A geometric construction of exceptional Lie algebras

José Figueroa-O'Farrill
Maxwell Institute & School of Mathematics
THE UNIVERSITY OF EDINBURGH

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2007 will be known as the year E8 made it to the mainstream!

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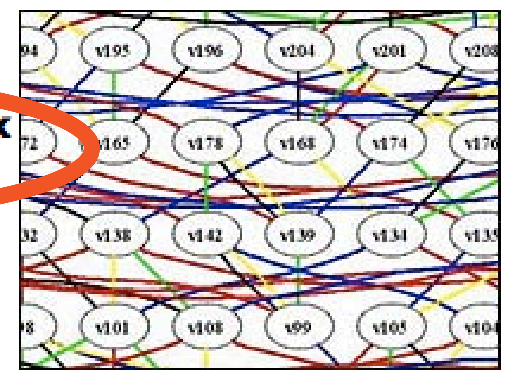
248-dimension maths puzzle solved

An international team of mathematicians has detailed a vast complex numerical "structure" which was described more than a century ago.



The structure is described in the form of a vast matrix

Mapping the 248-dimensional structure, called E8, took four years of work and produced more data than the Human Genome Project, researchers said.

E8 is a "Lie group", a means of describing symmetrical objects.

The team said their findings may assist fields of physics which use more than four dimensions, such as string theory.

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Surfer dude stuns physicists with theory of everything

By Roger Highfield, Science Editor
Last Updated: 6:01pm GMT 14/11/2007

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An impoverished surfer has drawn up a new theory of the universe, seen by some as the Holy Grail of physics, which has received rave reviews from scientists.

The E8 pattern (left), Garrett Lisi surfing (middle) and out of the water (right)

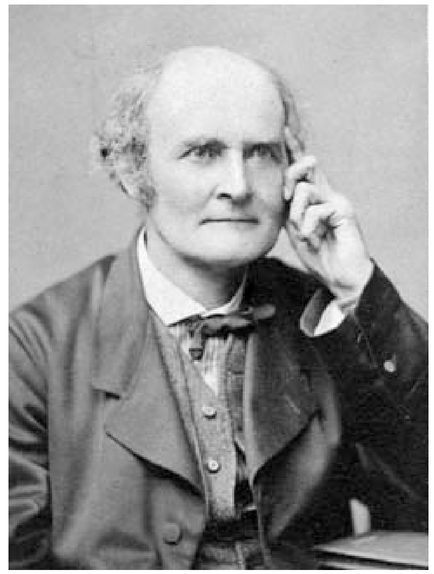
Garrett Lisi, 39, has a doctorate but no university affiliation and spends most of the year surfing in Hawaii, where he has also been a hiking guide and bridge builder (when he slept in a jungle yurt).

In winter, he heads to the mountains near Lake Tahoe, Nevada, where he snowboards. "Being poor sucks," Lisi says. "It's hard to figure out the secrets of the universe when you're trying to figure out where you and your girlfriend are going to sleep next month."

Despite this unusual career path, his proposal is remarkable because, by the arcane standards of particle physics, it does not require highly complex mathematics.

What would they think?

Hamilton



Cayley

Lie

Killing

É. Cartan

Hurwitz

Hopf

J.F. Adams

This talk is about a relation between **exceptional** objects:

- **Hopf bundles**
- exceptional **Lie algebras**

using a **geometric** construction familiar from **supergravity**: the **Killing (super)algebra**.

Real division algebras

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\geq$			
$ab = ba$	$ab = ba$	$ab \neq ba$	
$(ab)c = a(bc)$	$(ab)c = a(bc)$	$(ab)c = a(bc)$	$(ab)c \neq a(bc)$

These are all the euclidean normed real division algebras. [Hurwitz]

Hopf fibrations

$$S^1 \xrightarrow{S^0} S^1 \qquad S^3 \xrightarrow{S^1} S^2 \qquad S^7 \xrightarrow{S^3} S^4 \qquad S^{15} \xrightarrow{S^7} S^8$$

$$S^0 \subset \mathbb{R} \qquad S^1 \subset \mathbb{C} \qquad S^3 \subset \mathbb{H} \qquad S^7 \subset \mathbb{O}$$

$$S^1 \subset \mathbb{R}^2 \qquad S^3 \subset \mathbb{C}^2 \qquad S^7 \subset \mathbb{H}^2 \qquad S^{15} \subset \mathbb{O}^2$$

$$S^1 \cong \mathbb{RP}_1 \qquad S^2 \cong \mathbb{CP}_1 \qquad S^4 \cong \mathbb{HP}_1 \qquad S^8 \cong \mathbb{OP}_1$$

These are the only examples of fibre bundles where all three spaces are spheres. [Adams]

Simple Lie algebras

(over $\mathbb{C}$)

4 classical series:

$A_{n\geq 1}$	$SU(n+1)$
$B_{n\geq 2}$	$SO(2n+1)$
$C_{n\geq 3}$	$Sp(n)$
$D_{n\geq 4}$	$SO(2n)$

[Lie]

5 exceptions:

G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

[Killing, Cartan]

Supergravity

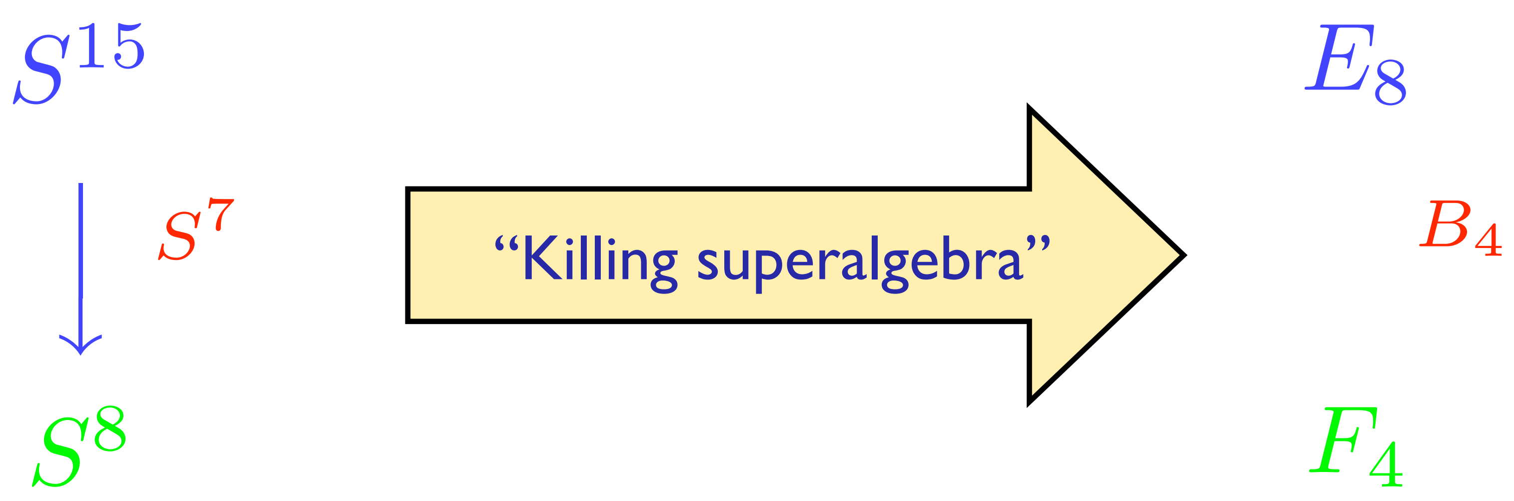
Supergravity is a nontrivial generalisation of Einstein's theory of General Relativity.

In supergravity, the universe is described by a **spin manifold** with a notion of **Killing spinor**.

These spinors generate the **Killing superalgebra**, which is a **useful invariant** of the universe.

Today we will apply this idea to a classical geometric situation.

Applying the Killing superalgebra construction to the **exceptional Hopf fibration**, one obtains a triple of **exceptional Lie algebras**:

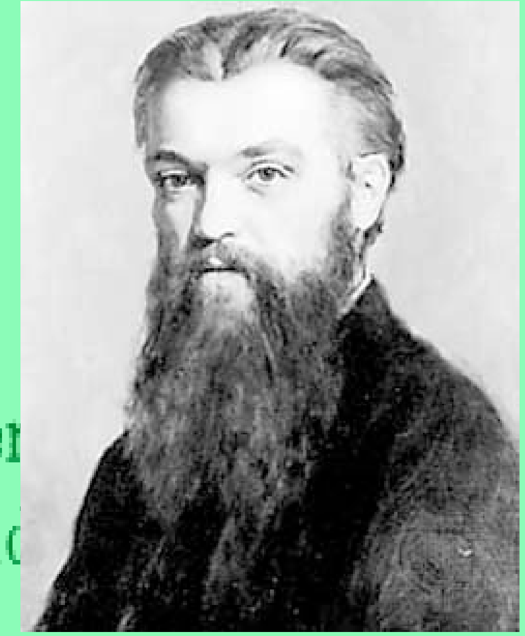


Spinors

closely analogous to it; of these I would point out particularly that of two units, belonging to the elliptic geometry of one dimension or to the theory of vectors in a plane. Let the units be ι_2, ι_3; then a product of any even number of linear functions must be of the form $a + b\iota_2\iota_3$. Let $i = \iota_2\iota_3$, then $i^2 = -1$; and such an even product is the ordinary complex number $a + bi$. In the method of Gauss every vector in the plane is represented by means of its ratio to the unit vector ι_2, that is to say, ι_2 and ι_3 are replaced by 1 and i. This gives an artificial but highly useful value for the product of two vectors. We might apply a similar interpretation to the algebra of four units, denoting the points $\iota_0, \iota_1, \iota_2, \iota_3$ by the symbols ω, i, j, k, and consequently their polar planes $\omega\iota_0, \omega\iota_1, \omega\iota_2, \omega\iota_3$ by the symbols $1, \omega i, \omega j, \omega k$; but I am not aware that any useful results would follow from this imitation of Gauss's plane of numbers.

Rules of Multiplication in an Algebra of n units.

In general, if we consider an algebra of n units, $\iota_1, \iota_2, \ldots \iota_n$,
$\iota_r^2 = -1$, $\iota_r\iota_s = -\iota_s\iota_r$, a product of m linear factors will contain ter
are all of even order if m is even, and all of odd order if m is odd



Clifford

Clifford algebras

$V^n \qquad \langle -, - \rangle$ real euclidean vector space

$$C\ell(V) = \frac{\bigotimes V}{\langle \boldsymbol{v} \otimes \boldsymbol{v} + |\boldsymbol{v}|^2 \mathbf{1} \rangle}$$ filtered associative algebra

$$C\ell(V) \cong \Lambda V$$ (as vector spaces)

$$C\ell(V) = C\ell(V)_0 \oplus C\ell(V)_1$$

$$C\ell(V)_0 \cong \Lambda^{\text{even}} V \qquad C\ell(V)_1 \cong \Lambda^{\text{odd}} V$$

orthonormal frame $\quad \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$

$$\boldsymbol{e}_i \boldsymbol{e}_j + \boldsymbol{e}_j \boldsymbol{e}_i = -2\delta_{ij}\mathbf{1} \qquad C\ell\left(\mathbb{R}^n\right) =: C\ell_n$$

Examples:

$$C\ell_0 = \langle \mathbf{1} \rangle \cong \mathbb{R}$$

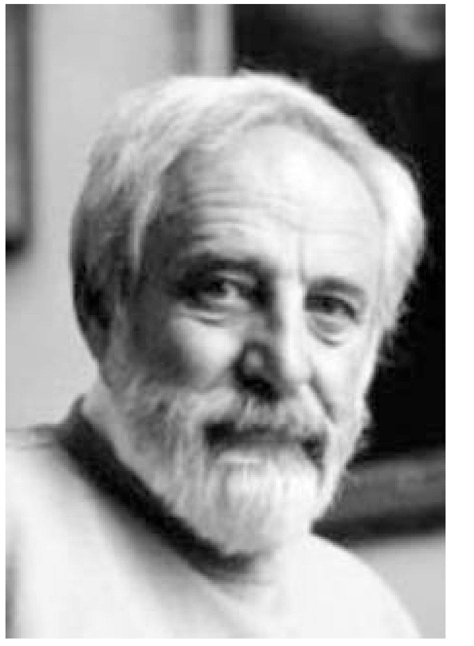
$$C\ell_1 = \left\langle \mathbf{1}, \boldsymbol{e}_1 \middle| \boldsymbol{e}_1^2 = -\mathbf{1} \right\rangle \cong \mathbb{C}$$

$$C\ell_2 = \left\langle \mathbf{1}, \boldsymbol{e}_1, \boldsymbol{e}_2 \middle| \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = -\mathbf{1}, \boldsymbol{e}_1 \boldsymbol{e}_2 = -\boldsymbol{e}_2 \boldsymbol{e}_1 \right\rangle \cong \mathbb{H}$$

Classification

n	$C\ell_n$
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$

Bott periodicity:



$$C\ell_{n+8} \cong C\ell_n \otimes \mathbb{R}(16)$$

e.g.,

$$C\ell_9 \cong \mathbb{C}(16)$$

$$C\ell_{16} \cong \mathbb{R}(256)$$

From this table one can read the type and dimension of the irreducible representations.

$C\ell_n$ has a **unique** irreducible representation if n is even and **two** if n is odd.

They are distinguished by the action of

$$\boldsymbol{e}_1\boldsymbol{e}_2\cdots\boldsymbol{e}_n$$

which is **central** for n odd.

Notation : $\mathfrak{M}_n$ or $\mathfrak{M}_n^{\pm}$ **Clifford modules**

$$\dim \mathfrak{M}_n = 2^{\lfloor n/2 \rfloor}$$

Spinor representatinos

$$\mathfrak{so}_n \to C\ell_n$$
$$\boldsymbol{e}_i \wedge \boldsymbol{e}_j \mapsto -\tfrac{1}{2}\boldsymbol{e}_i\boldsymbol{e}_j$$

exp ⇒

$$\mathrm{Spin}_n \subset C\ell_n$$

$$s \in \mathrm{Spin}_n, \quad \boldsymbol{v} \in \mathbb{R}^n \quad \Longrightarrow \quad s\boldsymbol{v}s^{-1} \in \mathbb{R}^n$$

which defines a 2-to-1 map $\mathrm{Spin}_n \to \mathrm{SO}_n$

with archetypical example $\mathrm{Spin}_3 \cong \mathrm{SU}_2 \subset \mathbb{H}$

$$\big\downarrow$$

$$\mathrm{SO}_3 \cong \mathrm{SO}(\mathrm{Im}\mathbb{H})$$

By restriction, every representation of $C\ell_n$ defines a representation of Spin_n :

$$C\ell_n \supset \mathrm{Spin}_n$$

$\mathfrak{M} = \Delta = \Delta_+ \oplus \Delta_-$	$\Delta_\pm$	**chiral spinors**
$\mathfrak{M}^\pm = \Delta$	Δ	**spinors**

One can read off the type of representation from

$$\mathrm{Spin}_n \subset (C\ell_n)_0 \cong C\ell_{n-1}$$

$$\dim \Delta = 2^{(n-1)/2} \qquad \dim \Delta_\pm = 2^{(n-2)/2}$$

Spinor inner product

$(-,-)$ bilinear form on Δ

$$(\varepsilon_1, \varepsilon_2) = \overline{(\varepsilon_2, \varepsilon_1)}$$

$$(\varepsilon_1, \boldsymbol{e}_i \cdot \varepsilon_2) = -(\boldsymbol{e}_i \cdot \varepsilon_1, \varepsilon_2) \qquad \forall \varepsilon_i \in \Delta$$

$$\Longrightarrow (\varepsilon_1, \boldsymbol{e}_i \boldsymbol{e}_j \cdot \varepsilon_2) = -(\boldsymbol{e}_i \boldsymbol{e}_j \cdot \varepsilon_1, \varepsilon_2)$$

which allows us to define $[-,-] : \Lambda^2 \Delta \to \mathbb{R}^n$

$$\langle [\varepsilon_1, \varepsilon_2], \boldsymbol{e}_i \rangle = (\varepsilon_1, \boldsymbol{e}_i \cdot \varepsilon_2)$$

Spin geometry

Spin manifolds

M^n differentiable manifold, orientable, spin

g riemannian metric

$$\begin{array}{ccccccc} \mathrm{GL}(M) & \longleftarrow & \mathrm{O}(M) & \overset{w_1 = 0}{\longleftarrow} & \mathrm{SO}(M) & \overset{w_2 = 0}{\longleftarrow} & \mathrm{Spin}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & & M & & M & & M \end{array}$$

$$\mathrm{GL}_n \hookleftarrow \mathrm{O}_n \hookleftarrow \mathrm{SO}_n \twoheadleftarrow \mathrm{Spin}_n$$

Possible $\mathrm{Spin}(M)$ are classified by $H^1(M;\mathbb{Z}/2)$.

e.g., $M = S^n \subset \mathbb{R}^{n+1}$

$$\mathrm{O}(M) = \mathrm{O}_{n+1}$$
$$\mathrm{SO}(M) = \mathrm{SO}_{n+1}$$
$$\mathrm{Spin}(M) = \mathrm{Spin}_{n+1}$$

$$S^n \cong \mathrm{O}_{n+1}/\mathrm{O}_n \cong \mathrm{SO}_{n+1}/\mathrm{SO}_n \cong \mathrm{Spin}_{n+1}/\mathrm{Spin}_n$$

$\pi_1(M) = \{1\} \implies$ unique spin structure

Spinor bundles

$$\begin{array}{c} C\ell(TM) \\ \downarrow \\ M \end{array}$$

Clifford bundle

$$C\ell(TM) \cong \Lambda TM$$

$$S(M) := \mathrm{Spin}(M) \times_{\mathrm{Spin}_n} \Delta$$

$$S(M)_\pm := \mathrm{Spin}(M) \times_{\mathrm{Spin}_n} \Delta_\pm$$

(chiral) spinor bundles

We make $S(M)$ into a $C\ell(TM)$-module.

The Levi-Cività connection allows us to differentiate spinors

$$\nabla : S(M) \to T^*M \otimes S(M)$$

which in turn allows us to define

parallel spinor $\nabla \varepsilon = 0$

Killing spinor $\nabla_X \varepsilon = \lambda X \cdot \varepsilon$

Killing constant → λ

If (M,g) admits

parallel spinors $\Rightarrow$ (M,g) is **Ricci-flat**

Killing spinors $\Rightarrow$ (M,g) is **Einstein**

$$R = 4\lambda^2 n(n-1)$$

$$\Longrightarrow \quad \lambda \in \mathbb{R} \cup i\mathbb{R}$$

Today we only consider **real** λ.

Killing spinors have their origin in **supergravity**.

The name stems from the fact that they are "**square roots**" of Killing vectors.

$$\left(V \in \Gamma(TM) \quad \text{is } \textbf{Killing} \text{ if} \quad \mathcal{L}_V g = 0 \right)$$

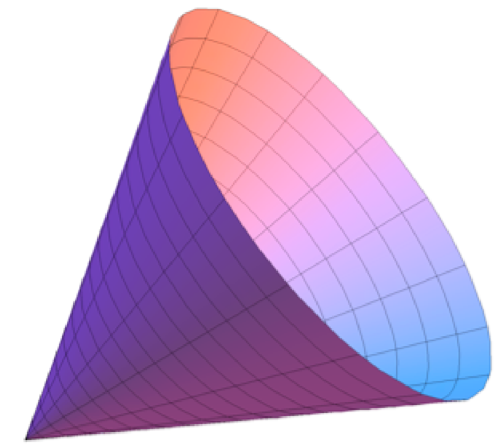
$\varepsilon_1, \varepsilon_2$ Killing ⇒ $[\varepsilon_1, \varepsilon_2]$ Killing

Which manifolds admit real Killing spinors?

Ch. Bär

(M, g)

$(\overline{M}, \overline{g})$ **metric cone**



$\overline{M} = \mathbb{R}^+ \times M$ $\qquad \overline{g} = dr^2 + r^2 g$

Killing spinors in (M, g)

$(\lambda = \pm\frac{1}{2})$

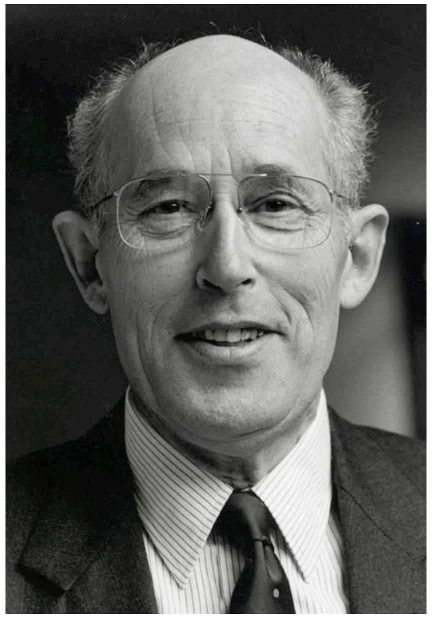
parallel spinors in the cone

More precisely...

If n is **odd**, Killing spinors are in one-to-one correspondence with **chiral** parallel spinors in the cone: the chirality is the **sign** of λ.

If n is **even**, Killing spinors with **both** signs of λ are in one-to-one correspondence with the parallel spinors in the cone, and the sign of λ enters in the relation between the Clifford bundles.

This reduces the problem to one (already solved) about the holonomy group of the cone.


M. Berger

n	Holonomy
n	SO_n
$2m$	U_m
$2m$	SU_m
$4m$	$\mathrm{Sp}_m \cdot \mathrm{Sp}_1$
$4m$	Sp_m
7	G_2
8	Spin_7


M. Wang

Or else the cone is flat and M is a sphere.

Killing superalgebra

Construction of the algebra

(M, g) riemannian spin manifold

$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$

$\mathfrak{k}_1 = \{$ Killing spinors $\}$

(with $\lambda = \frac{1}{2}$)

$\mathfrak{k}_0 = [\mathfrak{k}_1, \mathfrak{k}_1] \subset \{$ Killing vectors $\}$

$[-,-] : \Lambda^2 \mathfrak{k} \to \mathfrak{k}$?

$[-,-] : \Lambda^2 \mathfrak{k}_0 \to \mathfrak{k}_0$ ✓ $[-,-]$ of vector fields

$[-,-] : \Lambda^2 \mathfrak{k}_1 \to \mathfrak{k}_0$ ✓ $g([\varepsilon_1, \varepsilon_2], X) = (\varepsilon_1, X \cdot \varepsilon_2)$

$[-,-] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$? spinorial Lie derivative!

Kosmann

Lichnerowicz

$X \in \Gamma(TM)$ Killing $\Leftrightarrow$ $\mathcal{L}_X g = 0$

$\Leftrightarrow$ $A_X := Y \mapsto -\nabla_Y X$ $\Cap$ $\mathfrak{so}(TM)$

$\varrho : \mathfrak{so}(TM) \to \mathrm{End}\, S(M)$ spinor representation

$$\mathcal{L}_X := \nabla_X + \varrho(A_X)$$

spinorial Lie derivative

cf. $\mathfrak{L}_X Y = \nabla_X Y + A_X Y = \nabla_X Y - \nabla_Y X = [X, Y]$ ✓

Properties

$$\forall X, Y \in \mathfrak{k}_0, \quad Z \in \Gamma(TM), \quad \varepsilon \in \Gamma(S(M)), \quad f \in C^\infty(M)$$

$$\mathfrak{L}_X(Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathfrak{L}_X \varepsilon$$

$$\mathfrak{L}_X(f\varepsilon) = X(f)\varepsilon + f\mathfrak{L}_X \varepsilon$$

$$[\mathfrak{L}_X, \nabla_Z]\varepsilon = \nabla_{[X,Z]}\varepsilon$$

$$[\mathfrak{L}_X, \mathfrak{L}_Y]\varepsilon = \mathfrak{L}_{[X,Y]}\varepsilon$$

$\Rightarrow$

$$\forall \varepsilon \in \mathfrak{k}_1, X \in \mathfrak{k}_0$$

$$\mathcal{L}_X \varepsilon \in \mathfrak{k}_1$$

$$[-,-] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$$

$$[X, \varepsilon] := \mathcal{L}_X \varepsilon \quad ✓$$

The Jacobi identity

Jacobi: $\Lambda^3 \mathfrak{k} \longrightarrow \mathfrak{k}$

$$(X, Y, Z) \mapsto [X, [Y, Z]] - [[X, Y], Z] - [Y, [X, Z]]$$

4 components :

$\Lambda^3 \mathfrak{k}_0 \longrightarrow \mathfrak{k}_0$ ✓ Jacobi for vector fields

$\Lambda^2 \mathfrak{k}_0 \otimes \mathfrak{k}_1 \longrightarrow \mathfrak{k}_1$ ✓ $[\mathfrak{L}_X, \mathfrak{L}_Y]\varepsilon = \mathfrak{L}_{[X,Y]}\varepsilon$

$\mathfrak{k}_0 \otimes \Lambda^2 \mathfrak{k}_1 \longrightarrow \mathfrak{k}_0$ ✓ $\mathfrak{L}_X(Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathfrak{L}_X \varepsilon$

$\Lambda^3 \mathfrak{k}_1 \longrightarrow \mathfrak{k}_1$? but $\mathfrak{k}_0$ – equivariant

Exceptional spheres

$S^7 \subset \mathbb{R}^8$	$\mathfrak{k}_0 = \mathfrak{so}_8$	$\mathfrak{k}_1 = \Delta_+$	28 + 8 = 36	$\mathfrak{so}_9$
$S^8 \subset \mathbb{R}^9$	$\mathfrak{k}_0 = \mathfrak{so}_9$	$\mathfrak{k}_1 = \Delta$	36 +16 = 52	$\mathfrak{f}_4$
$S^{15} \subset \mathbb{R}^{16}$	$\mathfrak{k}_0 = \mathfrak{so}_{16}$	$\mathfrak{k}_1 = \Delta_+$	120+128 = 248	$\mathfrak{e}_8$

In all cases, the Jacobi identity follows from

$$\left(\mathfrak{k}_1 \otimes \Lambda^3 \mathfrak{k}_1^*\right)^{\mathfrak{k}_0} = \mathbf{0}$$

A sketch of the proof

Two observations:

1) The bijection between Killing spinors and parallel spinors in the cone is **equivariant** under the action of isometries.

⇨ Use the cone to calculate $\mathcal{L}_X \varepsilon$.

2) In the cone, $\mathcal{L}_X \varepsilon = \varrho(A_X)\varepsilon$ and since X is **linear**, the endomorphism A_X is constant.

⇨ It is the natural action on spinors.

We then compare with the known constructions.

Alternatively, we appeal to the classification of **riemannian symmetric spaces**.

These Lie algebras have the following form:

$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$	$\mathfrak{k}_0$	Lie algebra
	$\mathfrak{k}_1$	$\mathfrak{k}_0$-representation
	$(-,-)$	$\mathfrak{k}$-invariant inner product

$\Rightarrow$ K/K_0 symmetric space

Looking up the list, we find the following:

$$\mathrm{Spin}_9/\mathrm{Spin}_8$$

$$F_4/\mathrm{Spin}_9$$

$$E_8/\mathrm{Spin}_{16}$$

with the expected linear isotropy representations.

Open questions

- Other **exceptional** Lie algebras? **E6** follows from the 9-sphere by a similar construction; **E7** should follow from the 11-sphere, but this is still work in progress. **G2?**

- Are the Killing superalgebras of the Hopf spheres related?

- What structure in the 15-sphere has **E8** as **automorphisms**?