Why I like homogeneous manifolds

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- $H \to G$ is a principal H-bundle

$$\begin{array}{ccc} H & \to & G \\ & & \downarrow \\ & & M \end{array}$$

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- there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{Ad}(H)\text{-invariant} \\ \text{tensors on } \mathfrak{m}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}H\text{-invariant} \\ \text{tensors on } T_m M\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}G\text{-invariant} \\ \text{tensor fields on } M\end{matrix}\right\}$$

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- our (spatial) universe!

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The big questions

- What (topological) manifold is it? Is it compact? Simply-connected?
- We know it's expanding: but will it do so forever? or will it eventually contract?

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- riemannian: H compact, so G/H reductive

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- if $R = 0$ then (M, g) is either a Lie group with a bi-invariant metric or the round 7-sphere

  CARTAN–SCHOUTEN (1926); WOLF (1971-2)

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- Fubini–Study metric on complex projective space $\mathbb{C}\mathsf{P}^n = \mathsf{U}(n+1)/\mathsf{U}(n) \times \mathsf{U}(1)$

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- But on a homogeneous manifold, for $X, Y \in \mathfrak{m}$ Killing

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- curvature conditions (e.g., Einstein) $\implies$ **algebraic**

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But for more than century now, in Physics we work in *lorentzian* signature...

The birth of spacetime

Hermann Minkowski (1908)

Die Anschauungen über Raum und Zeit, die ich Ihnen entwicklen möchte, sind ***auf experimentell-physikalischem Boden erwachsen****. Darin liegt ihre Stärke. Ihre Tendenz ist eine radikale. Von Stund' an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbständigkeit bewahren.*

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> *The views of space and time that I wish to lay before you have* ***sprung from the soil of experimental physics****, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of both will retain an independent reality.*

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- Galilean group: translations, rotations, boosts

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- **Minkowski spacetime**: a very accurate model of the universe (at some scales)
- It is consistent with quantum theory (RQFT) and is *spectacularly* successful:

$$\left(\frac{g-2}{2}\right) = \begin{cases} 1\,159\,652\,182.79(7.71) \times 10^{-12} & \text{theory} \\ 1\,159\,652\,180.73(0.28) \times 10^{-12} & \text{experiment} \end{cases}$$

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GR mantra

Spacetime tells matter how to move, matter tells spacetime how to curve

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- $g, F, \ldots$ are subject to Einstein-like equations

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- Explicitly,

$$d \star F = \tfrac{1}{2} F \wedge F$$
$$\mathrm{Ric}(X, Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X, Y)|F|^2$$

together with $dF = 0$

Homogeneous supergravity backgrounds

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  - ...
- Many of them are homogeneous or of low cohomogeneity.

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- such spinor fields are called **Killing spinors**

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- a background is said to be ν-**BPS**, where $\nu = \frac{n}{32}$

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JMF+MEESSEN+PHILIP (2004)

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- It is called the **Killing superalgebra** of the supersymmetric background (M, g, F)

The homogeneity conjecture

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

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Theorem

Every ν-BPS background of eleven-dimensional supergravity with $\nu > \frac{3}{4}$ is homogeneous.

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What we proved is that the ideal $[\mathfrak{g}_1, \mathfrak{g}_1]$ of $\mathfrak{g}_0$ generated by the Killing spinors spans the tangent space to every point of M: local homogeneity

Generalisations

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Every ν-BPS background of type IIB supergravity with $\nu > \frac{3}{4}$ is homogeneous.
Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is homogeneous.

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JMF+Hackett-Jones+Moutsopoulos (2007)

The theorems actually suggest a stronger version of the conjecture: that the symmetries which are generated from the supersymmetries already act (locally) transitively.

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This would be **good** because

- the supergravity field equations for homogeneous backgrounds are algebraic and hence simpler to solve than PDEs
- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

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subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

▸ Skip technical details

Explicit expressions

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We raise and lower indices with γ_{ij}.

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Homogeneous Hodge/de Rham calculus

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  where $U_{ijk} = f_{i(jk)}$

Homogeneous Ricci curvature

Finally, the Ricci tensor for a homogeneous (reductive) manifold is given by

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It is now a matter of assembling these ingredients to write down the supergravity field equations in a homogeneous Ansatz.

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- Solve the equations!

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Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper. In particular, proper actions have compact stabilisers.

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If a semisimple Lie group acts transitively and non-properly on a lorentzian manifold (M, g), then (M, g) is locally isometric to the product of (anti) de Sitter spacetime and a riemannian homogeneous space.

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This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

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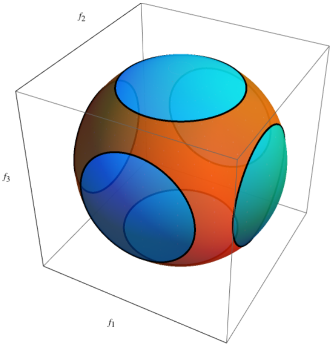
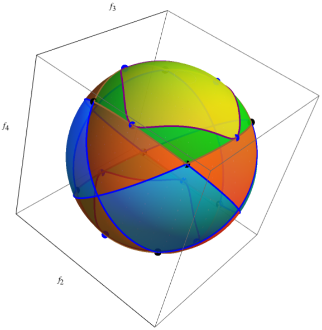
- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$
  JMF+Ungureanu (in preparation)

A moduli space of AdS_5 symmetric backgrounds



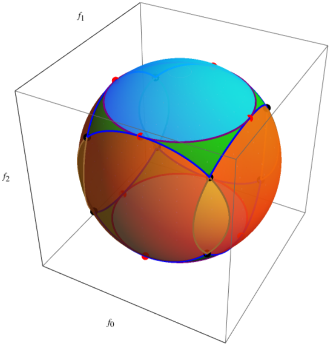
- $AdS_5 \times S^2 \times S^2 \times T^2$
- $AdS_5 \times S^2 \times S^2 \times H^2$
- $AdS_5 \times S^2 \times S^2 \times S^2$

A moduli space of AdS_3 symmetric backgrounds



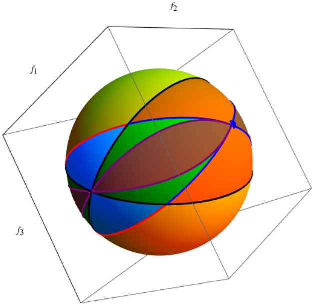
- $\mathrm{AdS}_3 \times S^2 \times T^6$ (blue dot)
- $\mathrm{AdS}_3 \times \mathbb{C}P^2 \times T^4$ (black dot)
- $\mathrm{AdS}_3 \times T^4 \times S^2 \times S^2$ (red line)
- $\mathrm{AdS}_3 \times \mathbb{C}P^2 \times S^2 \times T^2$ (brown line)
- $\mathrm{AdS}_3 \times \mathbb{C}P^2 \times H^2 \times T^2$ (blue line)
- $\mathrm{AdS}_3 \times \mathbb{C}P^2 \times H^2 \times H^2$ (cyan)
- $\mathrm{AdS}_3 \times \mathbb{C}H^2 \times S^2 \times S^2$ (yellow)
- $\mathrm{AdS}_3 \times \mathbb{C}P^2 \times S^2 \times H^2$ (orange)
- $\mathrm{AdS}_3 \times \mathbb{C}P^2 \times S^2 \times S^2$ (green)

A moduli space of AdS_2 symmetric backgrounds



- (red dot) $AdS_2 \times S^2 \times T^7$
- (black dot) $AdS_2 \times S^5 \times T^4$
- (crimson line) $AdS_2 \times T^5 \times S^2 \times S^2$
- (brown line) $AdS_2 \times S^5 \times H^2 \times T^2$
- (blue line) $AdS_2 \times S^5 \times S^2 \times T^2$
- (cyan square) $AdS_2 \times H^5 \times S^2 \times S^2$
- (yellow square) $AdS_2 \times S^5 \times H^2 \times H^2$
- (orange square) $AdS_2 \times S^5 \times S^2 \times H^2$
- (green square) $AdS_2 \times S^5 \times S^2 \times S^2$

And one final gratuitous pretty picture



- $\mathrm{AdS}_2 \times S^3 \times T^6$
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- $\mathrm{AdS}_2 \times \mathbb{C}P^2 \times H^3 \times T^2$
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Thank you for your attention