Lie superalgebras and lorentzian field equations

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Based on the following papers:

1511.08737 (CMP)
1511.09264 (JPA)
1608.05915 (ATMP) } w/ Andrea Santi

1605.00881 (JHEP) } w/ Paul de Medeiros & Andrea Santi
180? ———

Outline

1. The Spencer cohomology of the Poincaré superalgebra
2. Killing superalgebras
3. Field equations on homogeneous lorentzian mfds

1. The Spencer cohomology of the Poincaré superalgebra

The ur-example of a lorentzian manifold is (4-dim'l) Minkowski spacetime. In D dimensions it is a real affine space modelled on a vector space $V \cong \mathbb{R}^D$ with an indefinite inner product η with signature $(1,D-1)$ ("mostly minus"). The Lie algebra of isometries of Minkowski spacetime is the Poincaré Lie algebra $\mathfrak{p} \underset{\text{v.s.}}{=} \underset{0}{\mathfrak{so}(V,\eta)} \oplus \underset{-2}{V}$. It is a $\mathbb{Z}$-graded Lie algebra:

$$[A,B] = A\circ B - B\circ A \qquad \forall A,B \in \mathfrak{so}(V) \subset \operatorname{End}(V)$$
$$[A,v] = Av \qquad v,w \in V$$
$$[v,w] = 0$$

Associated to (V,η) we have the Clifford algebra $C\ell(V,\eta)$, generated by V subject to $v^2 = -\eta(v,v)\mathbb{1}$ $\forall v \in V$. The precise structure of $C\ell(V,\eta)$ depends on $\dim V$. If $\dim V = 4$, $C\ell(V,\eta) \cong \operatorname{Mat}(4,\mathbb{R})$ and if $\dim V = 11$, $C\ell(V,\eta) \cong 2\operatorname{Mat}(32,\mathbb{R})$. Let S be an irreducible real module of $C\ell(V,\eta)$. For example, if $\dim V = 4$, $S \cong \mathbb{R}^4$ and if $\dim V = 11$, $S \cong \mathbb{R}^{32}$ and we will choose the module on which the volume form in $C\ell(V,\eta)$ acts nontrivially. In both of these cases, Clifford action $V \times S \xrightarrow{c\ell} S$ is skewsymmetric relative to a $\mathfrak{so}(V)$-invariant symplectic structure $(-,-)$ on S. The transpose of $c\ell$ rel. to $(-,-)$ and η defines a symmetric bilinear map $\kappa : S \times S \to V$ known as the Dirac current. Explicitly, if $s \in S$, $\kappa(s) \in V$ is defined by

$$\eta(\kappa(s), v) = (s, v\cdot s) \qquad \forall v \in V.$$

Then $\kappa(s_1,s_2) = \frac{1}{2}\Big(\kappa(s_1+s_2) - \kappa(s_1) - \kappa(s_2)\Big)$.

The action of $\mathfrak{so}(V)$ on S is given by the embedding

$$\mathfrak{so}(V) \hookrightarrow C\ell(V)$$
$$v\wedge w \mapsto \tfrac{1}{4}[v,w]$$

where $v\wedge w \in \mathfrak{so}(V)$ is defined by $(v\wedge w)(x) = \eta(v,x)w - \eta(w,x)v$

The Poincaré superalgebra $\mathfrak{s} = \underline{\mathfrak{so}}(V) \oplus S \oplus V$ with brackets extending those of the Poincaré algebra above by
($\mathfrak{s}$: v.s.)

$$[A, s] = As$$
$$[s_1, s_2] = K(s_1, s_2)$$

It is a $\mathbb{Z}$-graded superalgebra $\underset{0}{\underline{\mathfrak{so}}(V)} \oplus \underbrace{\underset{-1}{S} \oplus \underset{-2}{V}}_{\mathfrak{s}_-}$

$\mathfrak{s}$ is a module over the subalgebra $\mathfrak{s}_-$. Therefore we can consider the complex $\partial : C^p(\mathfrak{s}_-; \mathfrak{s}) \longrightarrow C^{p+1}(\mathfrak{s}_-; \mathfrak{s})$

where $C^p(\mathfrak{s}_-; \mathfrak{s}) = \mathrm{Hom}(\Lambda^p \mathfrak{s}_-, \mathfrak{s})$ (in the super sense). Because $\mathfrak{s}$ is $\mathbb{Z}$-graded, the differential has $\deg \partial = 0$ and the complex admits a second grading so we can talk about $\partial : C^{p,d} \longrightarrow C^{p+1,d}$ where $C^{p,d} \subset C^p$ consists of those p-cochains of degree d. Also because $\mathfrak{s}$ is $\mathbb{Z}$-graded, $H^{p,d}$ is an $\underline{\mathfrak{so}}(V) = \mathfrak{s}_0$ - module. We are interested mostly in $H^{2,2}$, where $C^{2,2} = \{\alpha : \Lambda^2 V \longrightarrow V\} \oplus \{\beta : V \times S \longrightarrow S\} \oplus \{\gamma : S \times S \longrightarrow \underline{\mathfrak{so}}(V)\}$

eg: $\dim V = 11$,
$H^{2,2} \cong \Lambda^4 V$ as $\underline{\mathfrak{so}}(V)$-modules if $\varphi \in \Lambda^4 V$, $\beta_\varphi(v, s) = \frac{1}{24}(v \cdot \varphi \cdot s - 3 \varphi \cdot v \cdot s)$ — cf. D=11 SUGRA

$\dim V = 4$
$H^{2,2} \cong \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$ if $(a, A, b\,\mathrm{vol}) \in \Lambda^0 V \oplus V \oplus \Lambda^4 V$, $\beta_{a,A,b}(v, s) = -v \cdot (a + b\,\mathrm{vol}) \cdot s + (A \wedge v) \cdot \mathrm{vol} \cdot s - 2\eta(A, v)\,\mathrm{vol}\, s$ — cf. "old" minimal offshell N=1 D=4 SUGRA

2. Killing superalgebras (D=11 from now on)

Now let (M, g) be an 11-dim'l Lorentzian spin manifold and $F \in \Omega^4(M)$. Let $\$ \to M$ be the spinor bundle (actually a bundle of $C\ell(TM, g)$-modules) and define a connection D on $\$$ as follows:

$$D_X \varepsilon = \underset{\text{spin connection}}{\nabla_X} \varepsilon + \beta_F(X, \varepsilon)$$
$$\beta_F(X, \varepsilon) = \frac{1}{24}(X^\flat \cdot F \cdot \varepsilon - 3 F \cdot X^\flat \cdot \varepsilon)$$

We say $\varepsilon \in \Gamma(\$)$ is a Killing spinor if $D\varepsilon = 0$

If $\varepsilon \in \Gamma(\$)$ is a Killing spinor, then $K(\varepsilon)$ is a Killing vector field & if $dF = 0$, also $\mathcal{L}_{K(\varepsilon)} F = 0$.

(From now on assume $dF = 0$.)

Let $\mathfrak{k}_{\bar{0}} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0 \ \&\ \mathcal{L}_X F = 0\}$ and $\mathfrak{k}_{\bar{1}} = \{\varepsilon \in \Gamma(\$) \mid D\varepsilon = 0\}$.

$\boxed{\text{Thm (F + Meessen + Philip, '04) } \mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}} \text{ is a Lie superalgebra called the Killing superalgebra of } (M, g, F).}$

<u>conjecture</u> (Meessen, '04) If $\dim \mathfrak{k}_{\bar{1}} > \frac{1}{2}\operatorname{rnk} \$ $ $(=16)$ then $\exists$ a Lie group G acting transitively on M isometrically and preserving F.

<u>Theorem</u> (F+Hustler, '12) If $\dim \mathfrak{k}_{\bar{1}} > \frac{1}{2}\operatorname{rnk}\$ $, then the ideal $[\mathfrak{k}_{\bar{1}},\mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{0}}$ acts locally transitively on M:

$\forall p \in M$, $\operatorname{span}\{\xi(p) \mid \xi \in [\mathfrak{k}_{\bar{1}},\mathfrak{k}_{\bar{1}}]\} = T_pM$

<u>Theorem</u> (F+Santi, '16) The Killing superalgebra $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of (M,g,F) is a filtered Lie superalgebra and its associated graded algebra is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{s}$.

cf. (M,g) riemannian and $\mathfrak{g} = \{\xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = 0\}$ the Lie algebra of isometries.
Killing transport (Kostant, Geroch) $\Longrightarrow$ $\mathfrak{g}$ is filtered and $\operatorname{gr}\mathfrak{g}$ is a $\mathbb{Z}$-graded subalgebra of the euclidean Lie algebra

<u>ie</u>: isometry Lie algebra of the flat model

3. Lorentzian field equations on homogeneous lorentzian manifolds

<u>Theorem</u> (F+Santi, '16) If $\dim \mathfrak{k}_{\bar{1}} > \frac{1}{2}\operatorname{rnk}\$ $, then g & F obey the (bosonic) field equations of $d=11$ supergravity:

$$d \star F = -\tfrac{1}{2} F \wedge F \qquad \text{and} \qquad \operatorname{Ric}(g) = T(F,g)$$

"Maxwell" "Einstein"

Furthermore, we can reconstruct (M,g,F) up to local 'isometry' from $\mathfrak{k}$.

This 'reduces' the local classification of bosonic supergravity backgrounds preserving $> \frac{1}{2}$ of the supersymmetry to the classification of certain filtered deformations of $\mathbb{Z}$-graded subalgebras of $\mathfrak{s}$.

<u>State of the art</u> Let $n = \dim \mathfrak{k}_{\bar{1}}$. Then $n \in [0,32] \cap \mathbb{Z}$.

- $n=32$ classification (F+Papadopoulos '01) recovered algebraically (F+Santi, '15)
- $n=31$ $\nexists$ (Gran+Gutowski+Papadopoulos+Roest '07, F+Gadhia '07)
- $n=30$ $\nexists$ (Gran+Gutowski+Papadopoulos '10)
- $27 \leq n \leq 29$?
- $n = 26, 24, 22, 20, 18$ $\exists$ but no classification
- $n = 17, 19, 21, 23, 25$?

4. Other dimensions

One can play this game in any dimension and with other ($\mathbb{Z}$-graded) Lie superalgebras:
- (extended) Poincaré superalgebras
- "non-relativistic" superalgebras
- conformal superalgebras
- ...

We have already analysed this problem in $d=4$ and $d=6$.

Theorem (de Medeiros + F + Santi '16)

Let $\mathfrak{s} = \mathfrak{so}(V) \oplus S \oplus V$ be the 4d $N=1$ Poincaré superalgebra. Then $H^{2,2}(\mathfrak{s}_-;\mathfrak{s}) \underset{\mathfrak{so}(V)\text{-mod}}{\cong} \overset{a}{\Lambda^0 V} \oplus \overset{\varphi}{\Lambda^1 V} \oplus \overset{b}{\Lambda^4 V}$.

The corresponding Killing spinor equation is

$$D_X \varepsilon = \nabla_X \varepsilon - X \cdot (a + b\,\mathrm{vol}) \cdot \varepsilon + (\varphi \wedge X) \cdot \mathrm{vol} \cdot \varepsilon - 2 g(\varphi, X)\,\mathrm{vol} \cdot \varepsilon$$

minimal (old) off-shell SUGRA

and Killing spinors generate a Lie superalgebra.

The geometries for which D is flat are:

① $a = b = \varphi = 0 \implies$ Minkowski spacetime

② $\varphi = 0$, $a^2 + b^2 > 0 \implies AdS_4$ & $\mathrm{Ric} = -12(a^2+b^2) g$

③ $a = b = 0$, $\varphi \neq 0$

(a) φ timelike $\implies \mathbb{R} \times S^3$

(b) φ spacelike $\implies AdS_3 \times \mathbb{R}$

(c) φ lightlike $\implies$ Nappi-Witten pp wave

Theorem (de Medeiros + F + Santi, '18)

Let $\mathfrak{s} = \mathfrak{so}(V) \oplus S \oplus V$ be the $(1,0)$ $d=6$ Poincaré superalgebra, and let $\hat{\mathfrak{s}} = (\mathfrak{so}(V) \oplus \underset{\cong\, \mathfrak{sp}(1)\ \text{R-symmetry}}{\mathfrak{r}}) \oplus S \oplus V$ be the extension by R-symm.

Here $S \otimes \mathbb{C} = \underset{\substack{\text{half-spinor}\\ \text{rep of } Spin(V) \cong SL(2,\mathbb{H})}}{\Sigma_+} \otimes_{\mathbb{C}} \underset{\text{fundamental rep of } Sp(1)\ \text{R-symmetry}}{\Delta}$

Then $H^{2,2}(\mathfrak{s}_-;\mathfrak{s})$ and $H^{2,2}(\hat{\mathfrak{s}}_-;\hat{\mathfrak{s}})$ are $\mathfrak{so}(V) \oplus \mathfrak{sp}(1)$-mods.

Then $H^{2,2}(\mathfrak{s}_-;\mathfrak{s}) \cong \overset{H}{(\Lambda^3_+ V \otimes \Lambda^2 \Delta)} \oplus \overset{\varphi}{(V \otimes \mathrm{Sym}^2 \Delta)}$ and $H^{2,2}(\hat{\mathfrak{s}}_-;\hat{\mathfrak{s}}) \cong \overset{H}{(\Lambda^3 V \otimes \Lambda^2 \Delta)} \oplus \overset{\varphi}{(V \otimes \mathrm{Sym}^2 \Delta)}$

self-dual 3-form $\longleftarrow$ difference $\longrightarrow$ general 3-form

Killing spinor equation:

$$D_X \varepsilon := \nabla_X \varepsilon - \iota_X H \cdot \varepsilon + 3\,\varphi(X) \cdot \varepsilon - X \wedge \varphi \cdot \varepsilon \quad \text{(this goes beyond SUGRA)}$$

Hamburg talk (summary)

- Minkowski spacetime ⟶ Poincaré algebra ⟶ Poincaré superalgebra ($d=4$, $d=11$)
- Spencer cohomology ⟶ $\beta : V \otimes S \to S$ ($d=4$, $d=11$)
- Killing spinors ⟶ Killing superalgebra ⟶ Homogeneity ⟶ Algebraic structure
- $> \frac{1}{2}$-BPS $\Rightarrow$ field equations
- Applications to classification of $> \frac{1}{2}$-BPS sugra backgrounds
- Application to constructing rigidly supersymmetric field theories