Homogeneous kinematical spacetimes (EMPG 23/1/2019)

Based on **1711.06111**
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Main motivation: Given the rôle played by Minkowski and (anti) de Sitter spacetimes in modern theoretical physics: GR, QFT and the gauge/gravity correspondence, and wanting to explore 'non-relativistic' theories, a natural question is what are the non-relativistic analogues of these spacetimes?

Outline

We begin by introducing the cast of characters. Starting from the de Sitter spacetimes and taking the flat limit we arrive at Minkowski spacetime. We then take "non-relativistic" and "ultra-relativistic" limits of these lorentzian symmetric spacetimes and arrive at galilean and carrollian* spacetimes. We observe that these are all symmetric homogeneous spaces of kinematical Lie groups.

Natural question: are these all the "kinematical spacetimes"?

We answer this question by classifying them. Interestingly, there are infinitely many!

I will summarise these results by progressively completing a picture showing the spacetimes and their interrelations.

* The Red Queen said to Alice: "Now, here, you see, it takes all the running you can do, to keep in the same place."

Dramatis personae

AdS_{D+1} (D+1)-dimensional anti de Sitter spacetime is the simply-connected (ie: universal) cover of the quadric

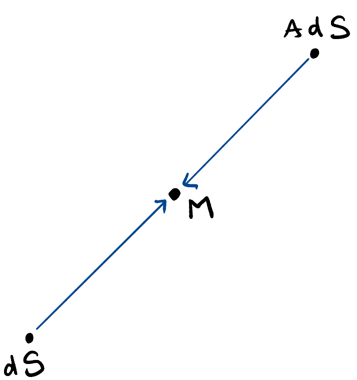
$$x_1^2 + x_2^2 + \cdots + x_D^2 - x_{D+1}^2 - x_{D+2}^2 = -R^2 \quad \text{in } \mathbb{R}^{D,2}$$

dS_{D+1} (D+1)-dimensional de Sitter spacetime is the universal cover of the quadric

$$x_1^2 + \cdots + x_{D+1}^2 - x_{D+2}^2 = R^2 \quad \text{in } \mathbb{R}^{D+1,1}$$

They are maximally symmetric spacetimes, so that the isometry group has dimension $(D+1)(D+2)/2$. The above embeddings show that the groups $SO(D,2)$ and $SO(D+1,1)$ act isometrically and transitively on the quadrics. The isometry groups of their universal covers are covering groups of these, but the Lie algebras remain isomorphic, so to simplify, we will concentrate on the Lie algebras of isometries, here $\underline{so}(D,2)$ and $\underline{so}(D+1,1)$.

The parameter R is a radius of curvature and the curvature behaves like $1/R^2$. The flat limit is taking $R \to \infty$ and this results in Minkowski spacetime $\mathbb{R}^{D,1}$, whose Lie algebra of isometries is the Poincaré algebra. The flat limit induces contractions $\underline{so}(D,2) \rightsquigarrow \mathfrak{p}$ and $\underline{so}(D+1,1) \rightsquigarrow \mathfrak{p}$. Conversely one can exhibit $\underline{so}(D,2)$ and $\underline{so}(D+1,1)$ as filtered deformations of $\mathfrak{p}$, which is $\mathbb{Z}$-graded.

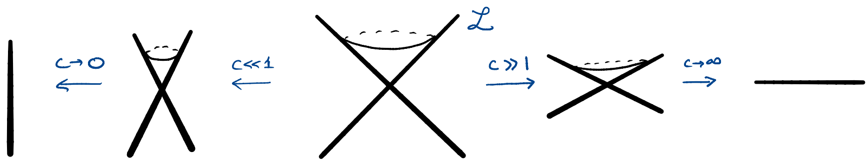


Minkowski spacetime is an affine space $\mathbb{A}^{D+1}$ with metric

$$dx_1^2 + \cdots + dx_D^2 - c^2\, dx_{D+1}^2$$

in affine coordinates and where we have introduced the speed of light. We may now consider limits where $c \to 0$ and $c \to \infty$.

To see what these limits to, we look at the lightcone $\mathcal{L}$ defined by $x_1^2 + \cdots + x_D^2 - c^2 x_{D+1}^2 = 0$.

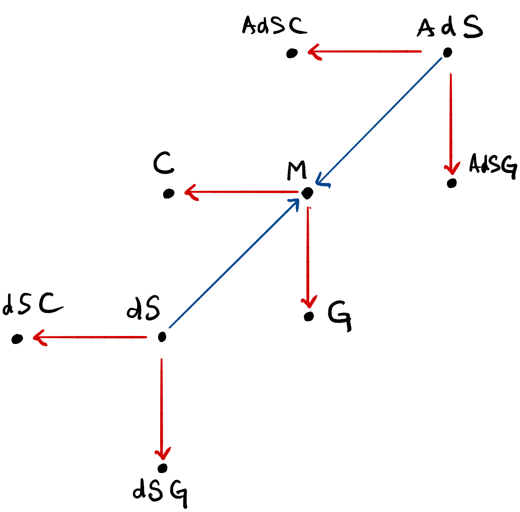


The $c \to \infty$ limit of Minkowski spacetime is the galilean spacetime, which is homogeneous under a contraction of the Poincaré group called the Galilean group. It's lie algebra is given sketchily by

$$[R,R]=R \qquad [H,B]=P$$
$$[R,B]=B \qquad \left(\begin{array}{l}[B,B]=R\\ [B,P]=H\end{array}\right) \leftarrow \text{missing from } \mathfrak{p}$$
$$[R,P]=P$$

The $c \to 0$ limit is the carrollian spacetime which is homogeneous under another contraction of the Poincaré group called the Carroll group, whose lie algebra takes the form

$$[R,R]=R \qquad [B,P]=H$$
$$[R,B]=B \qquad \left(\begin{array}{l}[B,B]=R\\ [H,B]=P\end{array}\right) \leftarrow \text{missing from } \mathfrak{p}$$
$$[R,P]=P$$



Not only Minkowski spacetime, but any lorentzian manifold has a lightcone in the tangent space at any point. So we can take the $c \to \infty$ and $c \to 0$ limits there as well. Doing so for (A)dS, we arrive at galilean and carrollian versions of (A)dS. These limits induce contractions of the isometry lie algebras, but this shows that (A)dSC, (A)dSG, C, G cannot admit invariant metrics, because by dimension they'd have to be maximally symmetric & hence (A)dS or M.

Nevertheless they are all symmetric homogeneous spaces of "kinematical Lie algebras".

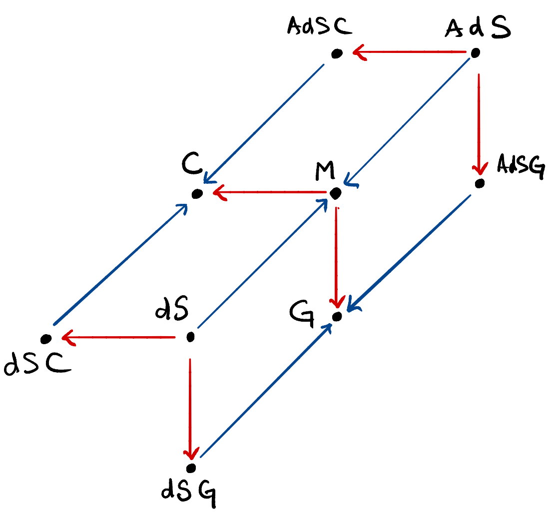
Definition A KLA (w/ D-dim'l space isotropy) is a real LA $\mathfrak{k}$ of dim $= \frac{1}{2}(D+1)(D+2)$ s.t. it has a subalgebra $\mathfrak{r} \cong \mathfrak{so}(D)$ and such that under $\mathfrak{r}$, $\mathfrak{k} = \mathfrak{r} \oplus 2V \oplus S$ ← scalar rep; V ← vector rep

If $\{R\}$ span $\mathfrak{r}$, $\{B,P\}$ $2V$ and H S,

$$[R,R] = R \quad [R,B] = B \quad [R,P] = P$$

(Simply-connected) kinematical spacetimes are of the form K/H where K is a kinematical Lie group and H is a closed subgroup whose Lie algebra consists of $\mathfrak{r} \oplus V$ ($\mathfrak{r}$: $\mathfrak{so}(D)$; V: vector rep ("boosts"))

Let $(\mathfrak{k},\mathfrak{h})$ be the infinitesimal description of a kinematical spacetime. As a vector space $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. If $\mathfrak{m}$ can be chosen so that $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ we say the spacetime is reductive. And if, in addition, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, the spacetime is symmetric. All the spaces in the diagram so far are symmetric. Symmetric spaces admit a canonical invariant connection w/out torsion. For riemannian (lorentzian, ...) symmetric spaces this is the LC connection.



(A)dSC, (A)dSG are not lorentzian, but the canonical connection has curvature and taking the flat limit gives C & G.

In fact carrollian spaces can be embedded as null hypersurfaces in lorentzian manifolds, and dSC, AdSC and C are null hypersurfaces in dS, AdS and M, respectively.

Question: what other (spatially isotropic) homogeneous kinematical spacetimes are there?

Equivalent algebraic question: classify (effective, geometrically realisable) pairs $(\mathfrak{k},\mathfrak{h})$ where $\mathfrak{k}$ is a KLA and $\mathfrak{h} = \{\text{rot'ns, boosts}\}$

<u>Step 1</u> Classify the KLAs $\mathfrak{k}$

D	0	1	2 ($\Lambda^2 V \to S$)	3 ($\Lambda^2 V \to V$)	>3
$\mathfrak{k}$	∃!	Bianchi	TA + JMF	Bacry Nuyts	JMF
		1898		'86	

In generic (D>3) dimension, apart from $\mathfrak{so}(D+2)$, $\mathfrak{so}(D+1,1)$, $\mathfrak{so}(D,2)$, $\mathfrak{p}, \mathfrak{e}, \mathfrak{g}, \mathfrak{c}, \mathfrak{n}_+, \mathfrak{n}_-$ (Newton–Hooke), we find

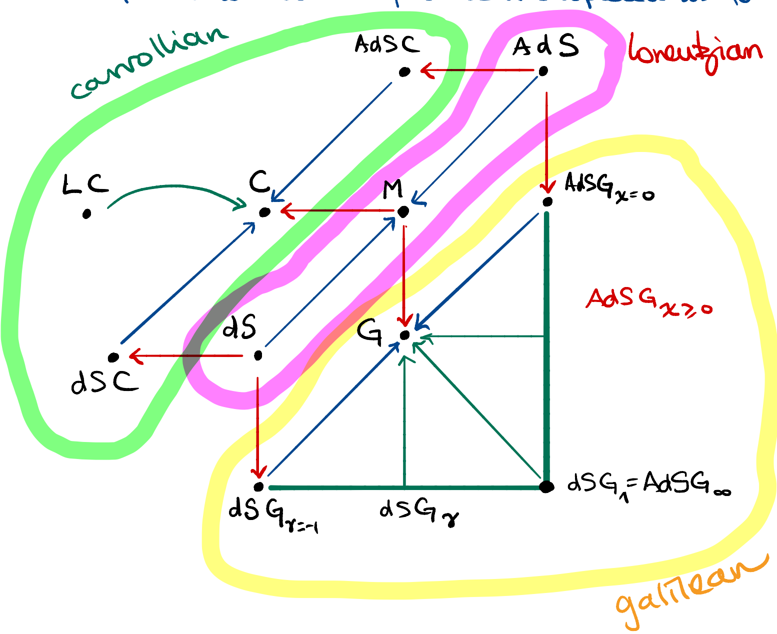
$[H,B] = \gamma B \quad [H,P] = P \qquad \gamma \in (-1,1] \quad (\gamma = -1 \text{ is } \mathfrak{n}_-)$

$[H,B] = B + P \quad [H,P] = P$

$[H,B] = \chi B + P \quad [H,P] = \chi P - B \qquad \chi > 0 \quad (\chi = 0 \text{ is } \mathfrak{n}_+)$

<u>Step 2</u> Classify the effective Lie pairs $(\mathfrak{k},\mathfrak{h})$ and select those which are geometrically realisable. (They all turn out to be.)

The upshot is that the picture is completed as follows:



* missing from the picture are S, H, E, the riemannian maximally symmetric spaces, as well as some aristotelian spacetimes corresponding to non-effective LPs.

LC is the lightcone in M^{D+2} and is a non-reductive carrollian spacetime

$AdSG_\chi$, dSG_γ are reductive galilean spacetimes: the canonical connection has torsion which vanishes for $AdSG_0$, dSG_{-1}.

Remarks ① All homogeneous kinematical spacetimes (except for some small number of exotic 2d spacetimes) fall into one of five classes:

- riemannian ($\exists$ an invariant riemannian metric)
- lorentzian ($\exists$ an invariant lorentzian metric)
- galilean ($\exists$ an invariant 1-form τ and an invariant degenerate cometric h with $h(\tau,-)=0$)
- carrollian ($\exists$ an invariant vector field ξ and an invariant degenerate metric g with $g(\xi,-)=0$.)
- aristotelian (simultaneously galilean/carrollian)

② Whereas the symmetry group of a lorentzian/riemannian structure is finite-dimensional (and bounded in dimension by $\frac{1}{2}(D+1)(D+2)$ for a $(D+1)$-dim'l space) the symmetry groups of galilean and/or carrollian structures can be ∞-dimensional.

For the galilean spacetimes in our classification, the Lie algebra of symmetries (vector fields ξ s.t. $\mathcal{L}_\xi \tau = \mathcal{L}_\xi h = 0$) is isomorphic to a semi-direct product $\mathbb{R} \ltimes C^\infty(\mathbb{R}, \mathfrak{e})$

↑ $\mathbb{R}$ action depends on spacetime ↑ D-dimensional euclidean algebra

For the carrollian spacetimes,

C	$\mathfrak{e} \ltimes C^\infty(\mathbb{E}^D)$
dS C	$\mathfrak{so}(D+1) \ltimes C^\infty(S^D)$
AdS C	$\mathfrak{so}(D,1) \ltimes C^\infty(H^D)$

whereas for LC it is finite-dim'l and $\cong \mathfrak{so}(D+1,1)$.