Generalised Spencer cohomology and supersymmetry (Freiburg, 27/5/2019)

(joint work with Andrea Santi & Paul de Medeiros 2016–⋯)

Plan

I. Classical prelude: the LA of isometries of a riemannian manifold
II. Super-analogue: the Killing superalgebra of a supergravity background
III. Generalised spencer cohomology of Poincaré superalgebras
IV. Some calculations

I. The Lie algebra of isometries of a riemannian mfd

(M^n, g) riemannian $\quad \underline{\mathfrak{iso}}(M,g) = \{ \xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = 0 \}$

real Lie algebra of dim $\leqslant \frac{n(n+1)}{2}$ $\quad (n = \dim M)$

ξ Killing $\Leftrightarrow$ $\nabla\xi : X \mapsto \nabla_X \xi$ is skewsymmetric: $g(\nabla_X\xi, Y) = -g(X, \nabla_Y \xi)$
Killing's identity: $\nabla^2 \xi = R(\xi, -)$ $\Rightarrow$ ξ is determined by $\xi(p), \nabla\xi(p)$ for any $p \in M$. Equivalently,

$$\underline{\mathfrak{iso}}(M,g) \cong \left\{ \begin{pmatrix} \xi \\ A \end{pmatrix} \in \begin{matrix} TM \\ \oplus \\ \underline{\mathfrak{so}}(TM) \end{matrix} \;\middle|\; \mathcal{D}\begin{pmatrix} \xi \\ A \end{pmatrix} = 0 \right\}$$

Killing transport connection

$$\left[\begin{pmatrix} \xi \\ A \end{pmatrix}, \begin{pmatrix} \eta \\ B \end{pmatrix} \right] = \begin{pmatrix} A\eta - B\xi \\ [A,B] - R(\xi,\eta) \end{pmatrix}$$

If $(M^n, g) = \mathbb{E}^n$, then $\underline{\mathfrak{iso}}(\mathbb{E}^n) \cong \underline{\mathfrak{so}}(n) \ltimes \mathbb{R}^n$ (euclidean Lie algebra)
The euclidean Lie algebra is $\mathbb{Z}$-graded: $\underset{0}{\underline{\mathfrak{so}}(n)} \oplus \underset{-1}{\mathbb{R}^n}$ and hence trivially filtered: $E^0 = \underline{\mathfrak{so}}(n)$, $E^{-1} = \underline{\mathfrak{so}}(n) \oplus \mathbb{R}^n$.
In general, $\underline{\mathfrak{iso}}(M^n, g)$ is filtered but not graded: $\mathfrak{k} = \mathfrak{k}^{-1} \supset \mathfrak{k}^0 \supset 0$ and $\mathfrak{k}$ is a linear subspace of $\mathfrak{e}$ but not a filtered subalgebra. However, the associated graded $\mathrm{gr}(\mathfrak{k})$ is a graded subalgebra of $\mathrm{gr}(\mathfrak{e}) \cong \mathfrak{e}$.
The curvature is the obstruction to $\mathfrak{k}$ being a filtered subalgebra of $\mathfrak{e}$.

II. The Killing superalgebra of a supergravity background

For definiteness, let's consider $d=11$ SUGRA. A $d=11$ SUGRA background consists of a lorentzian spin 11-manifold $(M, g, \$)$ and a closed 4-form F together with a connection D on $\$$:

rank 32 real symplectic spinor bundle $\$ \to M$

$$D_X \varepsilon = \nabla_X \varepsilon + \tfrac{1}{8} F \cdot X \cdot \varepsilon + \tfrac{1}{24} X \cdot F \cdot \varepsilon \qquad \forall \varepsilon \in \Gamma(\$)$$

whose curvature $R^D \in \Omega^2(M; \mathrm{End}\,\$)$ is "Clifford traceless":

$$\sum_i e^i \cdot R^D(e_i, -) \equiv 0 \quad \Leftrightarrow \quad \begin{cases} dF = 0 & \text{(Bianchi)} \\ d{*F} = \tfrac{1}{2} F \wedge F & \text{(Maxwell)} \\ \mathrm{Ric}_g = \mathcal{T}(F,g) & \text{(Einstein)} \end{cases}$$

$(M,g,F,\$)$ is **supersymmetric** if $\exists\, 0 \neq \varepsilon \in \Gamma(\$)$ such that $D\varepsilon = 0$.

$\dim \ker D \leqslant 32 = \mathrm{rank}\,\$$.

Associated to every supergravity background there is a Lie superalgebra:

$$\mathfrak{k}_{\bar 0} = \{ \xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = 0 \ \& \ \mathcal{L}_\xi F = 0 \}$$

$$\mathfrak{k}_{\bar 1} = \{ \varepsilon \in \Gamma(\$) \mid D\varepsilon = 0 \}$$

Killing superalgebra of $(M,g,F,\$)$

<u>Example</u> $(M,g) = \mathbb{M}^{11}$, $F = 0$, $\$ = \mathbb{M} \times S$ ← irred. $C\ell(1,10)$-module

$\mathfrak{k}$ = Poincaré superalgebra

$\mathfrak{p} = \underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V}$

$\mathbb{Z}$-graded Lie superalgebra

$\mathfrak{p}_{\bar 0} = \mathfrak{so}(V) \oplus V$ (Poincaré)

$\mathfrak{p}_{\bar 1} = S$

$\mathfrak{p}$ is also filtered: $0 \subset \underset{\mathfrak{so}(V)}{\mathfrak{p}^0} \subset \underset{\mathfrak{so}(V) \oplus S}{\mathfrak{p}^{-1}} \subset \underset{\mathfrak{so}(V)\oplus S \oplus V}{\mathfrak{p}^{-2}} = \mathfrak{p}$

<u>Theorem</u> (JMF + Santi '16)

The KSA $\mathfrak{k}$ of a SUGRA bg. $(M,g,F,\$)$ is filtered and $\mathrm{gr}(\mathfrak{k})$ is a graded subalgebra of $\mathrm{gr}(\mathfrak{p}) \cong \mathfrak{p}$. (The obstructions to $\mathfrak{k}$ being a filtered subalgebra of $\mathfrak{p}$ are the Riemann curvature of g and F.)

<u>Theorem</u> (JMF + Hustler '12)

If $\dim \mathfrak{k}_{\bar 1} > \frac{1}{2} \mathrm{rank}\,\$$, then $\mathrm{ev}_p : [\mathfrak{k}_{\bar 1}, \mathfrak{k}_{\bar 1}] \longrightarrow T_pM$ is surjective $\Rightarrow$ (M,g,F) is locally homogeneous. (Conjectured by Patrick Meessen in 2004)

A highly supersymmetric background ($\dim \mathfrak{k}_{\bar 1} > \frac{1}{2}\mathrm{rank}\,\$$) can be reconstructed (up to coverings) from its KSA. (JMF + Santi '16)

III. Generalised Spencer cohomology of Poincaré superalgebras

Algebraic deformations are typically governed by a cohomology theory. In the case of Lie (super)algebra deformations, it is Chevalley–Eilenberg cohomology with coefficients in the adjoint module. If the Lie (super)algebra is graded, the Lie brackets have degree 0 and so does the CE differential, being roughly its dual. This means that we may view $C^\bullet(\mathfrak{g};\mathfrak{g})$ as

$C^\bullet(\mathfrak{g};\mathfrak{g}) = \bigoplus_d C^{d,\bullet}(\mathfrak{g};\mathfrak{g})$ ← degree

Filtered deformations are controlled by $H^{+,\bullet}(\mathfrak{g};\mathfrak{g}) = \bigoplus_{d>0} H^{d,\bullet}(\mathfrak{g};\mathfrak{g})$

This cohomology theory is known as generalised Spencer cohomology & has been studied by Cheng & Kac. One starts by calculating $H^{+,2}(\mathfrak{g}_-;\mathfrak{g})$ where $\mathfrak{g}_- = \bigoplus_{d<0} \mathfrak{g}_d$ is the negatively graded part of $\mathfrak{g}$.

Let's consider the Poincaré superalgebra $\mathfrak{p} = \underset{\mathfrak{so}(V)}{\mathfrak{p}_0} \oplus \underset{S}{\mathfrak{p}_{-1}} \oplus \underset{V}{\mathfrak{p}_{-2}}$

If $A,B \in \mathfrak{so}(V)$, $s \in S$, $v,w \in V$,

$$[A,B] = AB - BA \qquad [s,s] = \kappa(s,s) \qquad \kappa : \odot^2 S \to V \quad \text{dual to Clifford } V\otimes S \to S.$$
$$[A,s] = As$$
$$[A,v] = Av$$

Dirac current

IV. Calculations

D=11 (JMF + Santi '15)

(V,η) Lorentzian 11-dim'l vector space

$\mathfrak{so}(V) \subset C\ell(V) \cong \mathrm{End}(S_+) \oplus \mathrm{End}(S_-)$, $S_\pm \cong \mathbb{R}^{32}$

$\mathfrak{p} = \underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V}$, S = any one of $S_\pm$.

$$C^{2,1}(\mathfrak{p}_-;\mathfrak{p}) = V^*\otimes \mathfrak{so}(V)$$
$$C^{2,2}(\mathfrak{p}_-;\mathfrak{p}) = (\Lambda^2 V^* \otimes V) \oplus (V^*\otimes S^* \otimes S) \oplus (\odot^2 S^* \otimes \mathfrak{so}(V))$$
$$C^{2,3}(\mathfrak{p}_-;\mathfrak{p}) = (\odot^2 S^* \otimes V^* \otimes V) \oplus (\odot^3 S^* \otimes S)$$

$$H^{2,2}(\mathfrak{p}_-;\mathfrak{p}) \cong \Lambda^4 V \longleftarrow \text{the 4-form of } D=11 \text{ SUGRA}$$

Given $\varphi \in \Lambda^4 V$, $\exists!$ cocycle $\beta^\varphi + \gamma^\varphi \in (V^*\otimes S^* \otimes S) \oplus (\odot^2 S^* \otimes \mathfrak{so}(V))$

and

$$\beta^\varphi(v,s) = \tfrac{1}{8}\varphi\cdot v \cdot s + \tfrac{1}{24} v\cdot\varphi\cdot s \longleftarrow \text{cf. spinor connection } D$$

Globalising, $\left.\begin{array}{l} \varphi \rightsquigarrow F \in \Omega^4(M) \\ \beta^\varphi \rightsquigarrow \beta_X = \tfrac{1}{8} F\cdot X \cdot + \tfrac{1}{24} X \cdot F \cdot \end{array}\right\} D_X = \nabla_X + \beta_X$, etc...

Upshot This could be considered a cohomological derivation of D=11 SUGRA, or a way to bypass the construction of the SUGRA in order to study the supersymmetric backgrounds.

D=4 (de Medeiros + JMF + Santi '16)

(V,η) lorentzian 4-dim'l vector space

$\mathfrak{so}(V) \subset C\ell(V) \cong \mathrm{End}(S)$, $S \cong \mathbb{R}^4$

$\mathfrak{p} = \underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V}$

$$C^{2,1}(\mathfrak{p}_-;\mathfrak{p}) = V^* \otimes \mathfrak{so}(V)$$
$$C^{2,2}(\mathfrak{p}_-;\mathfrak{p}) = (\Lambda^2 V^* \otimes V) \oplus (V^* \otimes S^* \otimes S) \oplus (\odot^2 S^* \otimes \mathfrak{so}(V))$$
$$C^{2,3}(\mathfrak{p}_-;\mathfrak{p}) = (\odot^2 S^* \otimes V^* \otimes V) \oplus (\odot^3 S^* \otimes S)$$

$$H^{2,2}(\mathfrak{p}_-;\mathfrak{p}) \cong \Lambda^0 V \oplus V \oplus \Lambda^4 V$$ ← "old minimal offshell" multiplet of D=4 N=1 SUGRA

Given $\Phi := (a,\varphi,\omega) \in \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$,

$\exists!$ cocycle $\beta^\Phi + \gamma^\Phi \in (V^* \otimes S^* \otimes S) \oplus (\odot^2 S^* \otimes \mathfrak{so}(V))$, and

$$\beta^\Phi(v,s) = -v\cdot(a+\omega)\cdot s + (\varphi\wedge v)\cdot \mathrm{vol}\cdot s - 2\eta(\varphi,v)\,\mathrm{vol}\cdot s$$

giving rise to a connection $D_X = \nabla_X + \beta_X$ on spinors ← old offshell minimal formulation of N=1 d=4 SUGRA

The backgrounds on which D is flat are:

– Minkowski	– $AdS_3 \times \mathbb{R}$
– AdS_4	– $S^3 \times \mathbb{R}$

– Nappi-Witten

(cf. Festuccia + Seiberg)

D=6 (de Medeiros + JMF + Santi '18)

We have the choice of including the Sp(1) R-symmetry in the (1,0) D=6 Poincaré superalgebra, and their Spencer cohomologies differ. Without R-symmetry, we obtain the Killing spinor equations of (1,0) D=6 SUGRA, but with R-symmetry we go beyond (?) SUGRA. The geometries admitting rigid D=6 (1,0) SUSY (but not nec. Poincaré) are:

– Minkowski	– $AdS_3 \times \mathbb{R}^3$
– $AdS_5 \times \mathbb{R}$	– $AdS_3 \times S^3$
– $S^5 \times \mathbb{R}$	– $\mathbb{R}^{2,1} \times S^3$
– pp-wave	– pp-wave

same geometry, but different SUSY algebra