Quotienting Freund-Rubin backgrounds

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- hep-th/0401206, and
- hep-th/0402094 [also with Owen Madden and Simon Ross (Durham)]

Motivation

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- supersymmetric Clifford–Klein space form problem

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- we are interested in the orbit space M/Γ

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  - ⋆ M supersymmetric, but M/Γ_L breaking all supersymmetry

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i.e., projectivised adjoint orbits of $\mathfrak{g}$

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In string/M-theory this cannot be the end of the story.

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$\mathbb{CP}^2$ is not even spin!

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- equivalently, the action of $\xi = \xi_X$ on spinors has integral weights

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  [Bär (1993), Kath (1999)]

- equivariant under the isometry group G of (M, g)

  [hep-th/9902066]

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- the problem reduces to one of flat spaces!

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- we need to determine the elementary blocks

Elementary blocks for $\mathfrak{so}(2, p)$

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$$B^{(0,2)}(\varphi) = B^{(2,0)}(\varphi) = \begin{bmatrix} 0 & \varphi \\ -\varphi & 0 \end{bmatrix}$$

- $(1, 1)$

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$$B_{\pm}^{(2,2)} = \begin{bmatrix} 0 & \mp 1 & 1 & 0 \\ \pm 1 & 0 & 0 & \mp 1 \\ -1 & 0 & 0 & 1 \\ 0 & \pm 1 & -1 & 0 \end{bmatrix}$$

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[Bañados–Henneaux–Teitelboim–Zanelli, gr-qc/9302012]

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- $(2,2)$, $\mu(x) = (x^2+\beta^2+\varphi^2) - 4\beta^2 x^2$, self-dual boost + antiself-dual rotation

$$B_{\pm}^{(2,2)}(\beta > 0, \varphi > 0) = \begin{bmatrix} 0 & \pm\varphi & 0 & -\beta \\ \mp\varphi & 0 & \pm\beta & 0 \\ 0 & \mp\beta & 0 & -\varphi \\ \beta & 0 & \varphi & 0 \end{bmatrix}$$

- $(2,3)$

- $(2,3), \ \mu(x) = x^5$

- $(2,3)$, $\mu(x) = x^5$, deformation of $B_+^{(2,2)}$ by a null rotation in a perpendicular direction

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$$B^{(2,3)}$$

- $(2,3)$, $\mu(x) = x^5$, deformation of $B_+^{(2,2)}$ by a null rotation in a perpendicular direction

$$B^{(2,3)} = \begin{bmatrix} 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- $(2, 4)$

- $(2, 4), \;\; \mu(x) = (x^2 + \varphi^2)^3$

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- and now we simply play LEGO !

Causal properties

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whose norms add

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- it is convenient to distinguish Killing vectors according to norm

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- norm bounded below:
  - ⋆ $B^{(2,0)}(\varphi) \oplus_i B^{(0,2)}(\varphi_i)$, if p is even

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- arbitrarily negative norm: the rest!

- ⋆ $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2) \oplus_i B^{(0,2)}(\varphi_i)$, unless $|\beta_1| = |\beta_2| > 0$
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Some of these give rise to higher-dimensional BTZ-like black holes

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Some of these give rise to higher-dimensional BTZ-like black holes: quotient only a part of AdS and check that the boundary thus introduced lies behind a horizon.

Discrete quotients with CTCs

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$$\gamma = \exp(LX)$$

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[FO–Leitner–Simón, to appear]

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- a number of maximally supersymmetric reductions of $\mathrm{AdS}_3 \times S^3$: near-horizon geometries of the supersymmetric rotating black holes, including over-rotating cases

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[Chamseddine–FO–Sabra, hep-th/0306278]

Thank you.