Homogeneous kinematical spacetimes (Bologna, 6/6/2018)

based on 1711.06111, 1711.07363, 1802.04048 with Tomasz Andrzejewski
work in progress with Stefan Prohazka and Ross Grassie

Several motivations:
- explore new holographic limits of AdS/CFT
- generalisation of supersymmetry and/or supergravity beyond the arena of lorentzian geometry

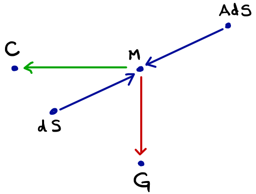
Some dramatis personae

We start with the ==de Sitter spacetimes== in $D+1$ dimensions: lorentzian manifolds with nonzero constant sectional curvature:

- $AdS_{D+1} \subset \mathbb{R}^{D,2}$ $\qquad x_1^2+\cdots+x_D^2-x_{D+1}^2-x_{D+2}^2 = -R^2 < 0$ $\qquad$ (anti de Sitter)
- $dS_{D+1} \subset \mathbb{R}^{D+1,1}$ $\qquad x_1^2+\cdots+x_D^2+x_{D+1}^2-x_{D+2}^2 = +R^2 > 0$ $\qquad$ (de Sitter)

The Lie algebra of isometries is $\mathfrak{so}(D,2)$ for AdS_{D+1} and $\mathfrak{so}(D+1,1)$ for dS_{D+1}.

Taking the curvature $\to 0$ limit (equivalently, $R\to\infty$) of AdS_{D+1} or dS_{D+1} results in ==Minkowski spacetime==, which is an affine space $\mathbb{A}^{D+1}$ with a flat metric $g = dx_1^2+\cdots+dx_D^2-c^2dx_{D+1}^2$ (introducing the speed of light c)



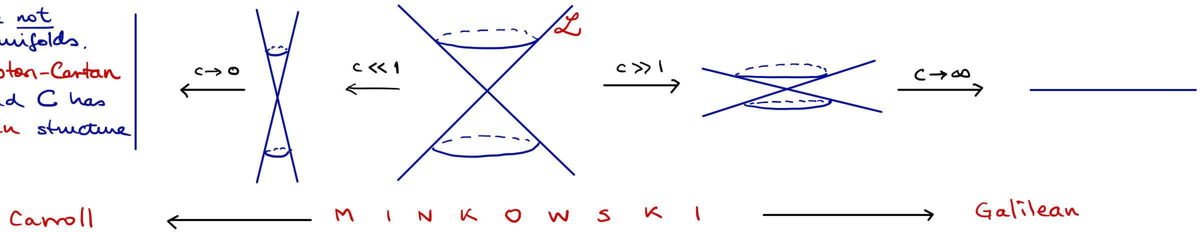
We can now take limits in which $c\to\infty$ (nonrelativistic) or $c\to 0$ (ultrarelativistic).

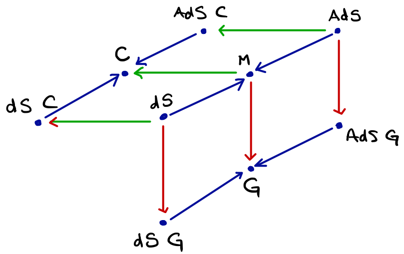
If $p \in \mathbb{A}^{D+1}$, $(T_p\mathbb{A}^{D+1}, g_p) \cong (\mathbb{R}^{D+1}, \eta) \supset \mathcal{L}$ (light cone)

lorentzian inner product

$\mathcal{L}$ $\qquad x_1^2+\cdots+x_D^2-c^2x_{D+1}^2 = 0$

NB: C and G are not lorentzian manifolds.
G has a Newton-Cartan structure and C has a carrollian structure





If $(\mathcal{M}, g)$ is a lorentzian manifold, $(T_p\mathcal{M}, g_p) \approx (\mathbb{R}^{D+1}, \eta)$ and so we have a lightcone and we can take the non- and ultra-relativistic limits. In particular for $(A)dS_{D+1}$ we obtain Carroll & galilean versions of (A)dS.

(A)dS C & (A)dS G are reductive homogeneous spacetimes of kinematical Lie groups. Their canonical connections (those with zero Nomizu maps) are not flat and the flat limit gives G & C.

All spacetimes in the above diagram are symmetric homogeneous spaces of kinematical Lie groups. The canonical connection is torsion-free, but only flat for M, C, G. We want to classify homogeneous spacetimes of kinematical Lie groups. We will work locally in terms of their Lie algebraic data.

Kinematical Lie algebras and their homogeneous spacetimes

Def. A **kinematical Lie algebra** (with D-dim'l space isotropy) is a real Lie algebra $\mathfrak{g}$ of dim $\frac{1}{2}(D+1)(D+2)$ satisfying

① $\mathfrak{g} > \mathfrak{r} \cong \mathfrak{so}(D)$

and ② $\mathfrak{g} \underset{\mathfrak{r}\text{-mod}}{=} \mathfrak{r} \oplus 2V \oplus S$

1-dim'l trivial r-mod

D-dim'l vector rep of r

Classification:		
	D = 0	∃! 1-dim'l LA
	D = 1	Bianchi (1898)
r-equiv. $\Lambda^2 V \longrightarrow S$ ←	D = 2	JMF + Andrzejewski ('18)
giving rise to more KLAs. $\Lambda^2 V \longrightarrow V$ ←	D = 3	Bacry + Nuyts + (Lévy-Leblond) ('86)
	D > 3	JMF ('17)

An **aristotelian Lie algebra** is a real LA of dim $\frac{1}{2}D(D+1)+1$ satisfying ① $\mathfrak{g} > \mathfrak{r} \cong \mathfrak{so}(D)$ and ② $\mathfrak{g} \underset{\mathfrak{r}\text{-mod}}{=} \mathfrak{r} \oplus V \oplus S$

Typically, aristotelian Lie algebras are quotients of KLAs by a vectorial ideal.
The classifications in $D \geq 2$ can be done via deformation theory. Every KLA is a deformation of the **static KLA**: the unique KLA where $2V \oplus S$ is an abelian ideal (and hence S is central.)

For generic D, one finds the following KLAs:

- static
- galilean
- Carroll
- Poincaré
- euclidean
- Newton–Hooke ($\mathfrak{n}_\pm$)
- $\mathfrak{so}(D,2)$
- $\mathfrak{so}(D+1,1)$
- $\mathfrak{so}(D+2)$
- a family $\gamma \in [-1,1]$ where $\gamma \to -1$ is $\mathfrak{n}_-$
- a family $\chi \geq 0$ where $\chi \to 0$ is $\mathfrak{n}_+$
- a KLA where $\mathrm{ad}[S,-]$ is not diagonalisable

We are interested in the "kinematical spacetimes": $(D+1)$-dimensional homogeneous spaces of kinematical Lie groups G of the form $\mathcal{M} = G/H$ where the LA $\mathfrak{g}$ of G is kinematical and the Lie subalgebra $\mathfrak{h} < \mathfrak{g}$ corresponding to the (closed) subgroup H is of the form $\mathfrak{h} = \mathfrak{r} \oplus V$ (as $\mathfrak{r}$-mod).

We work infinitesimally in terms of the Lie pair $(\mathfrak{g}, \mathfrak{h})$ and we classify these Lie pairs up to the natural equivalence: $(\mathfrak{g}_1, \mathfrak{h}_1) \sim (\mathfrak{g}_2, \mathfrak{h}_2)$ if there exists $\varphi : \mathfrak{g}_1 \xrightarrow{\cong} \mathfrak{g}_2$ with $\varphi|_{\mathfrak{h}_1} : \mathfrak{h}_1 \xrightarrow{\cong} \mathfrak{h}_2$.

In practice, we use the classification of KLAs to fix a KLA $\mathfrak{g}$ and classify Lie pairs $(\mathfrak{g}, \mathfrak{h})$ subject to the equivalence $(\mathfrak{g}, \mathfrak{h}_1) \sim (\mathfrak{g}, \mathfrak{h}_2)$ if $\exists\, \varphi \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\varphi|_{\mathfrak{h}_1} : \mathfrak{h}_1 \xrightarrow{\cong} \mathfrak{h}_2$.

A Lie pair $(\mathfrak{g}, \mathfrak{h})$ is effective if $\mathfrak{h}$ does not contain any nontrivial ideal of $\mathfrak{g}$.

Equivalence classes of effective Lie pairs are in bijective correspondence with local iso. classes of homogeneous spacetimes. Notice that there may be non-equivalent Lie pairs $(\mathfrak{g}, \mathfrak{h}_1), (\mathfrak{g}, \mathfrak{h}_2)$ with the same $\mathfrak{g}$. For example, $\mathfrak{g} = \mathfrak{so}(D+1,1)$ has at least two, because dS_{D+1} and H^{D+1} have $\mathfrak{g}$ as LA of isometries.

Non-effective kinematical Lie pairs have a vectorial ideal I and $\big(\underset{\mathfrak{r}\oplus V\oplus S}{\underset{\shortparallel}{\mathfrak{g}/I}}, \underset{\mathfrak{r}}{\underset{\shortparallel}{\mathfrak{h}/I}}\big)$ describes an aristotelian spacetime.

The result of the classification (for generic D) is obtained by adding some more spacetimes to our dramatis personae.

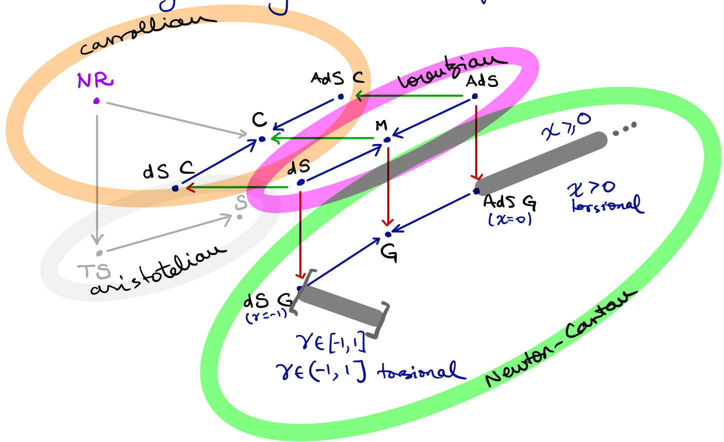
NR : non-reductive homogeneous spacetime of $\mathfrak{so}(D+1,1)$

S : static aristotelian (all limit to it)

TS : torsional static aristotelian

$(\mathrm{AdS}\,G)_\chi$: one-parameter family $\chi \geq 0$. ($\mathrm{AdS}\,G$ is $\chi = 0$)
all torsional except for $\chi = 0$

$(\mathrm{dS}\,G)_\gamma$: one-parameter family $\gamma \in [-1,1]$. ($\mathrm{dS}\,G$ is $\gamma = -1$)
all torsional except for $\gamma = -1$.



Some details not in the talk

In writing down KLAs, it is convenient to choose a standard basis $J_{ab} = -J_{ba}$ for r, and B_a, P_a for $2V$ and H for S. The $[J, *]$ brackets are the same for all KLAs and hence they are distinguished by the remaining brackets. When describing Lie pairs $(\mathfrak{g}, \mathfrak{h})$ we always choose a basis for the KLA in which $\mathfrak{h}$ is spanned by (J_{ab}, B_a) for kinematical spacetimes or (J_{ab}) for aristotelian spacetimes. The homogeneous spacetimes are then described by writing the brackets of B_a, P_a, H in this basis. The homogeneous spacetimes in the talk can be described locally as follows:

Aristotelian

- static $\quad [-,-] = 0$
- torsional static $\quad [H, P] = P$

Newton-Cartan

- galilean $\quad [H, B] = -P$
- $(\text{AdS G})_\chi$ $\quad [H, B] = -P \quad [H, P] = (1+\chi^2) B + 2\chi P$
- $(\text{dS G})_\gamma$ $\quad [H, B] = -P \quad [H, P] = \gamma B + (1-\gamma) P$

Carrollian

- Carroll $\quad [B, P] = H$
- AdS C $\quad [B, P] = H \quad [H, P] = B \quad [P, P] = J$
- dS C $\quad [B, P] = H \quad [H, P] = -B \quad [P, P] = -J$
- NR $\quad [B, P] = H + J \quad [H, P] = -P \quad [H, B] = B$

Lorentzian

- Minkowski $\quad [H, B] = -P \quad [B, B] = J \quad [B, P] = H$
- de Sitter $\quad [H, B] = -P \quad [B, B] = J \quad [B, P] = H \quad [H, P] = -B \quad [P, P] = -J$
- Anti de Sitter $\quad [H, B] = -P \quad [B, B] = J \quad [B, P] = H \quad [H, P] = B \quad [P, P] = J$