The gauged WZ term with boundary

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A quaternion of sigma models

WZ	bWZ
gWZ	gbWZ

… and one of cohomology theories

de Rham	relative de Rham
equivariant	relative equivariant

References

- hep-th/9407149
- *Phys. Lett.* **B341** (1994) 153–159, hep-th/9407196
- *JHEP* **01** (2001) 006, hep-th/0008038

with Sonia Stanciu

- hep-th/0506049

with Nouri Mohammedi

Sigma models

- Two oriented pseudo-riemannian manifolds: Σ^d, X^n
- $\partial\Sigma$ may or may not be empty
- $\varphi : \Sigma \to X$, $d\varphi \in \Omega^1(\Sigma, \varphi^* TX)$
- sigma model action

$$I_\sigma = \int_\Sigma \tfrac{1}{2} |d\varphi|^2$$

defines variational problem for harmonic maps $d \star_\Sigma d\varphi = 0$

$\square$

The Wess–Zumino term

- $H \in \Omega^{d+1}_{\text{closed}}(X)$
- assume $\partial\Sigma = \varnothing$ and $\varphi(\Sigma) \subset X$ bounds

$$\varphi(\Sigma) = \partial M \qquad \exists M \subset X$$

- Wess–Zumino term

$$I_{\text{WZ}} = \int_M H$$

- $dH = 0 \implies$ field equations for φ

$$\delta I_{\mathsf{WZ}} = \int_M \mathscr{L}_{\delta\varphi} H = \int_M d\imath_{\delta\varphi} H = \int_{\varphi(\Sigma)} \imath_{\delta\varphi} H$$

$\therefore$ classical theory is independent of choice of M

- quantum theory depends on I_{WZ} modulo $2\pi\mathbb{Z}$

$\therefore$ quantum theory is independent of M if $[\frac{1}{2\pi}H]$ is an integral class in $H^{d+1}_{\mathsf{dR}}(X)$

Example: the WZW model

- Σ two-dimensional
- X a compact simple Lie group
- $H_2(X) = 0$ so $\varphi(\Sigma)$ always bounds [Cartan]
- $[\frac{1}{2\pi}H]$ is k times the generator of $H^3(X;\mathbb{Z}) \cong \mathbb{Z}$, where k is the level of the WZW model

The presence of a boundary

- suppose $\partial\Sigma \neq \varnothing \implies$ need to specify boundary conditions
- let $i : Y \hookrightarrow X$, $i^* H = dB$ for some $B \in \Omega^d(Y)$
- BCs: $\varphi(\partial\Sigma) \subset Y$

$\implies$ a theory of relative maps

$$\varphi : (\Sigma, \partial\Sigma) \to (X, Y)$$

The boundary Wess–Zumino term

- $\varphi(\Sigma) \subset X$ is not a cycle, but it is a cycle modulo Y:

$$\partial\varphi(\Sigma) = \varphi(\partial\Sigma) \subset Y$$

so suppose that it bounds modulo Y:

$$\exists M \subset X \ , \quad D \subset Y \qquad \text{s.t.} \qquad \partial M = \varphi(\Sigma) + D$$

whence $\partial D = -\varphi(\partial\Sigma)$

- boundary Wess–Zumino term

$$I_{\mathsf{bWZ}} = \int_M H - \int_D B$$

- B only enters in the boundary conditions:

$$\delta I_{\mathsf{bWZ}} = \int_M d\imath_{\delta\varphi} H - \int_D \mathscr{L}_{\delta\varphi} B = \int_{\varphi(\Sigma)} \imath_{\delta\varphi} H + \int_{\varphi(\partial\Sigma)} \imath_{\delta\varphi} B$$

and field equations are not otherwise changed

$\Longrightarrow$ classical theory is again independent on choice of M and D, but quantum theory is not unless $[\frac{1}{2\pi}H]$ is an integral class in the relative de Rham cohomology $H^{d+1}_{\mathsf{dR}}(X,Y)$

Example: the boundary WZW model

- $Y \subset X$ a (twisted) conjugacy class
- integrality of $[\frac{1}{2\pi}H] \in H^3_{\mathsf{dR}}(X,Y)$ selects a discrete set of such conjugacy classes corresponding to unitary highest weight representations of the (twisted) affine Lie algebra at level k

Symmetries of sigma model with WZ term

- G a connected Lie group, with Lie algebra $\mathfrak{g}$ with basis Z_a
- G acts on X isometrically preserving H
- $Z_a \mapsto v_a$ a Killing vector, $[v_a, v_b] = f_{ab}{}^c v_c$
- let $\imath_a := \imath_{v_a}$ and $\mathscr{L}_a := \mathscr{L}_{v_a}$, then $\mathscr{L}_a = d\imath_a + \imath_a d$
- $\mathscr{L}_a H = 0$, equivalently $d\imath_a H = 0$

Gaugingasigma model

- means coupling it to a gauge field $A \in \Omega^1(\Sigma, \mathfrak{g})$ so that the action is invariant under (infinitesimal) gauge transformations:

$$\delta_\lambda A = d\lambda + [A, \lambda]$$

$$\delta_\lambda \omega = d\lambda^a \wedge \imath_a \omega + \lambda^a \mathscr{L}_a \omega$$

where $\omega \in \Omega(X)$ and $\lambda \in C^\infty(\Sigma, \mathfrak{g})$

- can write $\delta_\lambda A = d\lambda^a \wedge \imath_a A + \lambda^a \mathscr{L}_a A$ by defining

$$\imath_a A^b = \delta_a^b \qquad \imath_a F^b = 0 \qquad \text{and} \qquad \mathscr{L}_a = d\imath_a + \imath_a d$$

where $F = dA + \frac{1}{2}[A, A]$ is the field-strength

Minimal coupling

- $I_\sigma = \int_\Sigma \frac{1}{2}|d\varphi|^2$ can be gauged by minimal coupling:

$$d\varphi \mapsto d_A\varphi := d\varphi - A^a \imath_a d\varphi$$

but I_{WZ} is a different matter: the minimally coupled H need not be closed

Gauging the WZ term

- means extending H to a closed gauge-invariant form $\mathscr{H}$:

$$\mathscr{H} = H + \text{terms involving } A \text{ and } F$$

such that $d\mathscr{H} = 0$ and

$$\delta_\lambda \mathscr{H} = d\lambda^a \wedge \imath_a \mathscr{H} + \lambda^a \mathscr{L}_a \mathscr{H} = 0$$

equivalently $\imath_a \mathscr{H} = 0$ (and $\mathscr{L}_a \mathscr{H} = 0$)

Differential graded algebras

- a $\mathfrak{g}$-dga $\mathfrak{A}$:
  - ⋆ $\mathfrak{A} = \bigoplus_{i \geq 0} \mathfrak{A}^i$
  - ⋆ (associative, supercommutative) product

$$\wedge : \mathfrak{A}^i \otimes \mathfrak{A}^j \longrightarrow \mathfrak{A}^{i+j}$$

  - ⋆ differential $d : \mathfrak{A}^i \longrightarrow \mathfrak{A}^{i+1}$, $d^2 = 0$
  - ⋆ derivation $\imath_a : \mathfrak{A}^i \longrightarrow \mathfrak{A}^{i-1}$
  - ⋆ derivation $\mathscr{L}_a = d\imath_a + \imath_a d$ defines $\mathfrak{g}$-action

- $\Omega(X)$ is a $\mathfrak{g}$-dga
- so is the Weyl algebra

$$\mathfrak{W} = \Lambda\mathfrak{g}^* \otimes \mathfrak{S}\mathfrak{g}^*$$

with generators $\mathscr{A}^a \in \Lambda^1\mathfrak{g}^*$ and $\mathscr{F}^a \in \mathfrak{S}^1\mathfrak{g}^*$ and where $\imath_a\mathscr{A}^b = \delta_a^b$ and $\imath_a\mathscr{F}^b = 0$ and $d\mathscr{A}^a = \mathscr{F}^a - \frac{1}{2}f_{bc}{}^a\mathscr{A}^b \wedge \mathscr{A}^c$

- Weyl homomorphism $w : \mathfrak{W} \to \Omega(\Sigma, \mathfrak{g})$ defined by $\mathscr{A}^a \mapsto A^a$ and $\mathscr{F}^a \mapsto F^a$
- $\Omega(X) \otimes \mathfrak{W}$ is a $\mathfrak{g}$-dga

Equivariant forms

- $\mathfrak{A}$ a $\mathfrak{g}$-dga, then $\phi \in \mathfrak{A}$ is
  - ⋆ horizontal, if $\imath_a \phi = 0$
  - ⋆ invariant, if $\mathscr{L}_a \phi = 0$
  - ⋆ equivariant, if both
- $\Omega_{\mathfrak{g}}(X)$: subcomplex of equivariant elements of $\Omega(X) \otimes \mathfrak{W}$
- $\{w(\phi) \mid \phi \in \Omega_{\mathfrak{g}}(X)\}$ are gauge-invariant

$\therefore$ gauging the WZ term is finding an equivariant closed extension $\mathscr{H} \in \Omega_{\mathfrak{g}}^{d+1}(X)$ of $H \in \Omega^{d+1}(X)$

The Cartan model

- in a local gauge-invariant quantity, A only appears in minimally coupled expressions (or through F)

- this suggests defining

$$\Omega_C(X) := (\Omega(X) \otimes \mathfrak{S}\mathfrak{g}^*)^{\mathfrak{g}}$$

  called the Cartan model of $\Omega_{\mathfrak{g}}(X)$

- indeed, $\Omega_C(X) \cong \Omega_{\mathfrak{g}}(X)$

- $\pi : \Omega_{\mathfrak{g}}(X) \xrightarrow{\cong} \Omega_C(X)$ consists in setting $\mathscr{A} = 0$
- $\pi^{-1} : \Omega_C(X) \xrightarrow{\cong} \Omega_{\mathfrak{g}}(X)$ consists of minimal coupling
- induced differential $d_C = \pi \circ d \circ \pi^{-1} : \Omega_C^p(X) \to \Omega_C^{p+1}(X)$ is

  $$d_C \mathscr{F}^a = 0 \qquad \text{and} \qquad d_C \omega = d\omega - \imath_a \omega \mathscr{F}^a$$

  for $\omega \in \Omega(X)$
- gauging WZ term is equivalent to finding a d_C-closed extension $\mathscr{H}_C \in \Omega_C(X)$

The Hull–Spence obstructions

- write the most general $\mathscr{H}_C$ in the Cartan model

$$\mathscr{H}_C = H + \theta_a \mathscr{F}^a + \tfrac{1}{2}\theta_{ab}\mathscr{F}^a\mathscr{F}^b + \cdots$$

where $\theta_a \in \Omega^{d-1}(X)$, $\theta_{ab} \in \Omega^{d-3}(X)$,... satisfy

$$\mathscr{L}_a \theta_b = f_{ab}{}^c \theta_c \qquad \mathscr{L}_a \theta_{bc} = f_{ab}{}^d \theta_{dc} + f_{ac}{}^d \theta_{bd} \qquad \ldots$$

- splitting $d_C \mathscr{H}_C = 0$ into types:

$$\imath_a H = d\theta_a \qquad \imath_a \theta_b + \imath_b \theta_a = d\theta_{ab} \qquad \ldots$$

which are the first two Hull–Spence obstructions

- overcoming these obstructions yields $\mathscr{H}_C$ and minimal coupling yields $\mathscr{H}$ and the gauged WZ term

$$I_{\mathsf{gWZ}} = \int_M \mathscr{H}$$

The two-dimensional case

- $\mathscr{H}_C = H + \theta_a \mathscr{F}^a$
- $d_C \mathscr{H}_C = 0$ implies

$$\imath_a H = d\theta_a \qquad \text{and} \qquad \imath_a \theta_b + \imath_b \theta_a = 0$$

- $\mathscr{L}_a \mathscr{H}_C = 0$ implies

$$\mathscr{L}_a \theta_b = f_{ab}{}^c \theta_c$$

- the gauged WZ term is

$$I_{\mathsf{gWZ}} = \int_M H + \int_\Sigma \left(\varphi^*\theta_a \wedge A^a + \tfrac{1}{2}\varphi^*(\imath_a\theta_b)A^a \wedge A^b\right)$$

[Hull & Spence; Jack, Jones, Mohammedi & Osborn]

- to this action we can add a Yang–Mills term $\int_\Sigma \frac{1}{4}|F|^2$
- or other topological terms, corresponding to cocycles in $\Omega^3_{\mathfrak{g}}(X)$

Example: the gauged WZW model

- we try to gauge $\mathfrak{g} \subset \mathfrak{r} \oplus \mathfrak{r}$ defined by homomorphisms

$$\ell : \mathfrak{g} \longrightarrow \mathfrak{r} \qquad \text{and} \qquad r : \mathfrak{g} \longrightarrow \mathfrak{r}$$

- the only obstruction is

$$\ell^* \langle -, - \rangle = r^* \langle -, - \rangle$$

with $\langle -, - \rangle$ is the ad-invariant scalar product on $\mathfrak{r}$

- typical "anomaly-free" $\mathfrak{g}$:
  - ⋆ diagonal: $\ell = r$
  - ⋆ twisted diagonal: $\ell = r \circ \tau$ for some isometry $\tau \in \mathrm{Aut}(\mathfrak{x})$
  - ⋆ chiral: $r = 0$ and $\mathfrak{g} \subset \mathfrak{x}$ an isotropic subalgebra

The twisted Courant algebroid on $T \oplus \Lambda^{d-1}T^*$

- $TX \oplus \Lambda^{d-1}T^*X$ has a $\Lambda^{d-2}T^*X$-valued bilinear

$$\langle v+\alpha, w+\beta\rangle = \imath_v\beta + \imath_w\alpha \in \Lambda^{d-2}T^*X$$

- and also has a Courant bracket:

$$[v+\alpha, w+\beta] = [v,w] + \mathscr{L}_v\beta - \mathscr{L}_w\alpha - \frac{1}{2}d(\imath_v\beta - \imath_w\alpha)$$

- $H \in \Omega^{d+1}_{\text{closed}}(X)$ twists the bracket

$$[v+\alpha, w+\beta]_H = [v+\alpha, w+\beta] - \imath_v \imath_w H$$

- we say $L \subset TX \oplus \Lambda^{d-1}T^*X$ is isotropic if for all $v+\alpha \in C^\infty(L)$, $\imath_v \alpha \in \Omega^{d-2}_{\text{exact}}(X)$

- an isotropic, involutive L defines a Lie algebroid over X

The Lie algebroid of the gauged WZ term

- let L be the image of $\mathfrak{g} \to C^\infty(TX \oplus \Lambda^{d-1}T^*X)$ given by

$$Z_a \mapsto v_a + \theta_a$$

- $\imath_a\theta_b + \imath_b\theta_a = d\theta_{ab}$, whence L is isotropic
- $d\theta_a = \imath_a H$ and $\mathscr{L}_a\theta_b = f_{ab}{}^c\theta_c$ then imply that L is involutive
- so L defines a Lie algebroid isomorphic to $\mathfrak{g}$

[cf. Alekseev & Strobl ($d = 2$), Bonelli & Zabzine]

Gauging the boundary WZ term

- assume G acts preserving (Y, B)
- gauging I_{bWZ} consists in finding equivariant extensions $\mathscr{H}$ and $\mathscr{B}$ of H and B such that $d\mathscr{H} = 0$ and $i^*\mathscr{H} = d\mathscr{B}$ on Y
- or in the Cartan model finding $\mathscr{H}_C$ and $\mathscr{B}_C$ with $d_C\mathscr{H}_C = 0$ and $i^*\mathscr{H}_C = d_C\mathscr{B}_C$
- the gauged boundary WZ term is then

$$I_{\mathsf{gbWZ}} = \int_M \mathscr{H} - \int_D \mathscr{B}$$

Two-dimensional gauged boundary WZ term

- $\mathscr{B}_C = B + h_a \mathscr{F}^a$, where $h_a \in C^\infty(X)$ obeying

$$\mathscr{L}_a h_b = f_{ab}{}^c h_c$$

- $i^* \mathscr{H}_C = d_C \mathscr{B}_C$ is equivalent to

$$i^* \theta_a + i_a B = dh_a$$

and

$$I_{\mathsf{gbWZ}} = \int_M H - \int_D B \\ + \int_\Sigma \left(\varphi^*\theta_a \wedge A^a + \tfrac{1}{2}\varphi^*(\imath_a\theta_b)A^a \wedge A^b\right) + \int_{\partial\Sigma} \varphi^* h_a A^a$$

- one can add topological terms, corresponding to relative cocycles in $\Omega^3_{\mathfrak{g}}(X,Y)$

The boundary Lie algebroid

- the twisted Courant bracket restricts to $TY \oplus \Lambda^{d-1}T^*Y$, but since $i^*H = dB$, it is B-related to the untwisted Courant bracket:

$$[v+\alpha, w+\beta]_{dB} = [e^B(v+\alpha), e^B(w+\beta)]$$

- the image of the map $\mathfrak{g} \to C^\infty(TY \oplus \Lambda^{d-1}T^*Y)$ defined by

$$Z_a \mapsto e^B(v_a + i^*\theta_a) = v_a + \imath_a B + i^*\theta_a = v_a + dh_a$$

is isotropic and involutive: a Lie algebroid on Y isomorphic to $\mathfrak{g}$

Example: the gauged boundary WZW model

- possible boundary conditions are G-orbits:
  - ⋆ (twisted) diagonal gaugings: (twisted) conjugacy classes
  - ⋆ chiral gaugings: cosets
- boundary offers no new obstructions
- $dh_a = 0$ and $h_a \in [\mathfrak{g}, \mathfrak{g}]^o$
- boundary Lie algebroid is the action Lie algebroid of $\mathfrak{g}$ on Y

Open questions

- are there CFT constructions for the cosets?
- relation with boundary conditions on $X \times X$?
- relation with boundary integrable systems?
- (oidish) interpretation for ‘higher’ obstructions?

Thank you!