Time-dependent string backgrounds via quotients

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Based on work in collaboration with Joan Simón (Pennsylvania)

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Motivation

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- supersymmetric Clifford–Klein space form problem

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- we are interested in the orbit space M/Γ

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  - ⋆ M supersymmetric, but M/Γ_L breaking all supersymmetry

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i.e., projectivised adjoint orbits of $\mathfrak{g}$

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which we need to put in normal form.

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$$\mu(x) = \Big(\big(x^2 + \beta^2 + \varphi^2\big)^2 - 4\beta^2 x^2\Big)^n$$

Strategy

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- for each $\mu(x)$

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Elementary lorentzian blocks

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Play LEGO with the elementary blocks!

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to get rid of component of τ in the image of $[\lambda, -]$

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$\Longrightarrow$ half-BPS ten-dimensional nullbrane

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- some arithmetic issues remain

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In string/M-theory this cannot be the end of the story.

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$\mathbb{CP}^2$ is not even spin!

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- equivalently, the action of $\xi = \xi_X$ on spinors has integral weights

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End of first part.

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  [hep-th/9902066]

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- again the problem reduces to one of flat spaces!

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- one-parameter subgroups $\leftrightarrow$ projectivised adjoint orbits of $\mathfrak{so}(2,p)$ under $SO(2,p)$

Normal forms for $\mathfrak{so}(2,p)$

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We play LEGO again

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We play LEGO again but with a bigger set!

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$$B^{(0,2)}(\varphi) = B^{(2,0)}(\varphi) = \begin{bmatrix} 0 & \varphi \\ -\varphi & 0 \end{bmatrix}$$

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- $(1,1)$, $\mu(x) = x^2 - \beta^2$

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But there are also new ones:

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$$B_{\pm}^{(2,2)} = \begin{bmatrix} 0 & \mp 1 & 1 & 0 \\ \pm 1 & 0 & 0 & \mp 1 \\ -1 & 0 & 0 & 1 \\ 0 & \pm 1 & -1 & 0 \end{bmatrix}$$

- $(2, 2)$

- $(2,2)$, $\mu(x) = (x^2 - \beta^2)^2$

- $(2,2)$, $\mu(x) = (x^2 - \beta^2)^2$, deformation of $B_{\pm}^{(2,2)}$ by a (anti)selfdual boost

- $(2,2)$, $\mu(x)=(x^2-\beta^2)^2$, deformation of $B_{\pm}^{(2,2)}$ by a (anti)selfdual boost

$$B_{\pm}^{(2,2)}(\beta>0)$$

- $(2,2)$, $\mu(x) = (x^2 - \beta^2)^2$, deformation of $B_\pm^{(2,2)}$ by a (anti)selfdual boost

$$B_\pm^{(2,2)}(\beta > 0) = \begin{bmatrix} 0 & \mp 1 & 1 & -\beta \\ \pm 1 & 0 & \pm\beta & \mp 1 \\ -1 & \mp\beta & 0 & 1 \\ \beta & \pm 1 & -1 & 0 \end{bmatrix}$$

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The associated discrete quotient of AdS_3

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The associated discrete quotient of AdS_3 yields the extremal BTZ black hole

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The associated discrete quotient of AdS_3 yields the extremal BTZ black hole; the non-extremal black hole

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The associated discrete quotient of AdS_3 yields the extremal BTZ black hole; the non-extremal black hole is obtained from $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2)$

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$$B_{\pm}^{(2,2)}(\beta > 0) = \begin{bmatrix} 0 & \mp 1 & 1 & -\beta \\ \pm 1 & 0 & \pm\beta & \mp 1 \\ -1 & \mp\beta & 0 & 1 \\ \beta & \pm 1 & -1 & 0 \end{bmatrix}$$

The associated discrete quotient of AdS_3 yields the extremal BTZ black hole; the non-extremal black hole is obtained from $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2)$, for $|\beta_1| \neq |\beta_2|$

- $(2, 2)$

- $(2,2),\ \mu(x) = (x^2 + \varphi^2)^2$

- $(2,2)$, $\mu(x)=(x^2+\varphi^2)^2$, deformation of $B_{\pm}^{(2,2)}$ by a (anti)self-dual rotation

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$$B_{\pm}^{(2,2)}(\varphi)$$

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$$B_{\pm}^{(2,2)}(\varphi) = \begin{bmatrix} 0 & \mp 1 \pm \varphi & 1 & 0 \\ \pm 1 \mp \varphi & 0 & 0 & \mp 1 \\ -1 & 0 & 0 & 1+\varphi \\ 0 & \pm 1 & -1-\varphi & 0 \end{bmatrix}$$

- $(2, 2)$

- $(2,2)$, $\mu(x) = (x^2 + \beta^2 + \varphi^2) - 4\beta^2 x^2$

- $(2,2)$, $\mu(x) = (x^2 + \beta^2 + \varphi^2) - 4\beta^2 x^2$, self-dual boost + antiself-dual rotation

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$$B_{\pm}^{(2,2)}(\beta>0,\varphi>0) = \begin{bmatrix} 0 & \pm\varphi & 0 & -\beta \\ \mp\varphi & 0 & \pm\beta & 0 \\ 0 & \mp\beta & 0 & -\varphi \\ \beta & 0 & \varphi & 0 \end{bmatrix}$$

- $(2, 3)$

- $(2,3),\ \mu(x) = x^5$

- $(2,3)$, $\mu(x) = x^5$, deformation of $B_+^{(2,2)}$ by a null rotation in a perpendicular direction

- $(2,3)$, $\mu(x) = x^5$, deformation of $B_+^{(2,2)}$ by a null rotation in a perpendicular direction

$$B^{(2,3)}$$

- $(2,3)$, $\mu(x) = x^5$, deformation of $B_+^{(2,2)}$ by a null rotation in a perpendicular direction

$$B^{(2,3)} = \begin{bmatrix} 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- $(2,4)$

- $(2, 4), \ \mu(x) = (x^2 + \varphi^2)^3$

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$$B_{\pm}^{(2,4)}(\varphi)$$

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$$B_{\pm}^{(2,4)}(\varphi) = \begin{bmatrix} 0 & \mp\varphi & 0 & 0 & -1 & 0 \\ \pm\varphi & 0 & 0 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & \varphi & -1 & 0 \\ 0 & 0 & -\varphi & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & \varphi \\ 0 & \pm 1 & 0 & 1 & -\varphi & 0 \end{bmatrix}$$

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- and that's all!

Causal properties

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- Killing vectors on $\mathrm{AdS}_{1+p} \times S^q$ decompose

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whose norms add

$$\|\xi\|^2 = \|\xi_A\|^2 + \|\xi_S\|^2$$

- S^q is compact

- S^q is compact $\Longrightarrow$

$$R^2 M^2 \geq \|\xi_S\|^2$$

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- S^q is compact $\Longrightarrow$

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- ξ can be everywhere spacelike on $\mathrm{AdS}_{1+p} \times S^{2k+1}$, even if ξ_A is not spacelike everywhere

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- ξ can be everywhere spacelike on $\mathrm{AdS}_{1+p} \times S^{2k+1}$, even if ξ_A is not spacelike everywhere, provided that $\|\xi_A\|^2$ is bounded below

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- it is convenient to distinguish Killing vectors according to norm

- everywhere non-negative norm

- everywhere non-negative norm:
  - ⋆ $\oplus_i B^{(0,2)}(\varphi_i)$

- everywhere non-negative norm:
  - ⋆ $\oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2) \oplus_i B^{(0,2)}(\varphi_i)$

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  - ⋆ $\oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2) \oplus_i B^{(0,2)}(\varphi_i)$, if $|\beta_1| = |\beta_2|$

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  - ⋆ $B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B^{(1,2)} \oplus B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B_{\pm}^{(2,2)} \oplus_i B^{(0,2)}(\varphi_i)$

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  - ⋆ $B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B^{(1,2)} \oplus B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B^{(2,2)}_{\pm} \oplus_i B^{(0,2)}(\varphi_i)$
- norm bounded below

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  - ⋆ $B_{\pm}^{(2,2)} \oplus_i B^{(0,2)}(\varphi_i)$
- norm bounded below:
  - ⋆ $B^{(2,0)}(\varphi) \oplus_i B^{(0,2)}(\varphi_i)$

- everywhere non-negative norm:
  - ⋆ $\oplus_i B^{(0,2)}(\varphi_i)$
  - ⋆ $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2) \oplus_i B^{(0,2)}(\varphi_i)$, if $|\beta_1| = |\beta_2|$
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  - ⋆ $B_{\pm}^{(2,2)} \oplus_i B^{(0,2)}(\varphi_i)$
- norm bounded below:
  - ⋆ $B^{(2,0)}(\varphi) \oplus_i B^{(0,2)}(\varphi_i)$, if p is even

- everywhere non-negative norm:
  - $\star$ $\oplus_i B^{(0,2)}(\varphi_i)$
  - $\star$ $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2) \oplus_i B^{(0,2)}(\varphi_i)$, if $|\beta_1| = |\beta_2|$
  - $\star$ $B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
  - $\star$ $B^{(1,2)} \oplus B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
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- norm bounded below:
  - $\star$ $B^{(2,0)}(\varphi) \oplus_i B^{(0,2)}(\varphi_i)$, if p is even and $|\varphi_i| \geq \varphi > 0$ for all i

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- norm bounded below:
  - $\star$ $B^{(2,0)}(\varphi) \oplus_i B^{(0,2)}(\varphi_i)$, if p is even and $|\varphi_i| \geq \varphi > 0$ for all i
  - $\star$ $B_{\pm}^{(2,2)}(\varphi) \oplus_i B^{(0,2)}(\varphi_i)$

- everywhere non-negative norm:
  - $\star$ $\oplus_i B^{(0,2)}(\varphi_i)$
  - $\star$ $B^{(1,1)}(\beta_1) \oplus B^{(1,1)}(\beta_2) \oplus_i B^{(0,2)}(\varphi_i)$, if $|\beta_1| = |\beta_2|$
  - $\star$ $B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$
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- arbitrarily negative norm: the rest!

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Some of these give rise to higher-dimensional BTZ-like black holes: quotient only a part of AdS and check that the boundary thus introduced lies behind a horizon.

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$$\gamma = \exp(LX)$$

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- yields Freund–Rubin background of IIB

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Thank you.