Supersymmetric space forms

José Figueroa-O'Farrill

Edinburgh Mathematical Physics Group

School of Mathematics

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A cosmological motivation

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$\Longrightarrow$ spatial universe is a 'space form'

Space forms

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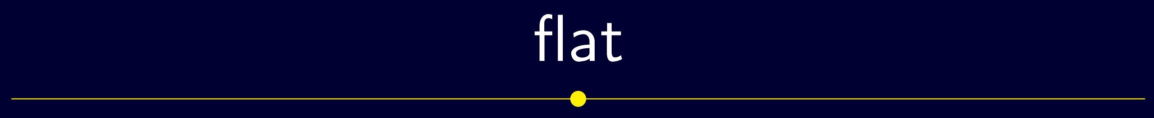

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- ‘maximally symmetric’

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where $\partial_i = \frac{\partial}{\partial x_i}$

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  i.e., $(\xi_p, \nabla\xi_p) \in T_pM \oplus \mathfrak{so}(T_pM)$

- $\dim\left(T_pM \oplus \mathfrak{so}(T_pM)\right)$

- $\dim\left(T_pM \oplus \mathfrak{so}(T_pM)\right) = n$

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- quadric is not simply-connected; its universal cover is AdS_n

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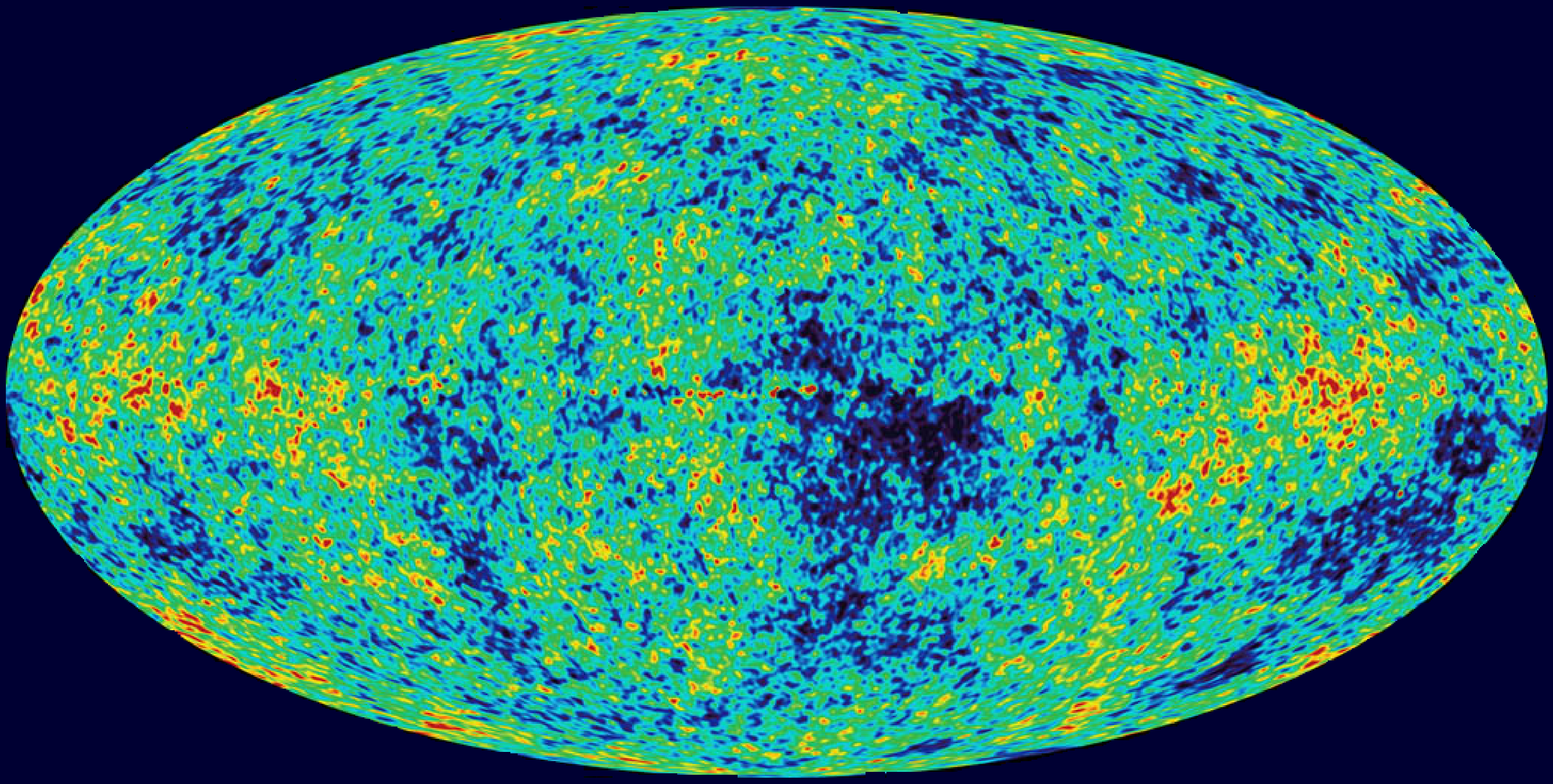

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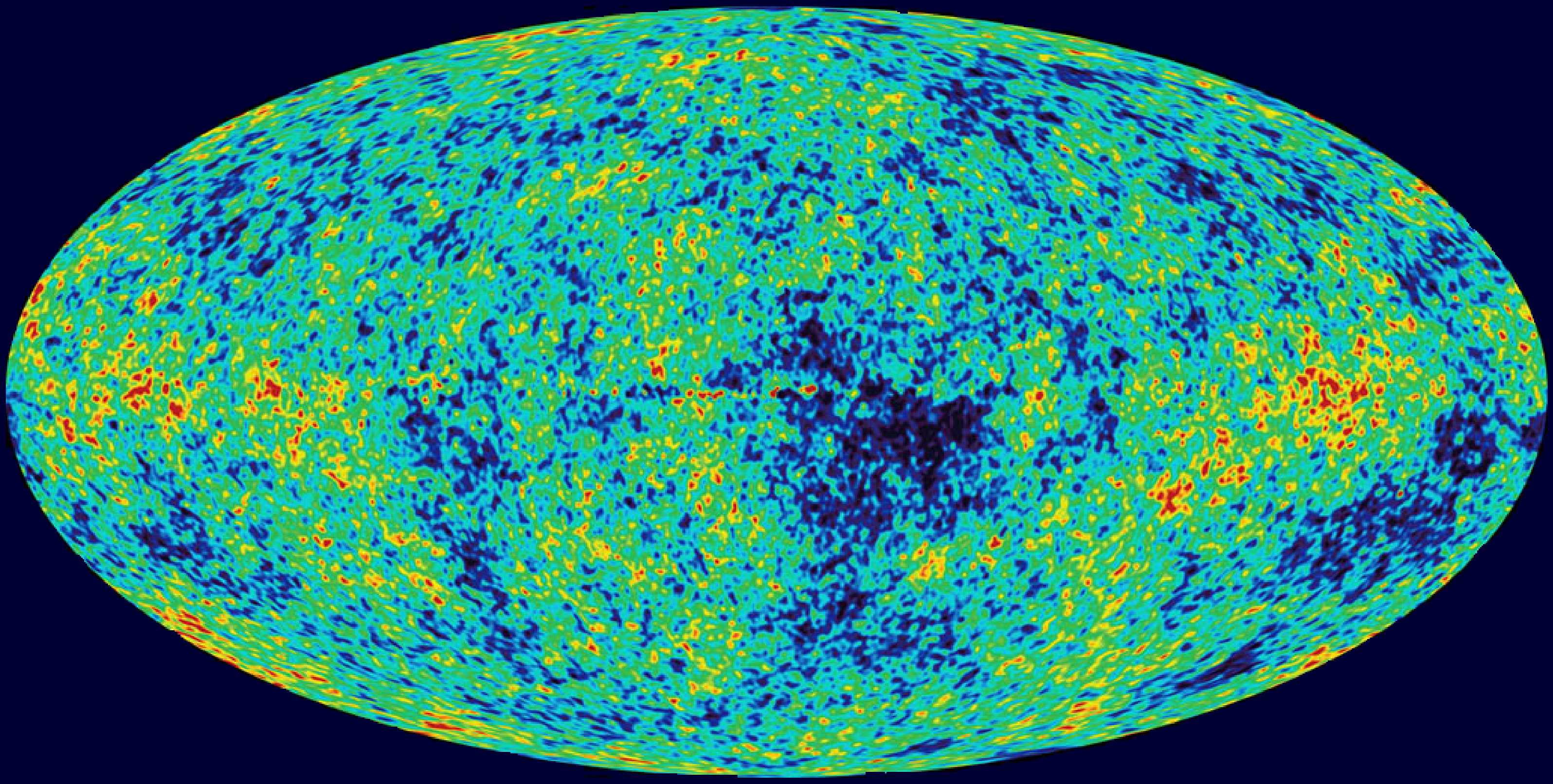

suggest that $g^{(3)}$ is the Poincaré dodecahedral space S^3/E_8

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Supergravities

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	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N=2$			$N=1$			
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7	$N=4$			$N=2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
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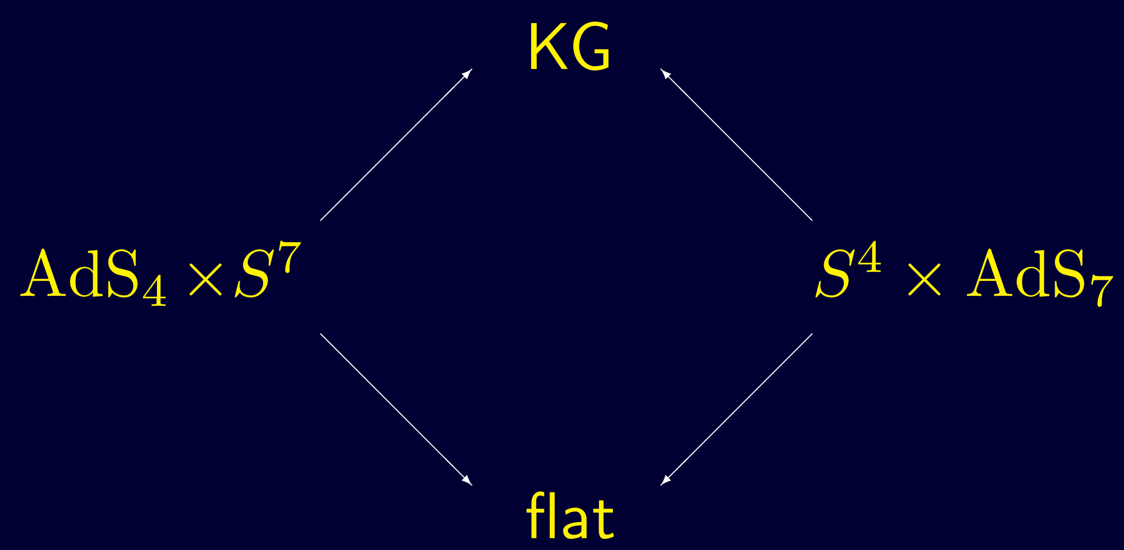

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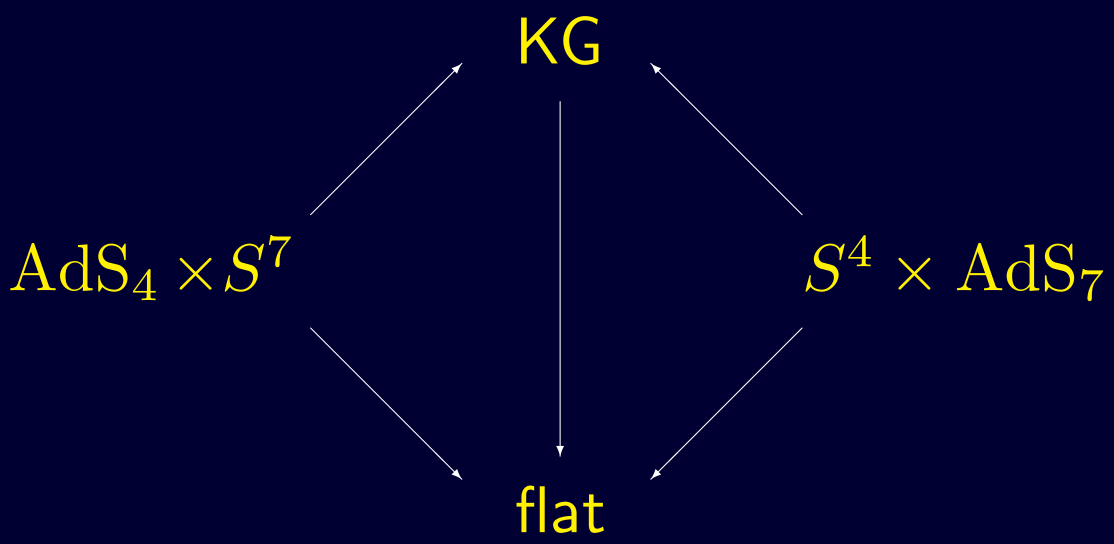

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BFHP

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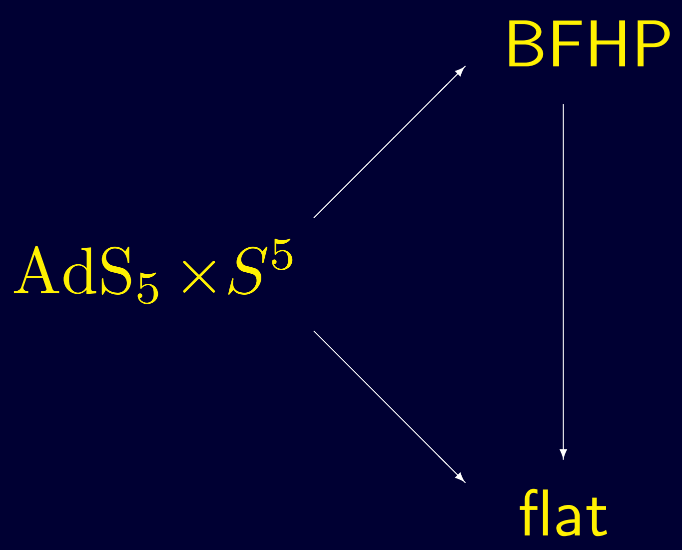

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Watch this space.