Supersymmetric space forms

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Based on work in collaboration with

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- George Papadopoulos (King's College, London)
  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170`

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  - `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - `math.AG/0211170`
- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
  - `in preparation`

A geometric motivation

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Equivalently, they are parallel sections of the bundle

$$\mathcal{E}(M) = TM \oplus \mathfrak{so}(TM)$$

relative to the connection

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi - A(X) \\ \nabla_X A - R(X, \xi) \end{pmatrix}$$

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Maximal symmetry $\implies \mathcal{E}(M)$ is flat
$\implies M$ has constant sectional curvature κ.

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The flat and spherical cases are solved (culminating in the work of Wolf in the 1970s), but the hyperbolic and lorentzian cases remain largely open despite many partial results.

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In this talk I will report on the solution of the local problem in several supergravity theories.

Note: A maximally supersymmetric supergravity background will be abbreviated *vacuum*.

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Extremals of this action—namely, Ricci-flat manifolds—are called *spacetimes*.

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The maximally symmetric solutions are the lorentzian space forms: smooth discrete quotients of Minkowski space and (the universal covers of) de Sitter and anti de Sitter spaces, depending on the sign of λ.

… and Supergravity

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What is so interesting about this action?

It is *invariant*

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Also this really only works as written in four dimensions. In other dimensions supergravity theories might have *other fields* and both the action and supersymmetry transformations become *more complicated*. **But** supergravity theories are *uniquely* determined by representation theory (of relevant superalgebras).

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N=2$			$N=1$			
8	$N=2$			$N=1$			
7	$N=4$			$N=2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
4	$N=8$	$N=6$	$N=5$	$N=4$	$N=3$	$N=2$	$N=1$

[Van Proeyen, hep-th/0301005]

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- classify vacua of theories at the top of each column, and
- investigate their possible Kaluza–Klein reductions.

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defined by the supersymmetry variation of the gravitino:

$$\delta_\varepsilon \Psi = D\varepsilon$$

(putting all fermions to zero)

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Typically $A = 0$ sets some fields to zero, and the flatness of D constrains the geometry and any remaining fields. The strategy is therefore to study the flatness equations for D.

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- $D = 6$ $(1, 0)$, $(2, 0)$ [Chamseddine–FO–Sabra]

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is a real 8-dimensional representation of $\mathrm{Spin}(1,5) \times \mathrm{Sp}(1)$.

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The solution to this problem is known.

Lorentzian Lie algebras

Which Lie algebras have an invariant metric?

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But there is a more general construction.

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- since $\mathfrak{h}$ preserves the metric on $\mathfrak{g}$, there is a linear map

$$\mathfrak{h} \longrightarrow \Lambda^2\mathfrak{g}$$

whose dual map

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This construction is due to Medina and Revoy.

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Every metric Lie algebra is obtained as an orthogonal direct sum of indecomposables.

[See also FO–Stanciu `hep-th/9506152`]

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(Any semisimple factors in $\mathfrak{a}$ factor out of the double extension. [FO–Stanciu hep-th/9402035])

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The third case is a six-dimensional version of the Nappi-Witten spacetime, NW_6, discovered by Meessen. [Meessen `hep-th/0111031`]

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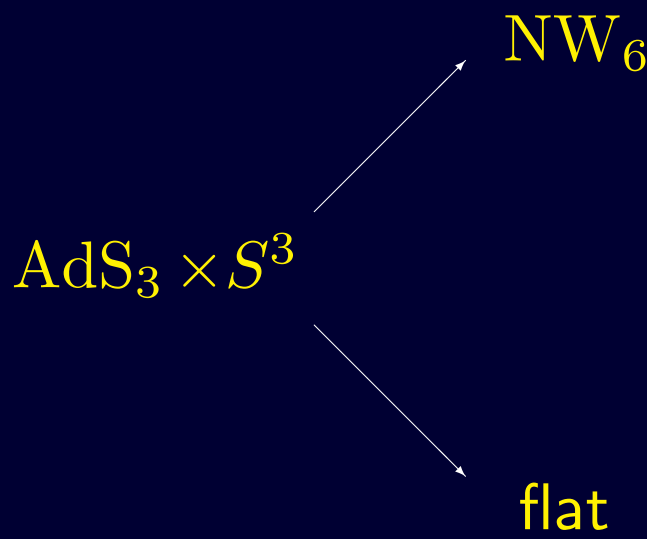

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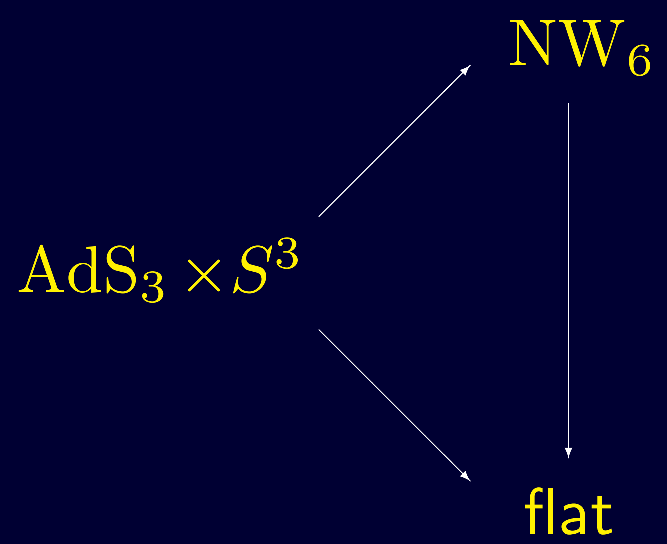

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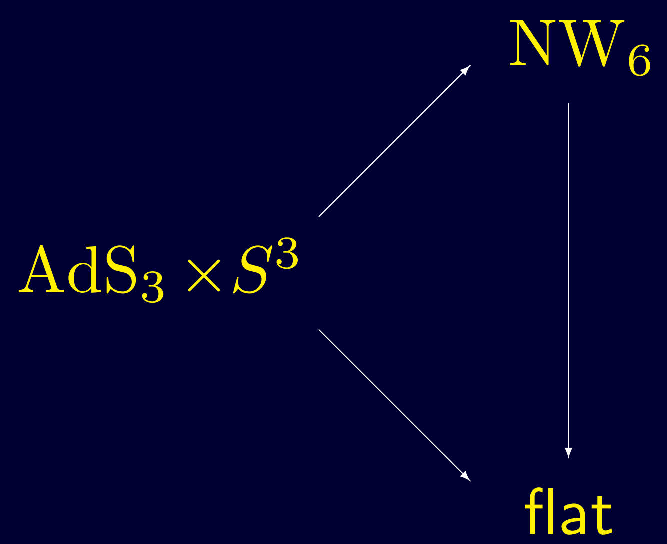

[Stanciu–FO hep-th/0303212]

[Back]

Vacua of eleven-dimensional supergravity

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[Cahen–Wallach (1970)]

$$g = 2dx^+dx^- - \tfrac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2 \right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

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KG

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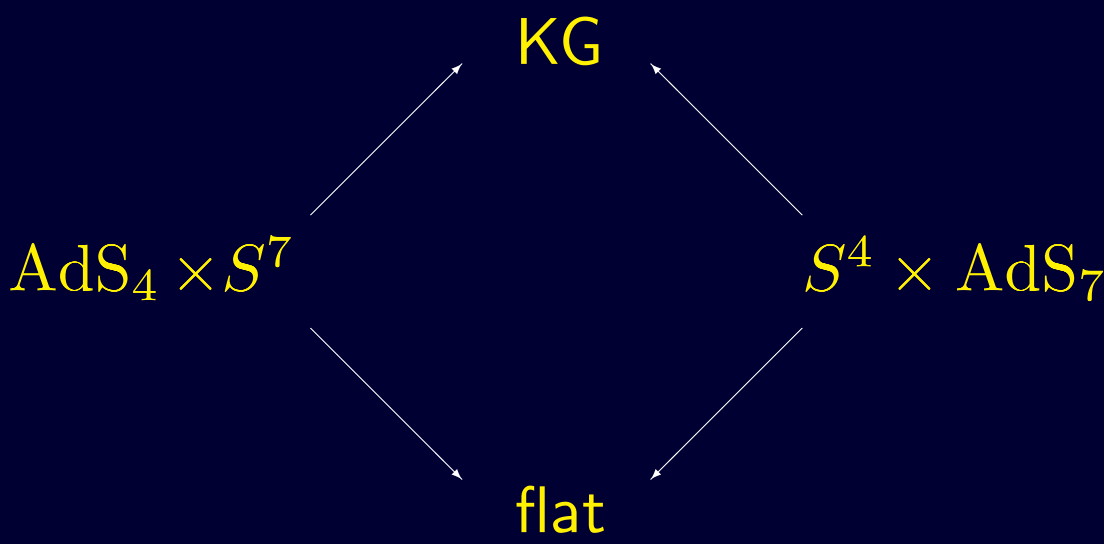

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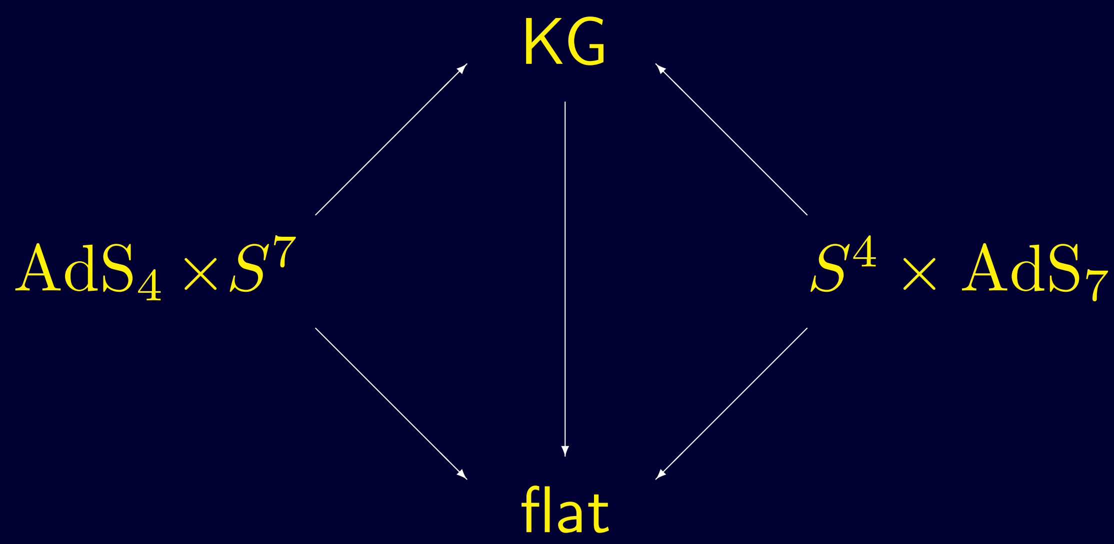

[Back]

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(Unfortunate notation: a 2-Lie algebra is a Lie algebra.)

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If F is totally antisymmetric then $\langle -, - \rangle$ is an *invariant metric*.

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$$F = G + \star G \qquad \text{where} \qquad G = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5$$

[FO–Papadopoulos math.AG/0211170]

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If $F = 0$ we recover the flat vacuum. Otherwise there are two possibilities: either one plane is lorentzian and the other euclidean, or both planes are null.

Knowing F we can determine $\mathrm{Riem}(g)$ and hence g, and one finds the following vacua (up to local isometry):

- F non-degenerate case: a one-parameter ($R > 0$) family of vacua

$$\mathrm{AdS}_5(-R) \times S^5(R) \qquad F = \sqrt{\frac{4R}{5}}\left(\mathrm{dvol}(\mathrm{AdS}_5) + \mathrm{dvol}(S^5)\right)$$

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[Blau–FO–Hull–Papadopoulos hep-th/0110242]

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The wave is isometric to a solvable lorentzian Lie group

[Stanciu–FO hep-th/0303212]

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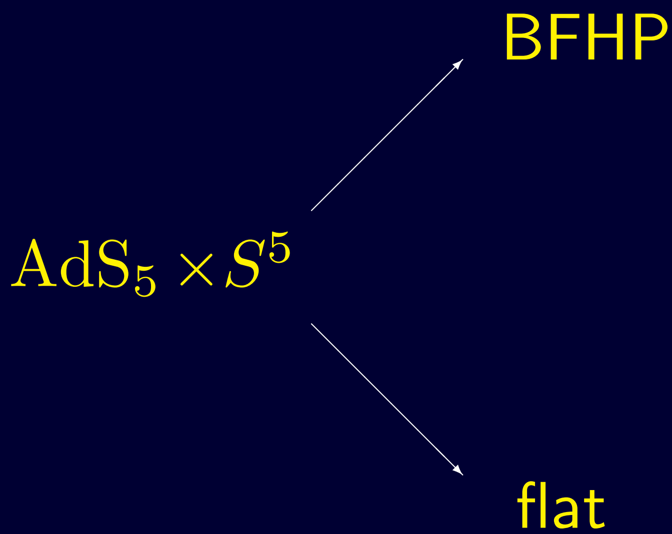

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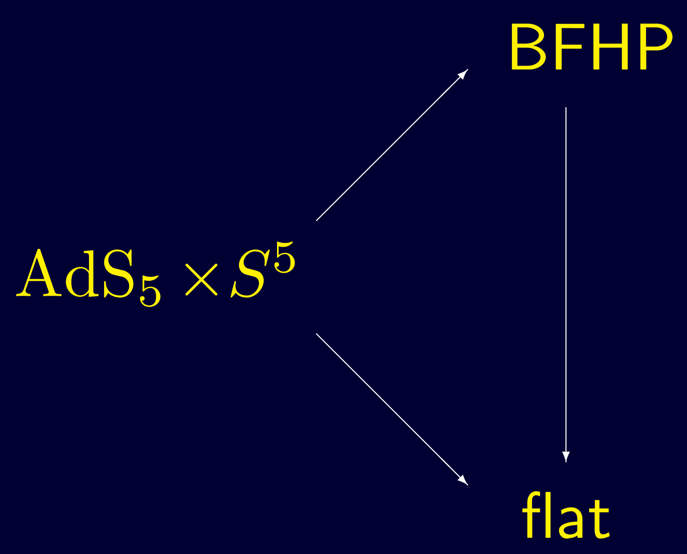

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[Gauntlett–Gutowsky–Hull–Pakis–Reall hep-th/0209114]

[Lozano-Tellechea–Meessen–Ortín hep-th/0206200]

Thank you.