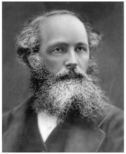
Supersymmetry and homogeneous spaces

José Miguel Figueroa O'Farrill



Colloquium "Analysis, Geometry and String Theory"
Hannover, 15 April 2013

Outline

1. Supergravity

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2. A geometrical interlude

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3. Homogeneity

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- $g, F, \ldots$ are subject to Einstein–Maxwell-like PDEs

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- Explicitly,

$$d \star F = \tfrac{1}{2} F \wedge F$$
$$\mathrm{Ric}(X, Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X, Y)|F|^2$$

together with $dF = 0$

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- It is convenient to organise this information according to how much “supersymmetry” the background preserves.

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- such spinor fields are called **Killing spinors**

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- a background is said to be ν-**BPS** if $n = 32\nu$

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- those in the 2nd row are now known to be **homogeneous**!

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- It is a very useful invariant of a supersymmetric supergravity background

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  for some $\lambda \in \mathbb{R} \cup i\mathbb{R}$
  - $\lambda \in \mathbb{R}$: **real** Killing spinors
  - $\lambda \in i\mathbb{R}$: **imaginary** Killing spinors
- they are special types of **twistor spinors**

Riemannian manifolds admitting Killing spinors

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  FRIEDRICH (1980)
- Bär's cone construction reduces the determination of which riemannian manifolds admit real Killing spinors to a holonomy problem: *which metric cones admit parallel spinors?*

  BÄR (1993)

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- This is the geometrization of Frank Adams's algebraic construction

JMF (2007)

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- By now many counterexamples are known which are not space forms
- Even today, all known simply-connected 6-dimensional riemannian manifolds admitting real Killing spinors (**nearly-Kähler 6-manifolds**) are homogeneous; although there are non-homogeneous quotients

1. Supergravity
2. A geometrical interlude
3. Homogeneity

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- This is the “right” working notion in supergravity

The homogeneity theorem

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

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MEESSEN (2004)

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The existence of non-homogeneous $\nu = \frac{1}{2}$ backgrounds shows that the theorem is sharp. But, **why** $\frac{1}{2}$?

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Generalisations

Theorem

Every ν-BPS background of type IIB supergravity with $\nu > \frac{1}{2}$ is locally homogeneous.

Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is locally homogeneous.

JMF+Hackett-Jones+Moutsopoulos (2007)

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The theorems actually prove the strong version of the conjecture: that the symmetries which are generated from the supersymmetries already act (locally) transitively.

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This actually only shows local homogeneity.

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The homogeneity theorem implies that classifying homogeneous supergravity backgrounds also classifies ν-BPS backgrounds for $\nu > \frac{1}{2}$.

This is **good** because

- the supergravity field equations for homogeneous backgrounds are algebraic and hence simpler to solve than PDEs
- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

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subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

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4. Solve the equations!

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Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper (i.e., inverse image of compact is compact). In particular, proper actions have compact stabilisers.

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This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

- Symmetric eleven-dimensional supergravity backgrounds
  JMF (2011)

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- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$
  JMF+Ungureanu (in preparation)

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- Checking supersymmetry is an additional problem, but perhaps it can be done simultaneously by classifying homogeneous supermanifolds

JMF+Santi+Spiro (in progress)