Supersymmetry and Geometry

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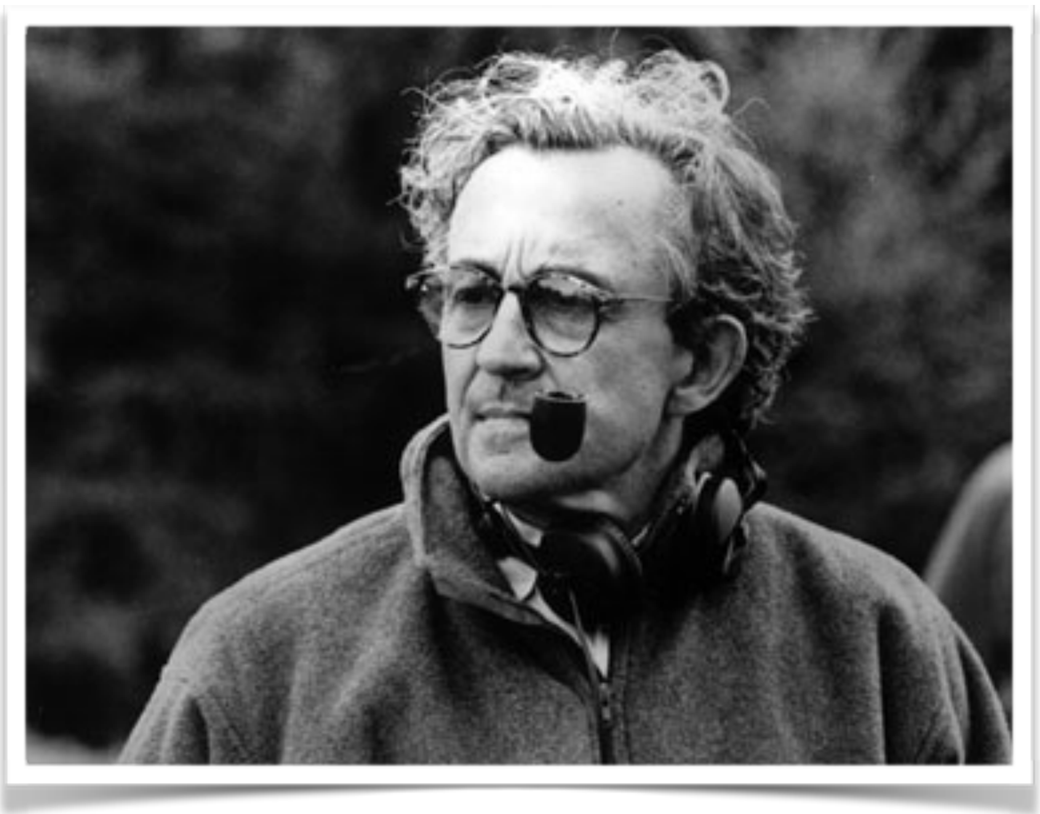
A legal precedent

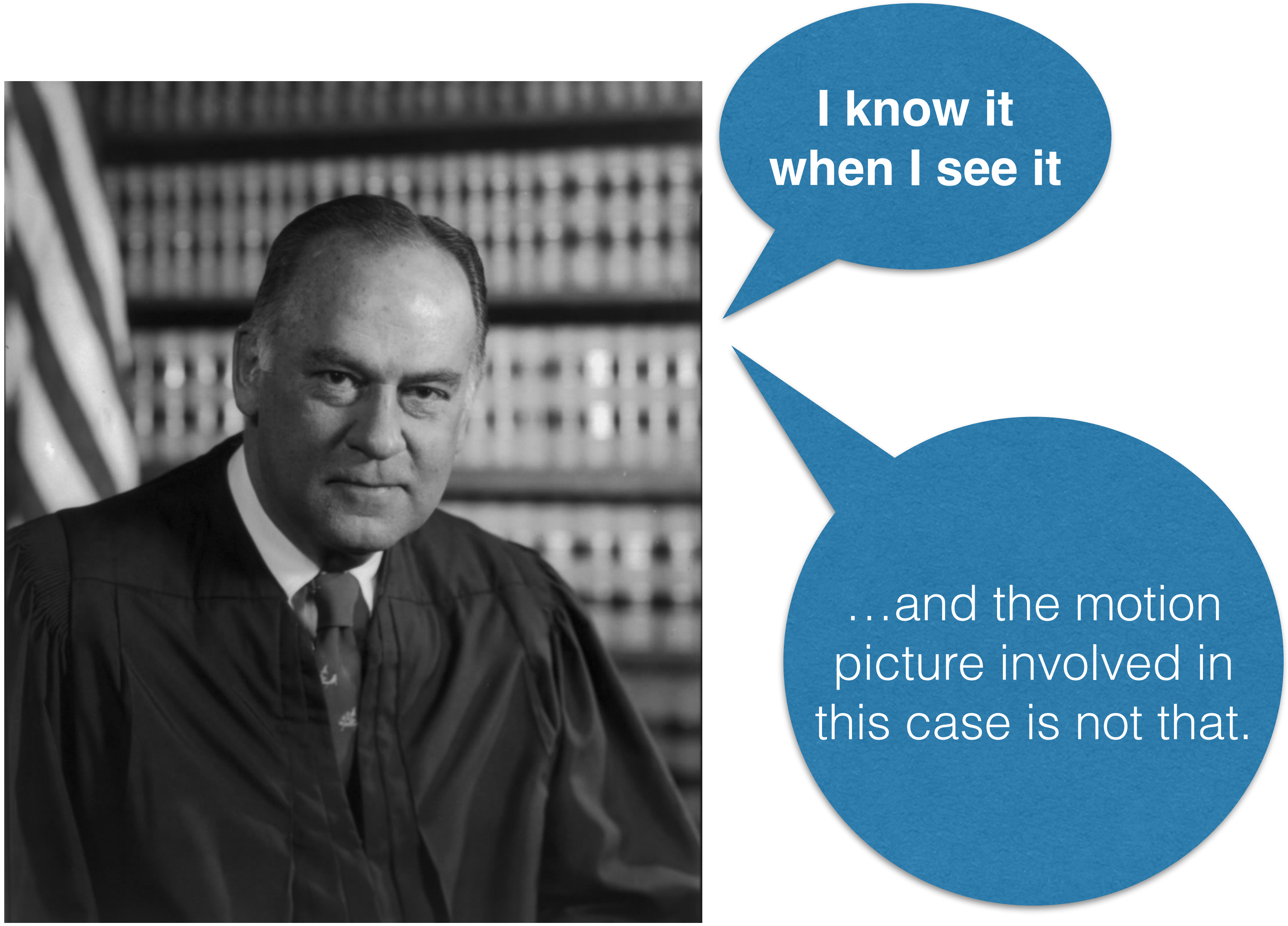


Louis Malle
(1932-1995)

Jeanne Moreau

A legal precedent

A legal precedent



Potter Stewart
(1915-1985)

Outline

- Supersymmetry in Physics
- Supersymmetry in Mathematics
    - Calibrated geometry
    - Gauge theory
    - Hodge theory

Supersymmetry in Physics

Two kinds of particles

- Elementary particles are described mathematically as unitary representations of the **Poincaré group**, the group of isometries of **Minkowski spacetime**

- The representations corresponding to massive particles are induced from finite-dimensional representations of the **spin group** Spin(3)

- Spin(3) has **centre** of order 2 and thus two kinds of representations, according to whether the centre acts trivially or not: **bosons** and **fermions**, respectively.

A new kind of group

- Supersymmetry is the (interesting) answer to the quest for the **relativistic quark model**: a “supergroup” with irreducible representations including *both* bosons *and* fermions

- The Coleman+Mandula theorem (1967) forbids the existence of such a Lie group

- Haag+Łopuszański+Sohnius (1975) got around the theorem by relaxing the notion of group to that of a **supergroup**

Supersymmetry algebras

- The infinitesimal object of a Lie supergroup is a **Lie superalgebra**
- The **supersymmetry algebras** are Lie superalgebras which have a Poincaré subalgebra
- Their irreducible representations are known as **supermultiplets** and they contain both bosons and fermions
- A **supersymmetric field theory** is one where the fields belong to supermultiplets of some supersymmetry algebra

Supersymmetry in nature?

- Except for some two-dimensional statistical mechanical systems at criticality, no supersymmetry has been found in nature
- In particular, there is no sign of supersymmetric particles in CERN
- But supersymmetry is alive and well in Mathematics!

Supersymmetry in Mathematics

Bosonic field equations are second-order PDEs:

Klein-Gordon	**Maxwell**	**Yang-Mills**
$(\Box + m^2)\varphi = 0$	$F = dA$	$F = dA + \frac{1}{2}[A, A]$
	$d \star F = 0$	$d^A \star F = 0$

Supersymmetry in Mathematics

Fermionic field equations are first-order PDEs:

Dirac

$$i\gamma^\mu \partial_\mu \psi = m\psi$$

Rarita-Schwinger

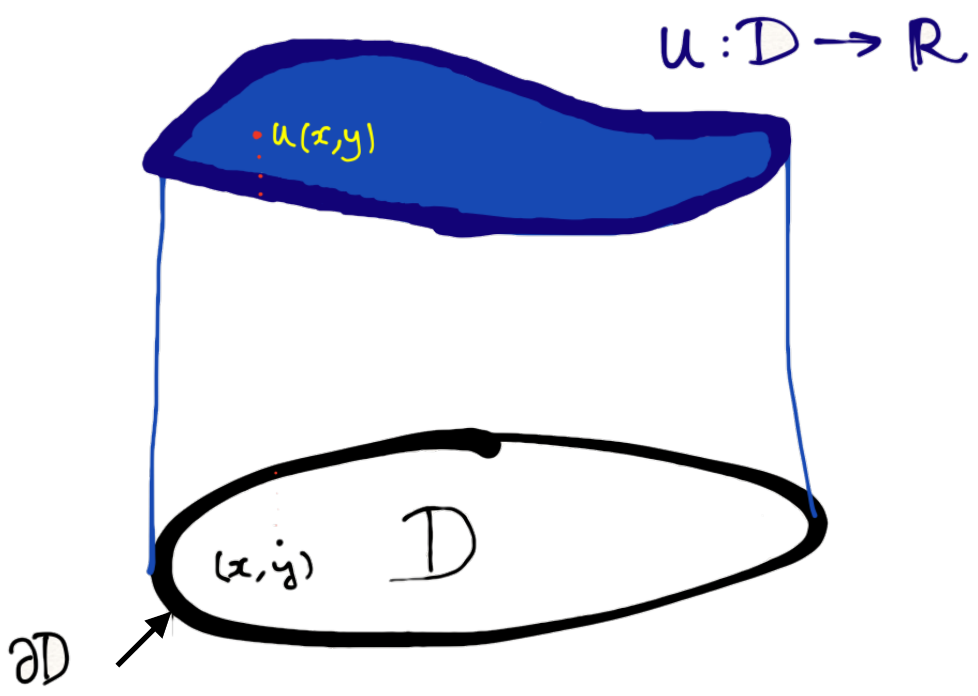
$$i\gamma^{\mu\nu\rho}\partial_\nu \psi_\rho = m\gamma^{\mu\nu}\psi_\nu$$

Supersymmetry in Mathematics

- Supersymmetry relates first-order PDEs to second-order PDEs
- There are many instances of this phenomenon in Differential Geometry, e.g.,
  - Calibrated Geometry
  - Gauge Theory
  - Hodge Theory
- Supersymmetry is intimately related to each one!

Calibrated Geometry

When is a graph a minimal surface?



$$\text{Area} = \int_D \sqrt{1 + u_x^2 + u_y^2}\, dx dy$$

Area is **extremal** iff $\Delta u = 0$

but then $u = \operatorname{Re} f$ $\quad \bar{\partial} f = 0$

Cauchy—Riemann

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Longrightarrow$

Laplace

$$\Delta u = 0 = \Delta v$$

Minimal surfaces

$\mathbf{(M,g)}$ riemannian, orientable

$$f : D \to M \qquad D \subset \mathbb{R}^p$$

Pull back the metric to D

$$ds^2 = \langle df, df \rangle = \sum_{i,j,a,b} \underbrace{g_{ij}\partial_a f^i \partial_b f^j}_{G_{ab}} dx^a dx^b$$

$$\text{Area} = \int_D \sqrt{\det G}\, d^p x$$

Minimality is a *2nd-order* PDE

Calibrations

$\omega \in \Omega^p(M)$ is a **calibration** if

$$\begin{cases} \omega(\pi) \leq \mathrm{dvol}(\pi) \qquad \forall \pi \in \mathrm{Gr}(p, T_x M) \qquad \forall x \in M \\ d\omega = 0 \end{cases}$$

e.g., $\omega = \frac{i}{2}\left(dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n\right) \in \Omega^2(\mathbb{C}^n)$

$\frac{1}{k!}\omega^k \in \Omega^{2k}(\mathbb{C}^n)$

are calibrations thanks to **Wirtinger's inequality**

Calibrated submanifolds

$f : N^p \to M^n$ compact, oriented submanifold

N is **calibrated** (by ω) if

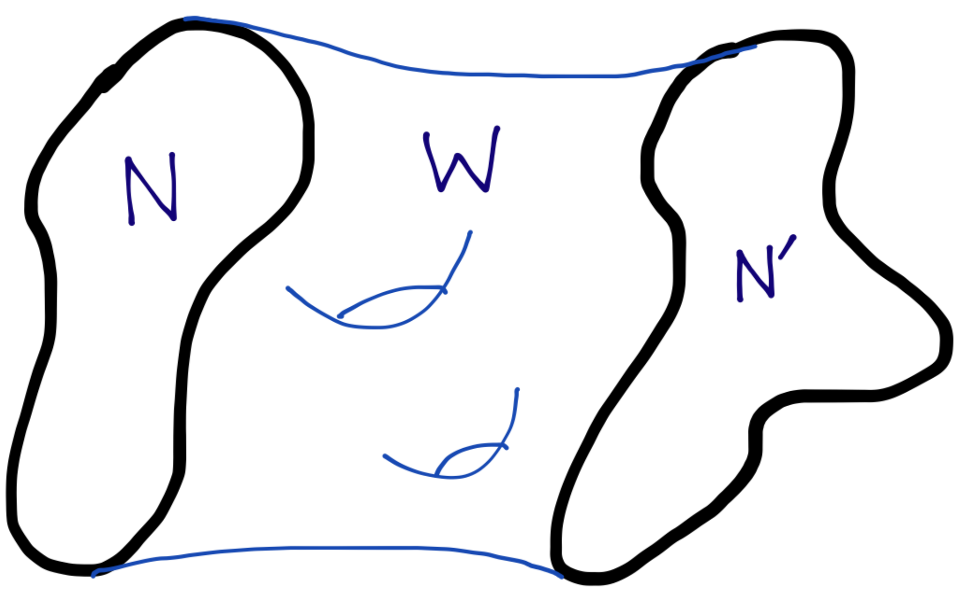
$$\omega(T_x N) = \mathrm{dvol}(T_x N) \qquad \forall x \in N$$

$$\mathrm{vol}(N) = \int_N \omega$$

This condition is *first-order* in the embedding:

$T_x N$ is spanned by $f_*(\partial_a) = \partial_a f^i \partial_i$

Calibrated ⇒ Minimal



$$N - N' = \partial W$$

$$\therefore \quad \int_N \omega = \int_{N'} \omega$$

$$\mathrm{vol}(N) = \int_N \omega = \int_{N'} \omega \leq \mathrm{vol}(N')$$

Calibrated submanifolds are **volume-minimizing** in their homology class!

What about Cauchy-Riemann?

$D \subset \mathbb{C}^k \qquad \varphi : D \to \mathbb{C}^{n-k} \qquad N^{2k} = \mathrm{graph}\,\varphi$

$\boldsymbol{N}$ is calibrated by $\frac{1}{k!}\omega^k$ iff $T_x N \in \mathrm{Gr}_{\mathbb{C}}(k, T_x\mathbb{C}^n)$

iff φ_* is complex-linear

iff $\overline{\partial}\varphi = 0$

When $k = n = 1$ we recover the classical example.

Calibrated geometries

Harvey+Lawson (1980) developed the subject of calibrated geometry and discovered natural calibrations on a large class of manifolds:

- Kähler (complex)
- **hyperkähler** (complex*)
- quaternion Kähler (quaternionic)
- **Calabi-Yau** (complex and special lagrangian)
- **Spin(7)** (Cayley)
- $\mathbf{G_2}$ (associative and coassociative)

Some are intimately related to **spinors** and hence to supersymmetry!

Gauge Theory

Gauge fields

$\mathfrak{g}$ a (compact, simple) Lie algebra

e.g., $\mathfrak{su}(2)$ traceless skewhermitian 2×2 matrices

gauge field $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$

e.g., $A = \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix}$ $a, b, c \in \Omega^1(\mathbb{R}^4)$

field strength $F = dA + \frac{1}{2}[A, A] \in \Omega^2(\mathbb{R}^4, \mathfrak{g})$

$d_A : \Omega^p(\mathbb{R}^4, \mathfrak{g}) \to \Omega^{p+1}(\mathbb{R}^4, \mathfrak{g})$ $d_A\theta = d\theta + [A, \theta]$

$d_A^2\theta = [F, \theta]$

Bianchi identity $d_A F = 0$

Yang-Mills theory

$\star : \Omega^p(\mathbb{R}^4) \to \Omega^{4-p}(\mathbb{R}^4)$

Yang-Mills action functional $S_{\mathrm{YM}} = \int \mathrm{Tr}(F \wedge \star F)$

Yang-Mills equations $d_A \star F = 0$ (2nd-order)

$\star^2 = +1$ on $\Omega^2(\mathbb{R}^4)$

$$\Omega^2(\mathbb{R}^4) = \Omega^2_+(\mathbb{R}^4) \oplus \Omega^2_-(\mathbb{R}^4)$$

self-dual **anti self-dual**

Instantons

(anti) self-duality $\star F = \pm F$ (1st-order)

Bianchi identity ⇒ Yang-Mills equation

(Anti) self-dual gauge fields are called (anti) **instantons**.

$$S_{\mathrm{YM}} \geq \left|\int \mathrm{Tr} F \wedge F\right|$$

topological invariant

Equality if and only if A is an (anti) instanton.

Instanton moduli space

- Instantons make sense on any orientable riemannian 4-manifold, not just euclidean space
- Instantons are determined by their topological charge
- For a fixed charge, they are further determined by a finite number of parameter (**moduli**)
- The moduli spaces of instantons have interesting geometries: in some cases, it is a hyperkähler manifold
- Supersymmetry explains some geometrical features of the instanton moduli spaces.

Monopoles

Monopoles are translationally invariant instantons: $\partial_4 \equiv 0$

$$A = \boxed{A_1 dx^1 + A_2 dx^2 + A_3 dx^3} + \phi dx^4$$

Higgs field

3-d gauge field

4-d (anti) self-duality — **Bogomol'nyi equation**

$$d_A \phi = \pm \star F$$ (1st-order)

4-d Yang-Mills equation — **Yang-Mills-Higgs equation**

(2nd-order)

Monopole moduli space

- Although the Bogomol'nyi equation makes sense on any orientable three-dimensional riemannian manifold, monopoles have been studied mostly in euclidean and hyperbolic spaces
- Euclidean monopoles have a topological charge and once fixed, also come in **moduli** spaces
- These are hyperkähler manifolds, like the celebrated Atiyah-Hitchin manifold

Supersymmetry

- Supersymmetric Yang-Mills theories exist in various dimensions (≤ 10) and on various geometries
- Conjectures about higher-dimensional gauge theory
- Conjectures about **L^2 cohomology** of (certain) hyperkähler manifolds
- Seiberg-Witten invariants
- Explains the geometry of instanton and monopole moduli spaces

Hodge Theory

(M, g) a compact, orientable riemannian manifold

de Rham complex of differential forms

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

differential graded algebra

$$\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M) \qquad d : \Omega^p(M) \to \Omega^{p+1}(M)$$

$$d^2 = 0$$

Hodge star $\quad \star : \Omega^p(M) \to \Omega^{n-p}(M)$

inner product $$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \star\beta$$

formal adjoint $$d^\dagger = \delta = \star d \star$$

Hodge laplacian $$\triangle = d\delta + \delta d$$

"energy" $$\langle \alpha, \triangle\alpha \rangle = \|d\alpha\|^2 + \|\delta\alpha\|^2 \geq 0$$

harmonic $$\triangle\alpha = 0 \iff d\alpha = \delta\alpha = 0 \iff D\alpha = 0$$

$$D := d + \delta : \Omega^{\mathrm{even}/\mathrm{odd}}(M) \to \Omega^{\mathrm{odd}/\mathrm{even}}(M)$$

Hodge Theorem $$\mathcal{H}^p \cong H^p_{\mathrm{dR}}(M, \mathbb{R})$$

Supersymmetric Quantum Mechanics

Underlying supersymmetric theory is the one-dimensional **supersymmetric sigma model**

$\triangle$	hamiltonian	$\Omega^\bullet(M)$	Hilbert space
D	supercharge	$\mathcal{H}^\bullet$	ground states

Witten index $\mathrm{Tr}(-1)^F := \#\text{bosonic states} - \#\text{fermionic states}$

only ground states contribute

$$\begin{aligned}\mathrm{Tr}(-1)^F &= \dim \mathcal{H}^{\text{even}} - \dim \mathcal{H}^{\text{odd}} \\ &= \chi(M) \\ &= \mathrm{ind} D\end{aligned}$$

Index Theorem

- This result is *paradigmatic*
- It allows a supersymmetric proof of the **Atiyah-Singer index theorem**
- Worked out in detail by Álvarez-Gaumé,… from supersymmetric theories generalising the one-dimensional sigma model
- Mathematically rigorous proof by Getzler using these ideas

Complex geometry

- **Hodge-Lefschetz** theory for Kähler and hyperkähler manifolds also follows from supersymmetry, starting from a four- and six-dimensional supersymmetric sigma model, respectively
- Sigma model symmetries give rise to actions of SL(2,$\mathbb{C}$) and SO(4,1) on the cohomology rings.
- Moreover, the very notions of **Kähler** and **hyperkähler** can be defined in supersymmetric terms!
- This gave rise to the hyperkähler and quaternion Kähler **quotient constructions**!

Summary

First-order PDE	Second-order PDE
calibrated	**minimal**
(anti)self-duality	**Yang-Mills**
Bogomol'ny	**Yang-Mills-Higgs**

A heuristic principle?

Supersymmetry underlies any situation where a first-order PDE **implies** a second-order PDE and where the solutions of the first-order PDE are **optimal** (in some sense) among the solutions of the second-order PDE.

Exercise: find examples from other branches of Mathematics!