Equivariant cohomology & the moment map

G connected, compact Lie group, Lie algebra $\mathfrak{g}$

M C^∞ mfd and $G \circlearrowright M \Rightarrow \mathfrak{g} \to \mathfrak{X}(M)$ LA hom.

$$X \longmapsto \xi_X$$

The de Rham complex $(\Omega^\cdot(M), d)$ admits $\mathfrak{g}$-action:

$X \in \mathfrak{g}$

$$\left.\begin{aligned} \iota_X &:= \iota_{\xi_X} : \Omega^p(M) \longrightarrow \Omega^{p-1}(M) \\ \text{and} \quad \mathcal{L}_X &:= \mathcal{L}_{\xi_X} = d\iota_X + \iota_X d \end{aligned}\right\} \quad \Omega^\cdot(M) \text{ is a } \mathfrak{g}\text{-DGA}$$

If G acts freely on M, we get a ppal G-bundle $G \hookrightarrow M \xrightarrow{\pi} M/G$ and π^* embeds $\Omega^\cdot(M/G)$ as the subcomplex of $\Omega^\cdot(M)$ consisting of basic forms:

$$\alpha \in \Omega^p(M) \text{ is basic} \iff \begin{cases} \iota_X \alpha = 0 \quad \forall X \in \mathfrak{g} & \text{(horizontal)} \\ \mathcal{L}_X \alpha = 0 \quad \forall X \in \mathfrak{g} & \text{(invariant)} \end{cases}$$

What if the action is not free? M/G need not be a manifold. How do we make sense of $H^\cdot(M/G)$? The idea is to modify M as little as possible but admitting a free action. If G acts freely on E, it will act freely on $E \times M$, and if in addition E is contractible $E \times M$ is homotopy equivalent to M. If so we may define the homotopy quotient $M_G := (E \times M)/G$, whose de Rham cohomology is the cohomology of the basic forms on $E \times M$. It is called the G-equivariant cohomology of M and denoted $H^\cdot_G(M)$. If the G-action on M is free, $H_G(M) \cong H(M/G)$.

The problem with this idea is that E is an inductive limit of manifolds & hence infinite-dimensional.
eg: $G = U(1)$ E is an inductive limit of S^{2n+1} as $n \to \infty$.
In general, $E = EG \to BG$ is the total space of the universal ppal G-bundle over the classifying space BG, so called because ppal G-bundles over M are classified by $[M, BG]$. Not wanting to do analysis on E, our strategy is to model $\Omega^\cdot(E)$ algebraically.

The Weil algebra $W(\mathfrak{g}) := \Lambda^\cdot \mathfrak{g}^* \otimes \mathfrak{S}^\cdot \mathfrak{g}^*$ is a $\mathfrak{g}$-DGA defined as follows. Let (e_a) be a basis for $\mathfrak{g}$ & (θ^a) the canonical dual basis for $\mathfrak{g}^*$.
Let $[e_a, e_b] = f_{ab}{}^c e_c$. $\mathfrak{g}^* \xrightarrow{\cong} \Lambda^1 \mathfrak{g}^*$ sends θ^a to A^a and $\mathfrak{g}^* \xrightarrow{\cong} \mathfrak{S}^1 \mathfrak{g}^*$ sends θ^a to F^a. We let $\deg A^a = 1$ and $\deg F^a = 2$. The differential is

$$dA^a = -\tfrac{1}{2} f_{bc}{}^a A^b A^c + F^a \qquad \& \qquad dF^a = f_{bc}^{\ a} F^b A^c$$

and extending to $W(\mathfrak{g})$ as a deg 1 odd derivation.

Exercise Show $d^2 = 0$ and that $H^n_d \cong \begin{cases} \mathbb{R} & n = 0 \\ 0 & n > 0 \end{cases}$

On $W(\mathfrak{g})$ define for $X = X^a e_a \in \mathfrak{g}$,

$$\imath_X A^a = X^a \qquad \imath_X F^a = 0$$

and $\mathcal{L}_X = d\imath_X + \imath_X d$. This turns $W(\mathfrak{g})$ into a $\mathfrak{g}$-DGA.

Remark $H^\cdot_{basic}(W(\mathfrak{g})) \cong (\mathfrak{S}^\cdot \mathfrak{g}^*)^{\mathfrak{g}}$.

So $W(\mathfrak{g})$ & $\Omega^\cdot(M)$ are $\mathfrak{g}$-DGAs and hence so is $W^\cdot(\mathfrak{g}) \otimes \Omega^\cdot(M)$. The equivariant de Rham model for $H^\cdot_G(M)$ is the basic subcomplex $\{W(\mathfrak{g}) \otimes \Omega(M)\}_{basic}$.
There is another algebraic model for $H^\cdot_G(M)$ which is more convenient for calculations.

Let's define **minimal coupling**

$$K : \Omega^{\cdot}(M) \longrightarrow \Lambda^{\cdot} g^* \otimes \Omega^{\cdot}(M)$$

$$dx^i \longmapsto dx^i - A^a \xi^i_a \qquad \text{where} \quad \xi_{e_a} = \xi^i_a \frac{\partial}{\partial x^i}$$

relative to local coordinates. It is well-defined.

Now let $\varepsilon : \Lambda^{\cdot} g^* \otimes \Omega^{\cdot}(M) \longrightarrow \Omega^{\cdot}(M)$ be defined by simply putting $A^a = 0$. Then $\varepsilon \circ K = id_{\Omega^{\cdot}(M)}$ and hence $K \circ \varepsilon$ is idempotent. Since

$$\iota_X K(dx^i) = \iota_X(dx^i - A^a \xi^i_a) = \xi^i_X - X^a \xi^i_a = 0$$

one shows that $K \circ \varepsilon$ is the projector onto horizonal forms in $\Lambda^{\cdot} g^* \otimes \Omega^{\cdot}(M)$: we have isos

$$\Omega^{\cdot}(M) \underset{\varepsilon}{\overset{K}{\rightleftarrows}} \left(\Lambda^{\cdot} g^* \otimes \Omega^{\cdot}(M)\right)_{horz}$$

Since F^a are horizontal, we get

$$S^{\cdot} g^* \otimes \Omega^{\cdot}(M) \underset{\varepsilon}{\overset{K}{\rightleftarrows}} \left(W^{\cdot}(g) \otimes \Omega^{\cdot}(M)\right)_{horz}$$

and since ε, K are g-maps,

$$\Omega^{\cdot}_C := \left(S^{\cdot} g^* \otimes \Omega^{\cdot}(M)\right)^g \underset{\varepsilon}{\overset{K}{\rightleftarrows}} \left(W^{\cdot}(g) \otimes \Omega^{\cdot}(M)\right)_{basic}$$

We transport the differential: $d_C := \varepsilon \circ d \circ K$, ie:

$$d_C F^a = \varepsilon(dF^a) = 0 \qquad d_C \omega = \varepsilon\, d\,(\omega - A^a \iota_a \omega + \cdots)$$

$$= \varepsilon(d\omega - F^a \iota_a \omega + \cdots)$$

$$= d\omega - F^a \iota_a \omega$$

$(\Omega^{\cdot}_C, d_C)$ is the **Cartan model** for $H^{\cdot}_G(M)$.

What about moment maps? Let (M,ω) be symplectic and G acts on M preserving ω:

$$\mathcal{L}_X \omega = 0 \iff d\iota_X\omega = 0 \quad \text{since } d\omega = 0.$$

Let $\iota_X\omega = d\phi_X$ $\exists \phi_X \in C^\infty(M)$. The functions ϕ_X are defined up to constants. If these can be chosen so that

$$\mathcal{L}_X \phi_Y = \phi_{[X,Y]} \quad \forall X,Y \in \mathfrak{g}$$

then $\mathfrak{g} \to C^\infty(M)$ sending $X \longmapsto \phi_X$ is a Lie algebra homomorphism and $\mu : M \to \mathfrak{g}^*$ with $\mu(p)(X) = \phi_X(p)$ is a moment map.
The G action is said to be hamiltonian.

Since ω is $\mathfrak{g}$-invariant, $\omega \in \Omega^2_C$. We say that ω admits an equivariant closed extension if $\exists\ \hat{\omega} \in \Omega^2_C$, $d_C\hat{\omega} = 0$ and $\hat{\omega}|_{F=0} = \omega$.

Theorem (Atiyah-Bott)

A symplectic G-action on (M,ω) is hamiltonian iff ω admits an equivariant closed extension.

Proof $\hat{\omega} \in \Omega^2_C$

$$\hat{\omega}|_{F=0} = \omega \iff \hat{\omega} = \omega + \varphi_a F^a \quad \exists \varphi_a \in C^\infty(M)$$

$$\left.\begin{aligned} \mathcal{L}_b \hat{\omega} = 0 &\iff \mathcal{L}_b \varphi_c = f_{bc}{}^a \varphi_a \\ d_C \hat{\omega} = 0 &\iff d\varphi_a = \iota_a \omega \end{aligned}\right\} \quad \phi_X := X^a \varphi_a \text{ is a moment map. } \blacksquare$$

Exercise Show $H_d^n(W(\mathfrak{g})) \cong \begin{cases} \mathbb{R}, & n=0 \\ 0, & n>0 \end{cases}$

Define $k : W^p(\mathfrak{g}) \to W^{p-1}(\mathfrak{g})$ by $k(A^a)=0$ $k(F^a)=A^a$ and extending as an antiderivation.

$$(kd+dk)(A^a) = k\left(F^a - \tfrac{1}{2} f^a_{bc}A^bA^c\right) = A^a$$

$$\begin{aligned}(kd+dk)(F^a) &= k\left(f^a_{bc}F^bA^c\right) + dA^a \\ &= f^a_{bc}A^bA^c + dA^a \\ &= F^a - \tfrac{1}{2} f_{bc}{}^a A^bA^c + f^a_{bc}A^bA^c \\ &= F^a + \tfrac{1}{2} f_{bc}{}^a A^bA^c\end{aligned}$$

Filter $W(\mathfrak{g})$ by fdeg $F^a = 1$, fdeg $A^a = 0$.

Let $m \geq 1$ and $\alpha \in W^m(\mathfrak{g})$. Then $(dk+kd)\alpha \in W^m(\mathfrak{g})$.
Suppose fdeg $\alpha = q \geq 0$ (and $\alpha \neq 0$).
The fdeg of $\boxed{\beta = \alpha - \dfrac{1}{m-q}(dk\alpha + kd\alpha)}$ is $\leq q-1$.

If $d\alpha = 0$, β is cohomologous to α but has strictly smaller fdeg. Iterating this, we arrive at 0 cohomologous to α.

$$\alpha \in W^m(\mathfrak{g}) = \bigoplus_{r+2s=m} \Lambda^r\mathfrak{g}^* \otimes S^s\mathfrak{g}^*$$

$\&\ s \leq q \quad \Rightarrow \quad m-q \geq q$

$\therefore\ m-q=0 \iff q=0, m=0 \Rightarrow\Leftarrow$