \sqrt{\frac{4R}{5}} \left(\mathrm{dvol}(\mathrm{AdS}_5) - \mathrm{dvol}(S^5)\right)$$

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- $G = 0 \implies (M, g) = \mathbb{E}^{1,9}$
- G defines a parallel rank 5 sub-bundle $E \subset TM$:
  - ⋆ E non-degenerate: a one-parameter $(R > 0)$ family

$$\mathrm{AdS}_5(-R) \times S^5(R) \qquad F = \sqrt{\frac{4R}{5}}\left(\mathrm{dvol}(\mathrm{AdS}_5) - \mathrm{dvol}(S^5)\right)$$

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[Blau–FO–Hull–Papadopoulos hep-th/0110242]

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KG

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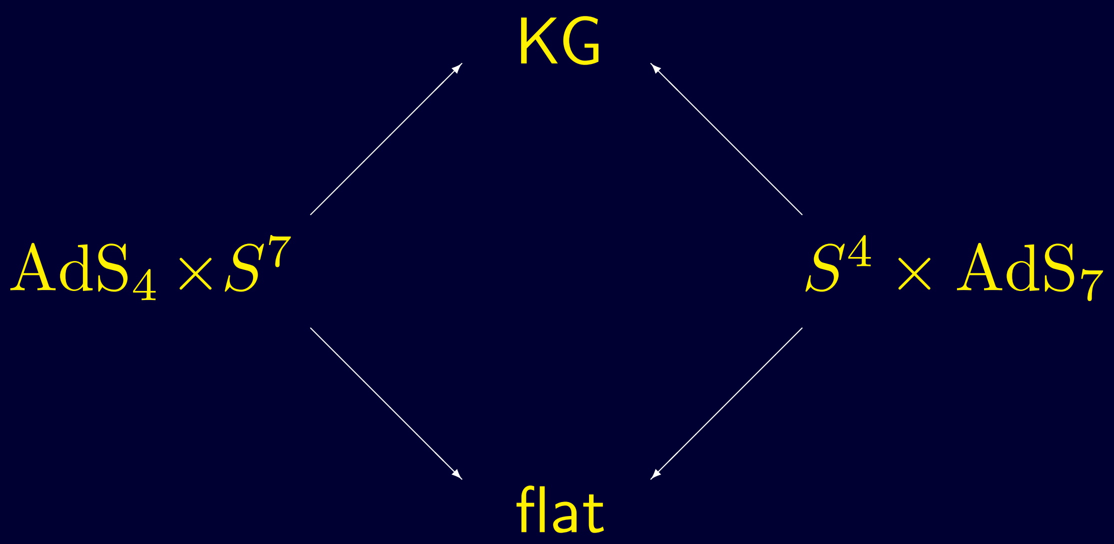
In eleven-dimensional supergravity:



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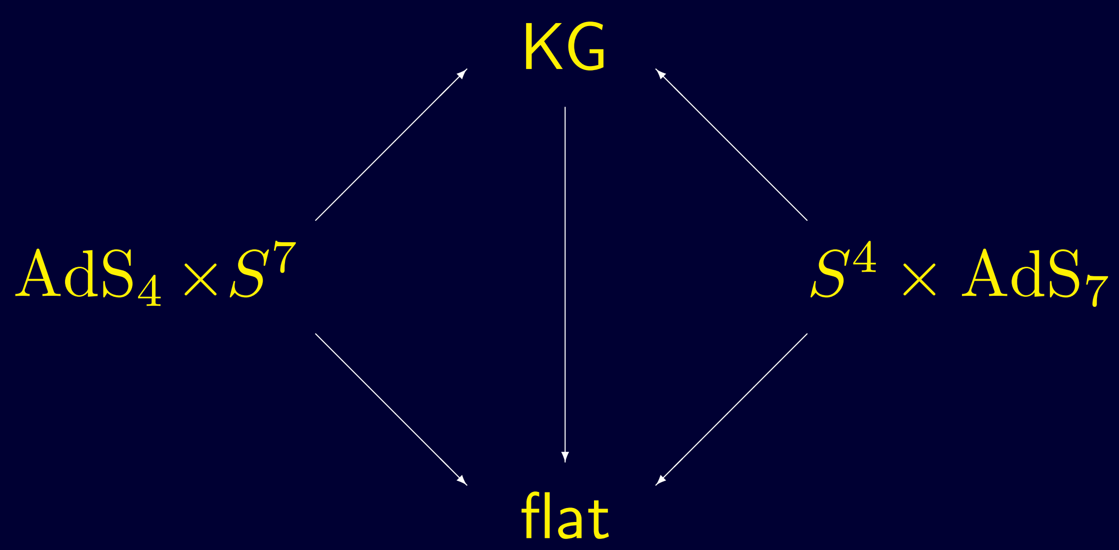

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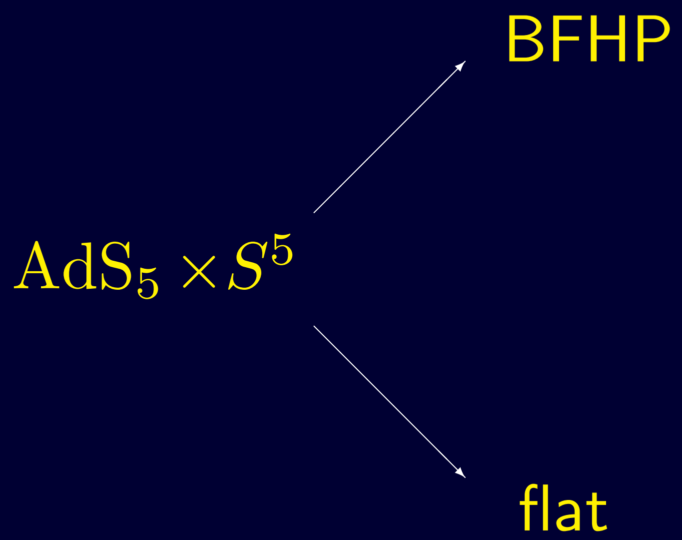

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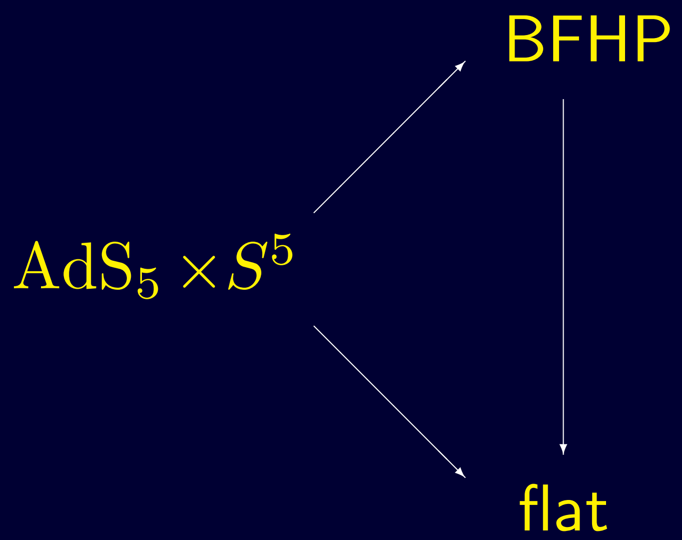

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Watch this space.