# Geometry and symmetries of homogeneous kinematical spacetimes

José Figueroa-O'Farrill

**DAMTP, 30 April 2019**

（令和時代の初セミナー?）

# Based on…

- **1711.06111**
- **1711.07363**
- **1802.04048** with Tomasz Andrzejewski
- **1809.01224** with Stefan Prohazka
- **1905.?????** with Ross Grassie and Stefan Prohazka

# Part 1
# Main results

# Motivation

Maximally symmetric lorentzian manifolds — (anti) de Sitter and Minkowski spacetimes — play an important rôle in contemporary theoretical physics: GR, QFT, AdS/CFT,…

There is a desire to explore “non-relativistic” limits of these theories, in view of its applications to flat space holography, condensed matter,…

**Natural question:** What are the “non-relativistic” analogues of these spacetimes?

# Dramatis personae

**Anti de Sitter spacetime (D+1 dimensional)**

$$x_1^2 + x_2^2 + \cdots + x_D^2 - x_{D+1}^2 - x_{D+2}^2 = -\ell^2 \qquad \text{in} \quad \mathbb{R}^{D,2}$$

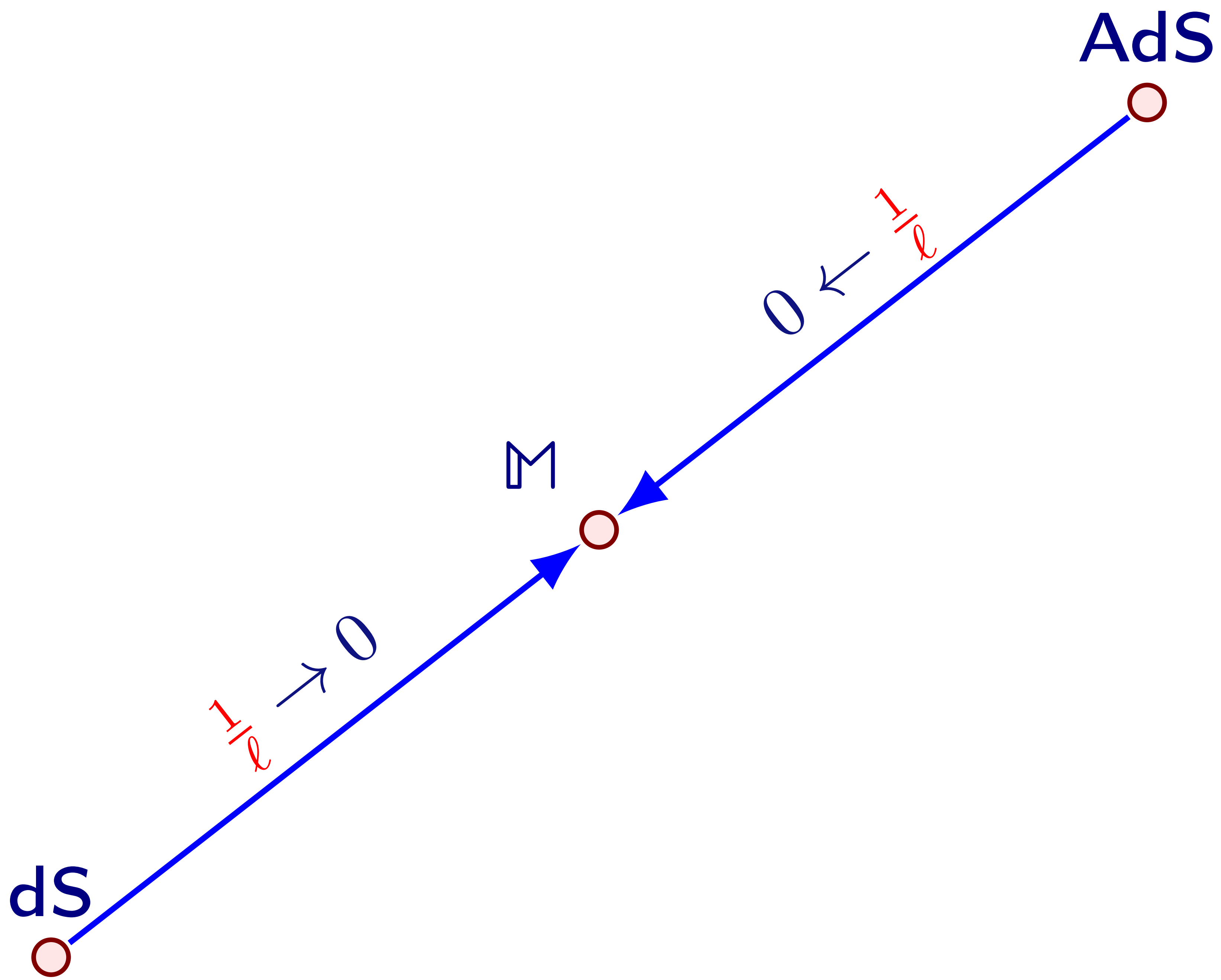
**de Sitter spacetime (D+1 dimensional)**

$$x_1^2 + x_2^2 + \cdots + x_D^2 + x_{D+1}^2 - x_{D+2}^2 = \ell^2 \qquad \text{in} \quad \mathbb{R}^{D+1,1}$$

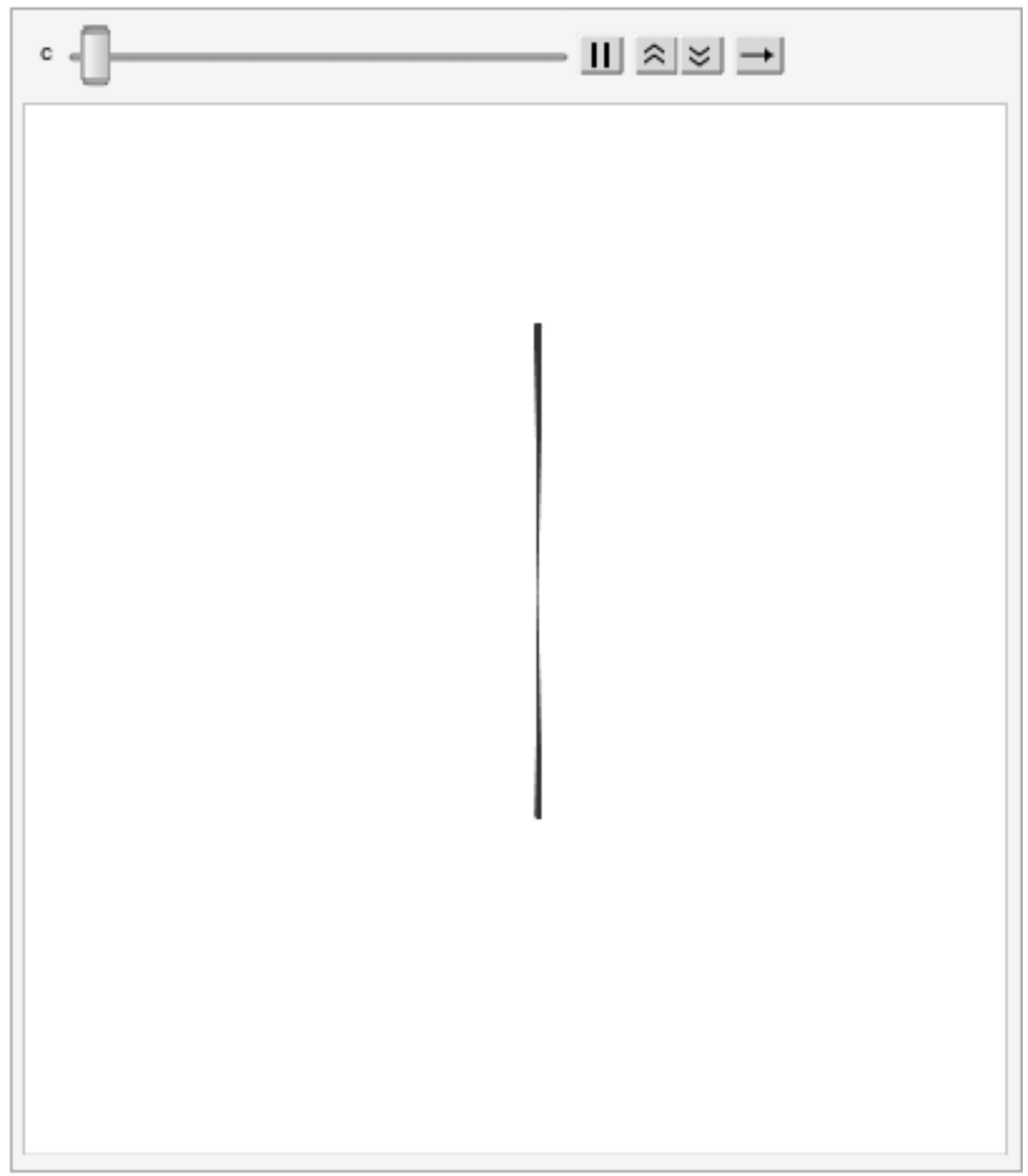
**Minkowski spacetime (D+1 dimensional)**

$$\mathbb{A}^{D+1} \qquad \text{with metric} \quad dx_1^2 + dx_2^2 + \cdots + dx_D^2 - c^2 dx_{D+1}^2$$

# Zero-curvature limits

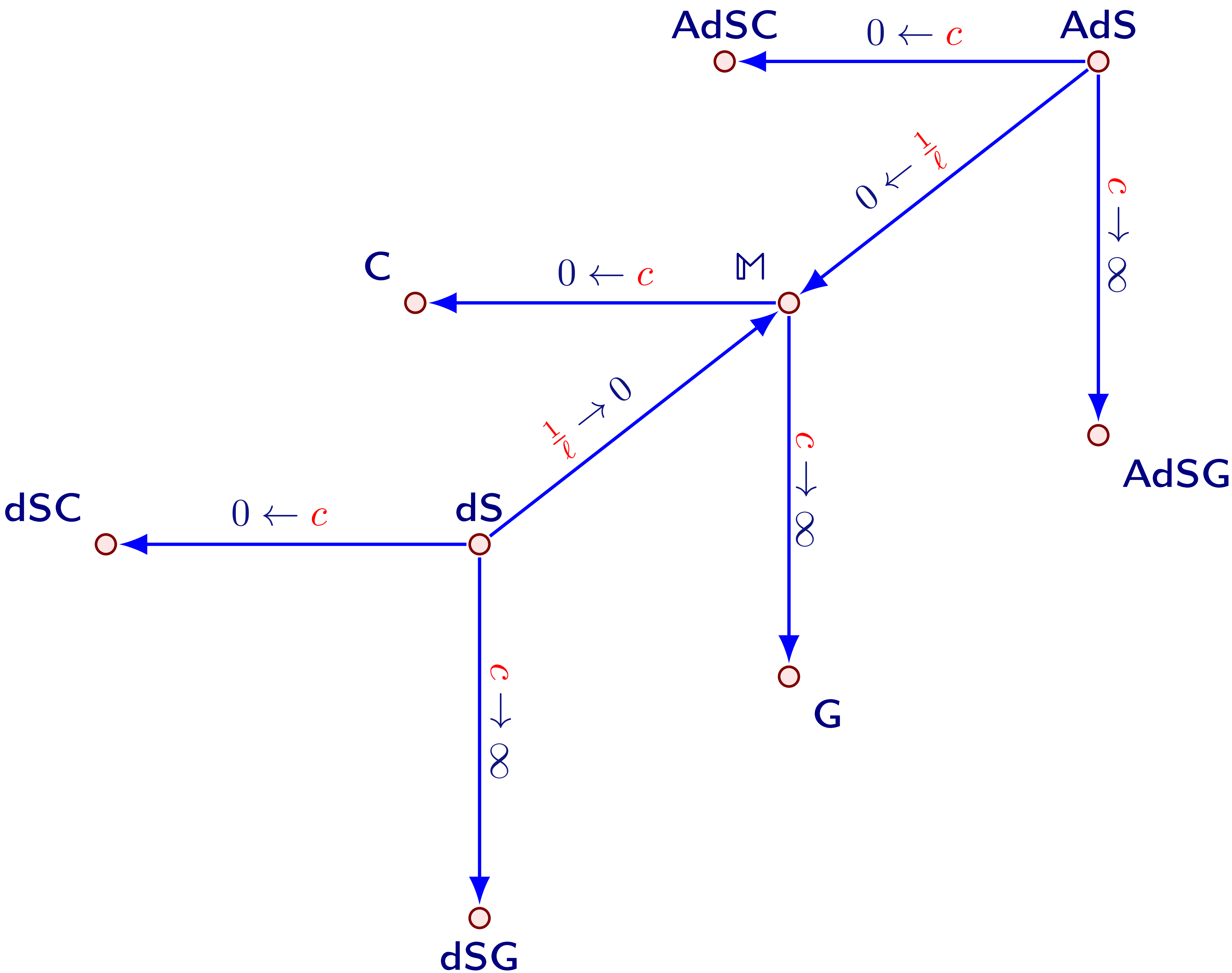


# Non- and ultra-relativistic limits



# Beyond lorentzian geometry

- The **non-relativistic limit of Minkowski spacetime is Galilean spacetime**
- **The ultra-relativistic limit of Minkowski spacetime is Carrollian spacetime** **[Lévy-Leblond 1965]**
- **(Anti) de Sitter spacetimes also have such limits: galilean (A)dS and carrollian (A)dS**
- **None of these spacetimes inherit a lorentzian metric in the limit!**

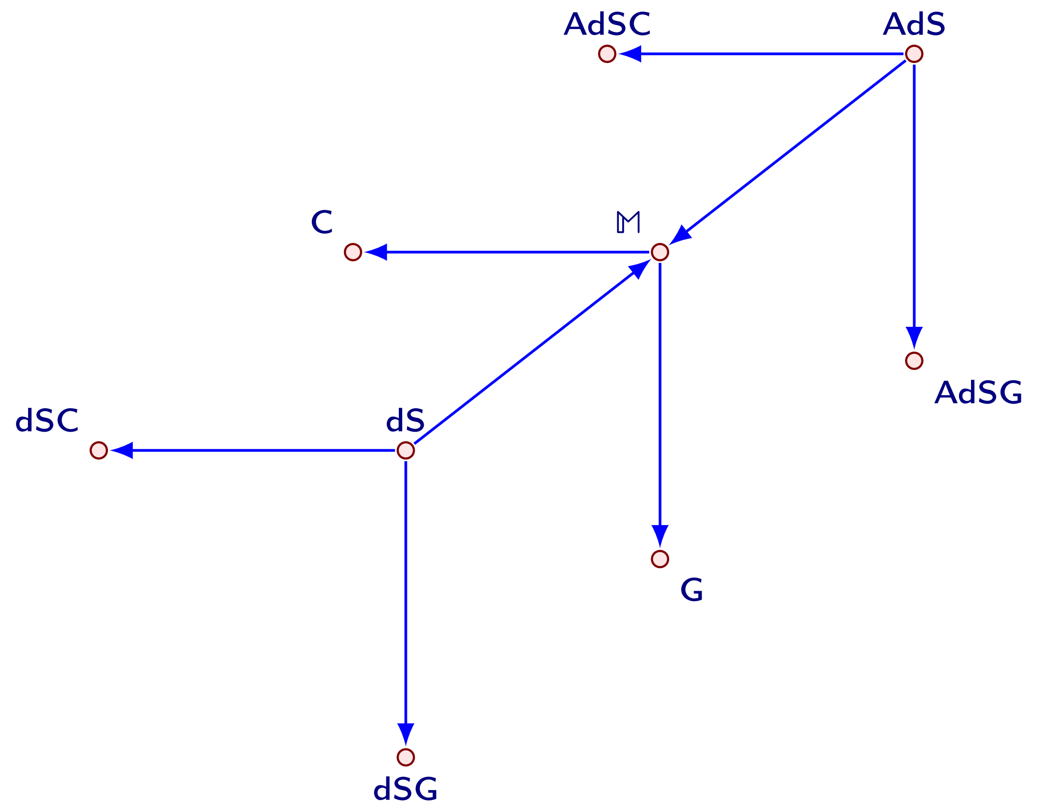


AdSC
$0 \leftarrow c$
AdS
$0 \leftarrow \frac{1}{\ell}$
$c \to \infty$
C
$0 \leftarrow c$
$\mathbb{M}$
AdSG
$\frac{1}{\ell} \to 0$
$c \to \infty$
dSC
$0 \leftarrow c$
dS
$c \to \infty$
G
dSG

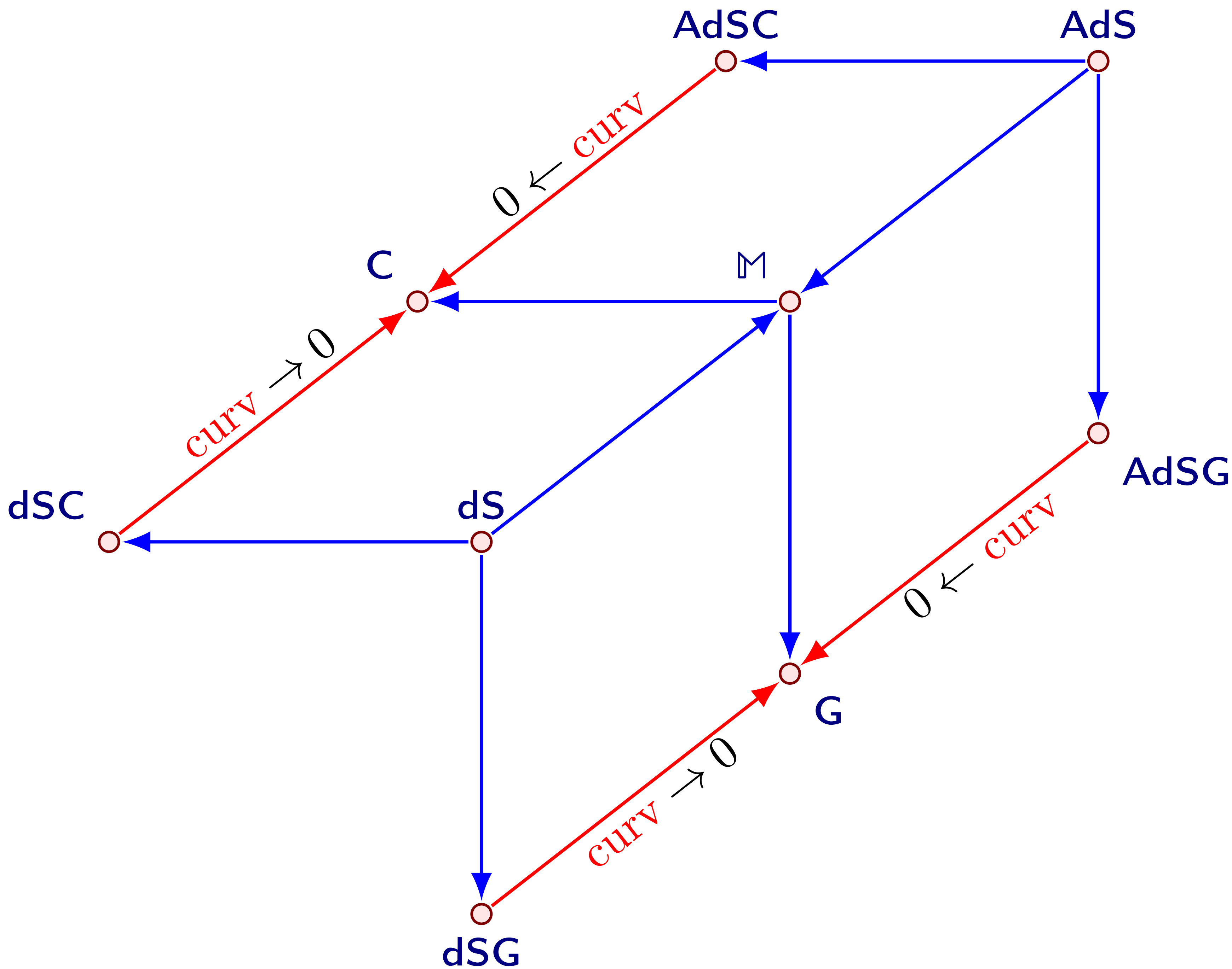
# Symmetric spaces of kinematical Lie groups

- These 9 spacetimes: **M, AdS, dS, C, G, dSC, AdSC, dSG, AdSG** (together with the riemannian symmetric spaces **E, S, H**) are symmetric homogeneous spaces of **kinematical Lie groups** (with **D**-dimensional space isotropy)

- Symmetric homogeneous spaces admit canonical torsion-free invariant connections. For **C, G, dSC, AdSC, dSG, AdSG** these are **not** metric connections. We can nevertheless still take the zero curvature limit of **dSC, AdSC, dSG, AdSG**. (**C** and **G** are already flat.)



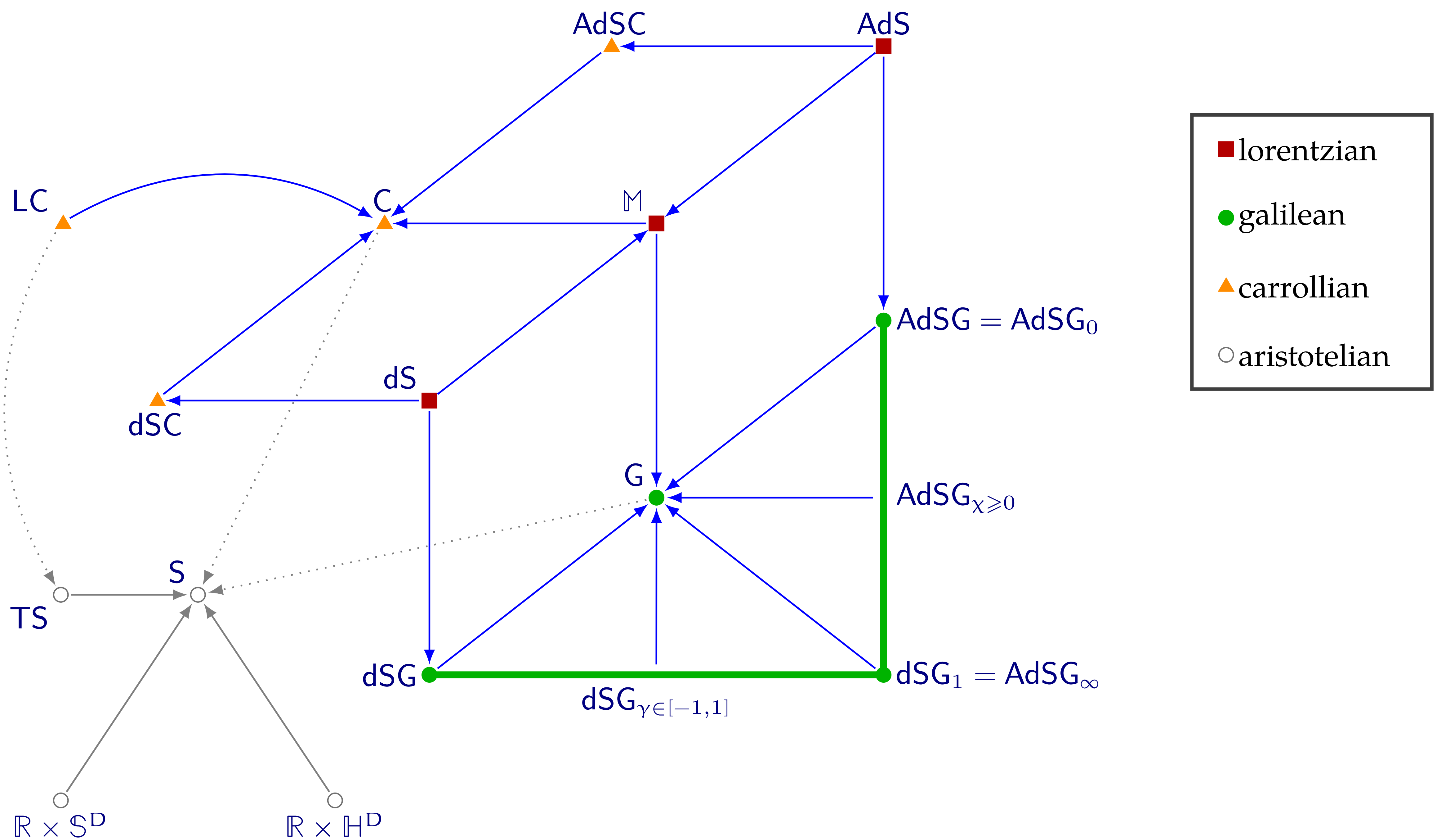
# State of (prior) art



# Highlights of new results

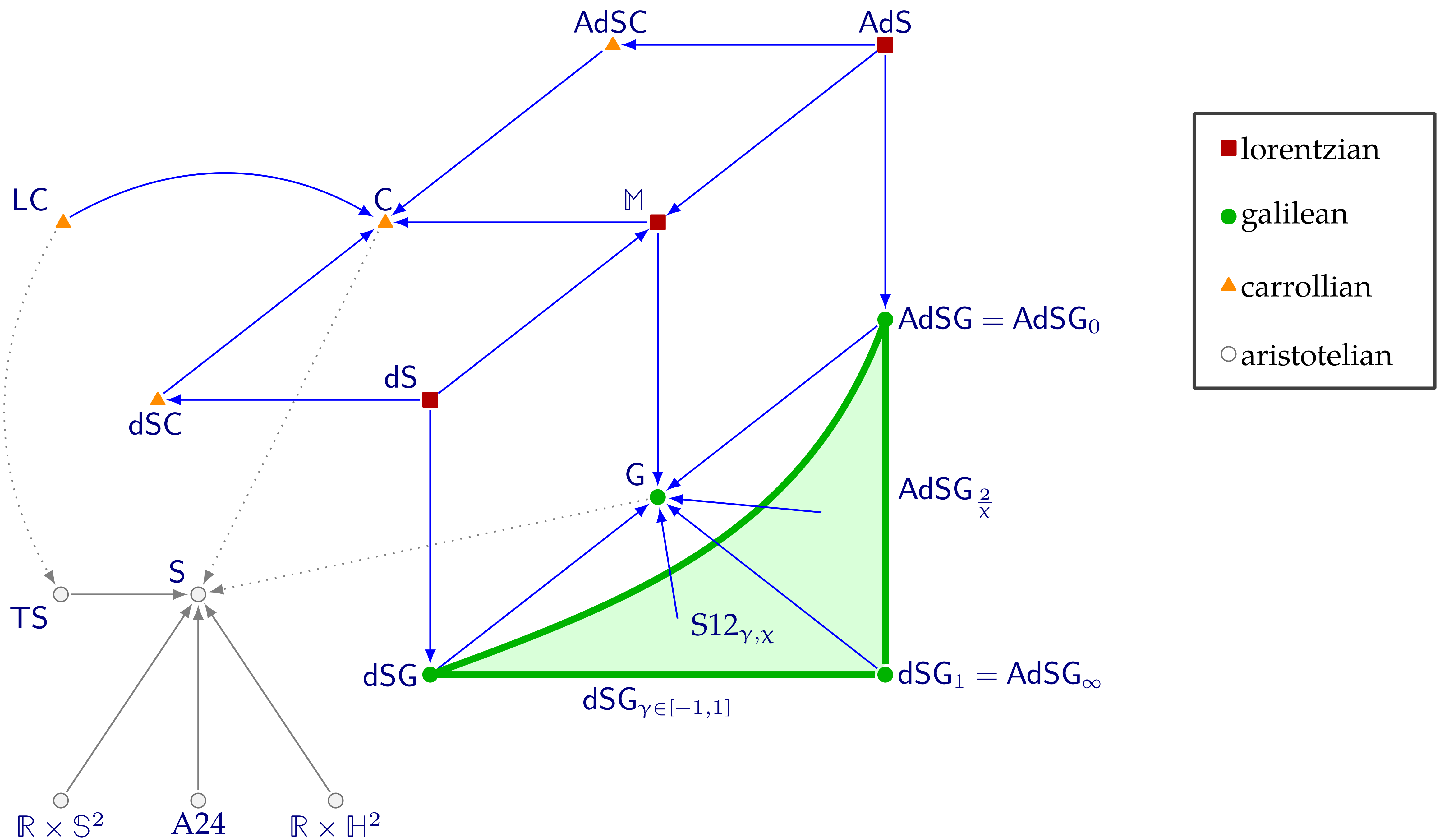
- Classification of (**D+1**)-dimensional, **simply-connected, spatially isotropic, homogeneous,** kinematical spacetimes
- Classification of (**D+1**)-dimensional **aristotelian** spacetimes
- Results differ for **$D \geq 3$, $D=2$** and **$D=1$**
- Limits between the spacetimes
- Proof that the orbits of boosts are generically non-compact (except in riemannian and aristotelian "spacetimes", of course)
- Determination of (infinite-dimensional) Lie algebras of (conformal) symmetries

# D≥3



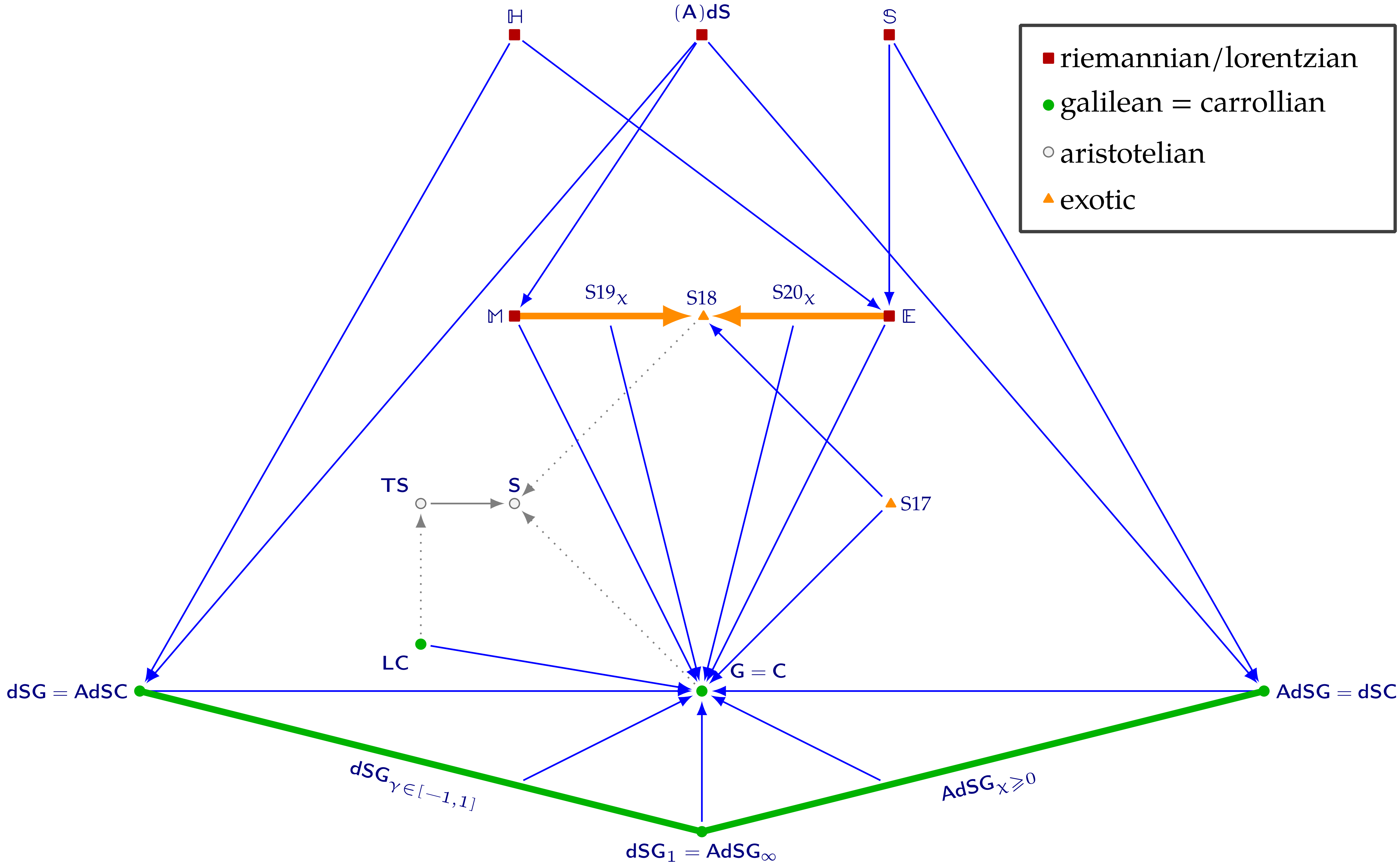
AdSC
AdS
LC
C
$\mathbb{M}$
dS
dSC
G
$\mathsf{AdSG} = \mathsf{AdSG}_0$
$\mathsf{AdSG}_{\chi \geqslant 0}$
S
TS
dSG
$\mathsf{dSG}_1 = \mathsf{AdSG}_\infty$
$\mathsf{dSG}_{\gamma \in [-1,1]}$
$\mathbb{R} \times \mathbb{S}^D$
$\mathbb{R} \times \mathbb{H}^D$
lorentzian
galilean
carrollian
aristotelian

# D=2



AdSC
AdS
LC
C
$\mathbb{M}$
dS
dSC
G
$\mathsf{AdSG} = \mathsf{AdSG}_0$
$\mathsf{AdSG}_{\frac{2}{\chi}}$
$\mathsf{S12}_{\gamma,\chi}$
S
TS
dSG
$\mathsf{dSG}_1 = \mathsf{AdSG}_\infty$
$\mathsf{dSG}_{\gamma\in[-1,1]}$
$\mathbb{R}\times\mathbb{S}^2$
A24
$\mathbb{R}\times\mathbb{H}^2$
lorentzian
galilean
carrollian
aristotelian

# D=1



ℍ
(A)dS
𝕊
riemannian/lorentzian
galilean = carrollian
aristotelian
exotic
S19$_\chi$
S18
S20$_\chi$
𝕄
𝔼
TS
S
S17
LC
G = C
dSG = AdSC
AdSG = dSC
dSG$_{\gamma\in[-1,1]}$
AdSG$_{\chi\geqslant 0}$
dSG$_1$ = AdSG$_\infty$

# Part 2
# Technical details

# Kinematical Lie algebras

- The Lie algebra of a kinematical Lie group (with **D**-dimensional space isotropy) is a **kinematical Lie algebra**
- A **kinematical Lie algebra** (with **D**-dimensional space isotropy) is a real Lie algebra $\mathfrak{k}$ of dimension **(D+1)(D+2)/2** such that
  - $\mathfrak{so}(D) \subset \mathfrak{k}$
  - under $\mathfrak{so}(D)$, $\mathfrak{k} = \mathfrak{so}(D) \oplus 2\,V \oplus S$
    - **V** = **D**-dimensional **vector** rep of $\mathfrak{so}(D)$
    - **S** = 1-dimensional **scalar** rep of $\mathfrak{so}(D)$

Typically, we write the generators as

$$J_{ab} \qquad B_a \qquad P_a \qquad H$$

$\mathfrak{so}(D)$

"boosts"

"spatial translations"

"time translation"

The reason for the " " is that the geometrical/physical interpretation can only be given when the Lie algebra acts on a spacetime.

$$[\boldsymbol{J}, \boldsymbol{J}] = \boldsymbol{J} \qquad [\boldsymbol{J}, \boldsymbol{B}] = \boldsymbol{B} \qquad [\boldsymbol{J}, \boldsymbol{P}] = \boldsymbol{P} \qquad [\boldsymbol{J}, H] = 0$$

# Examples

**Simple Lie algebras** $\mathfrak{so}(D+1,1)$ $\mathfrak{so}(D,2)$ $\mathfrak{so}(D+2)$

$$\mathfrak{p} = \mathfrak{so}(D,1) \ltimes \mathbb{R}^{D,1}$$

**Poincaré**

$$\mathfrak{e} = \mathfrak{so}(D+1) \ltimes \mathbb{R}^{D+1}$$

**Euclidean**

$$[H, \boldsymbol{B}] = \boldsymbol{P}$$

**Galilean**

$$[\boldsymbol{B}, \boldsymbol{P}] = H$$

**Carroll**

# More examples

$$[H, \boldsymbol{P}] = \boldsymbol{P} \qquad [H, \boldsymbol{B}] = \gamma \boldsymbol{B} \qquad \gamma \in [-1, 1]$$

$\gamma = -1$ **Newton-Hooke**

$$[H, \boldsymbol{B}] = \chi \boldsymbol{B} + \boldsymbol{P} \qquad [H, \boldsymbol{P}] = \chi \boldsymbol{P} - \boldsymbol{B} \qquad \chi \geq 0$$

$\chi = 0$ **Newton-Hooke**

These are all in **D>3.** For **D=3** and **D=2** there are others due to the existence of

$\epsilon_{abc}$ and $\epsilon_{ab}$

# Classifications

- **D=0** There is only one 1-dimensional Lie algebra
- **D=1** There are no rotations, so any 3-dimensional Lie algebra is kinematical **[Bianchi 1898]**
- **D=3** **[Bacry+Lévy-Leblond 1968] [Bacry+Nuyts 1986]**
- **D≥3** **[JMF 2017]**

**(using deformation theory)**

- **D=2** **[Andrzejewski+JMF 2018]**

# Homogeneous spacetimes

$\mathcal{K}/\mathcal{H}$ **Lie pair** $(\mathfrak{k}, \mathfrak{h})$

$\mathcal{K}$ Lie group with kinematical Lie algebra $\mathfrak{k}$

$\mathcal{H}$ closed Lie subgroup with Lie algebra $\mathfrak{h}$

$\mathrm{Span}\{\boldsymbol{J}, \boldsymbol{B}\}$

$\exists$ choice of basis

**Theorem** There is a one-to-one correspondence between (isomorphism classes of) simply-connected homogeneous spacetimes and (isomorphism classes of) *geometrically realisable*, *effective* Lie pairs.

# Caveat

- A kinematical Lie algebra $\mathfrak{k}$ may possess
  - **no** homogeneous spacetimes,
  - a **unique** homogeneous spacetime, or
  - **more than one** homogeneous spacetimes:

$\mathfrak{so}(D+1,1)$ {
**de Sitter**
**hyperbolic space**
**carrollian light cone**

$\mathfrak{p}$ {
**Minkowski**
**carrollian AdS**

$\mathfrak{e}$ {
**Euclidean**
**carrollian dS**

# Classifications

- Geometrically realisable, effective Lie pairs for kinematical Lie algebras **[JMF+Prohazka 2018]**

- Aristotelian ("no boosts") Lie algebras and their spacetimes **[JMF+Prohazka 2018]**

- Boosts act with generically non-compact orbits in all spacetimes except the aristotelian ($\nexists$ boosts) and the riemannian symmetric spaces ("boosts" = rotations) **[JMF+Grassie+Prohazka 2019]**

# Invariant structures

$\mathcal{M} = \mathcal{K}/\mathcal{H}$ simply-connected homogeneous kinematical spacetime

$\mathcal{K}$ simply-connected $\qquad$ $\mathcal{H}$ closed, connected

**<u>Theorem</u>** There is a one-to-one correspondence between $\mathcal{K}$-invariant tensor fields on $\mathcal{M}$ and $\mathfrak{h}$-invariant tensors on $\mathfrak{k}/\mathfrak{h}$ .

**invariant *metric*** $\left(S^2(\mathfrak{k}/\mathfrak{h})^*\right)^{\mathfrak{h}}$

**invariant *cometric*** $\left(S^2(\mathfrak{k}/\mathfrak{h})\right)^{\mathfrak{h}}$

**invariant one-form** $\left((\mathfrak{k}/\mathfrak{h})^*\right)^{\mathfrak{h}}$

**invariant vector field** $(\mathfrak{k}/\mathfrak{h})^{\mathfrak{h}}$

- With the exception of some “exotic” 2-dimensional spacetimes, the others fall into one of several classes determined by their invariant structure:

  - **riemannian:** invariant positive-definite metric
  - **lorentzian:** invariant lorentzian metric
  - **galilean:** invariant “clock” one-form $\tau$ and spatial cometric $h$, with $h(\tau,-)=0$
  - **carrollian:** invariant vector field $\kappa$ and spatial metric $b$, with $b(\kappa,-)=0$
  - **aristotelian:** simultaneously invariant galilean and carrollian structures: $\tau, \kappa, h, b$

# Null hypersurfaces

- Carrollian manifolds may be embedded as null hypersurfaces in a lorentzian manifold:
  - **C** embeds in Minkowski spacetime as $x^+ = 0$

    **[Duval+Gibbons+Horvathy+Zhang 2014]**
  - **LC** embeds in Minkowski spacetime as the future lightcone*

    * except in **D=1** since the lightcone is not simply-connected
  - **dSC** embeds in de Sitter spacetime
  - **AdSC** embeds in anti de Sitter spacetime

# Symmetries

- Symmetries of riemannian and lorentzian manifolds are always finite-dimensional and the same is true for aristotelian manifolds

- Symmetries of galilean and carrollian manifolds need not be finite-dimensional

- Same holds for conformal symmetries

# Galilean symmetries

$(\mathcal{M}, \tau, h)$ homogeneous galilean spacetime

A vector field $\xi$ is a **galilean Killing vector** if

$$\mathcal{L}_\xi \tau = 0 \qquad \text{and} \qquad \mathcal{L}_\xi h = 0$$

All the homogeneous galilean kinematical spacetimes have isomorphic Lie algebra $\mathfrak{gkv}$ of galilean Killing vectors, the semidirect product

$$0 \longrightarrow C^\infty(\mathbb{R}_t, \mathfrak{e}) \longrightarrow \mathfrak{gkv} \longrightarrow \mathbb{R} \longrightarrow 0$$

$$\frac{\partial}{\partial t} \leftarrow 1$$

euclidean Lie algebra in **D** dimensions

***Coriolis algebra*** [Duval 1993]

A vector field $\xi$ is a **galilean conformal Killing vector** if

$$\mathcal{L}_\xi \tau = \lambda \tau \qquad \text{and} \qquad \mathcal{L}_\xi h = -2\lambda h$$

$$\exists\, \lambda \in C^\infty(\mathcal{M})$$

All the homogeneous galilean kinematical spacetimes have isomorphic Lie algebra $\mathfrak{gckv}$ of galilean conformal Killing vectors:

$$f' r \frac{\partial}{\partial r} - \frac{f}{D} \frac{\partial}{\partial t} \leftarrow\!\mapsto f$$

$$0 \longrightarrow C^\infty(\mathbb{R}_t, \mathfrak{e}) \longrightarrow \mathfrak{gckv} \longrightarrow C^\infty(\mathbb{R}_t) \longrightarrow 0$$

"wronskian" Lie algebra

$$[f, g] = f g' - f' g$$

[Duval+Gibbons+Horvathy 2014]

# Carrollian symmetries

$(\mathcal{M}, \kappa, b)$ homogeneous carrollian spacetime

A vector field $\xi$ is a **carrollian Killing vector** if

$$[\xi, \kappa] = 0 \qquad \text{and} \qquad \mathcal{L}_\xi b = 0$$

A vector field $\xi$ is a **carrollian conformal Killing vector** if

$$[\xi, \kappa] = -\lambda\kappa \qquad \text{and} \qquad \mathcal{L}_\xi b = 2\lambda b$$

$$\exists\ \lambda \in C^\infty(\mathcal{M})$$

# Symmetries of C

The Lie algebra $\mathfrak{ckv}$ of carrollian Killing vectors of **C** is the semidirect product:

$$0 \longrightarrow C^\infty(\mathbb{E}^D) \longrightarrow \mathfrak{ckv} \longrightarrow \mathfrak{e} \longrightarrow 0$$

The Lie algebra $\mathfrak{cckv}$ of carrollian conformal Killing vectors of **C** depends on the dimension **D.**

density line bundle

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}_{D\geq 3} \longrightarrow \mathfrak{so}(D+1,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}_{D=2} \longrightarrow \mathscr{O}(\mathbb{C}) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}_{D=1} \longrightarrow C^\infty(\mathbb{R}) \longrightarrow 0$$

For **D=3** it is isomorphic to the BMS Lie algebra. **[Duval+Gibbons+Horvathy 2014]**

# Carrollian symmetries of the light-cone

For **D≥2** the Lie algebra of carrollian symmetries of the light-cone is just the finite-dimensional kinematical Lie algebra $\mathfrak{so}$**(D+1,1)**, but for **D=1** it is the "wronskian" Lie algebra

$$C^\infty(\mathbb{R}) \qquad [f,g] = fg' - f'g$$

The Lie algebra of conformal carrollian symmetries of the light-cone is a semidirect product of Lie algebra of carrollian symmetries by the abelian ideal of sections of the density line bundle:

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\mathsf{LC}}_{D\geq 2} \longrightarrow \mathfrak{so}(D+1,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\mathsf{LC}}_{D=1} \longrightarrow C^\infty(\mathbb{R}) \longrightarrow 0$$

**[Duval+Gibbons+Horvathy 2014]**

# Symmetries of (A)dSC

The Lie algebras of carrollian Killing vectors of **(A)dSC** are semidirect products:

$$0 \longrightarrow C^\infty(\mathbb{S}^D) \longrightarrow \mathfrak{ckv}_{\mathsf{dSC}} \longrightarrow \mathfrak{so}(D+1) \longrightarrow 0$$

$$0 \longrightarrow C^\infty(\mathbb{H}^D) \longrightarrow \mathfrak{ckv}_{\mathsf{AdSC}} \longrightarrow \mathfrak{so}(D,1) \longrightarrow 0$$

The Lie algebras of carrollian conformal Killing vectors of **(A)dSC** are semidirect products, which depend on **D**.

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{(A)dSC}}_{D\geq 3} \longrightarrow \mathfrak{so}(D+1,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{dSC}}_{D=2} \longrightarrow \mathfrak{so}(3,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{AdSC}}_{D=2} \longrightarrow \mathscr{O}(\mathbb{H}) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{dSC}}_{D=1} \longrightarrow C^\infty(S^1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{AdSC}}_{D=1} \longrightarrow C^\infty(\mathbb{R}) \longrightarrow 0$$

# Future directions

- Galilean spacetimes are null reductions of lorentzian manifolds. It would be interesting to exhibit the homogeneous galilean spacetimes in this light.
- We also classified the invariant connections and it would be interesting to explore their geodesics.
- The BMS-like Lie algebras associated to **AdSC** extend the Poincaré algebra. Do they play a rôle in flat space holography?
- There are limits from these spacetimes to spacetimes without spatial isotropy (but with “Lorentz” isotropy), resulting in “pseudo-carrollian” spacetimes such as Ashtekar—Hansen’s **Spi**. This landscape is largely unexplored. **[Gibbons 2019]**