(The algebraic structure of) Killing superalgebras

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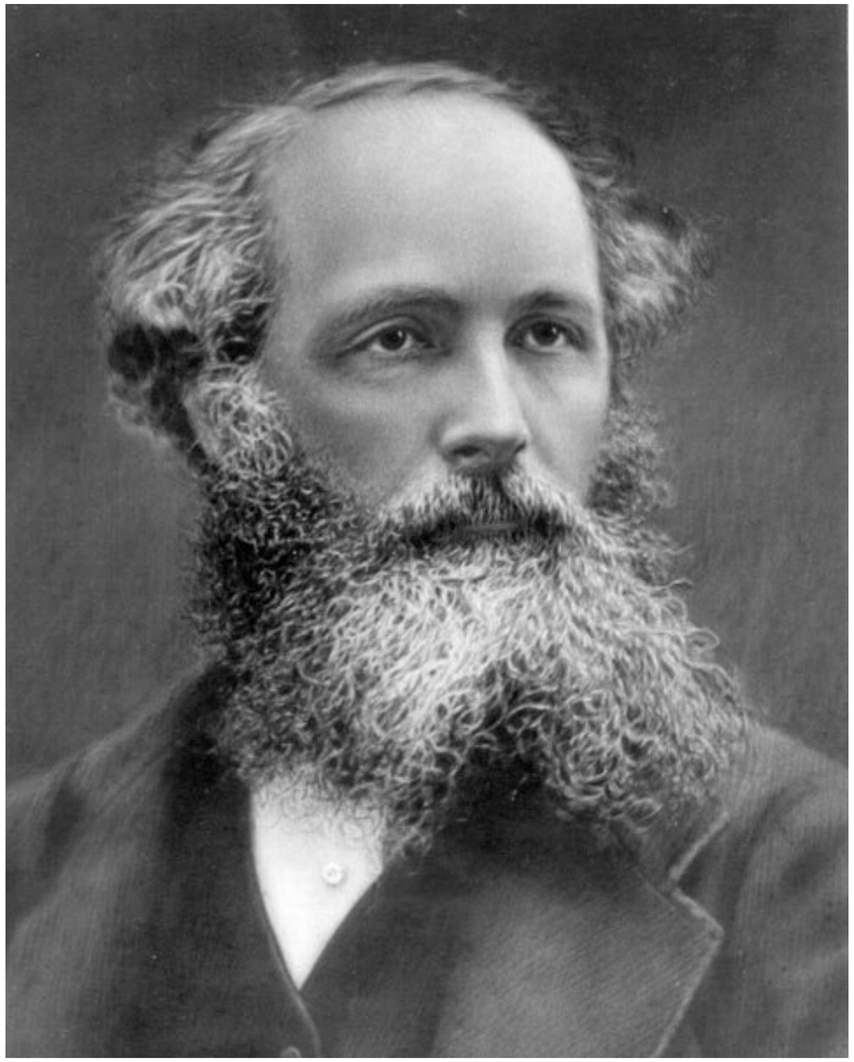
No superalgebras were harmed in the production of this talk.
Some were slightly deformed and/or lost some of their ideals.



Happy Birthday!

Abstract

One can associate a **Lie superalgebra** with every **solution** of the bosonic field equations of a **supergravity** theory which preserves some supersymmetry.

This **Killing superalgebra** is a very useful invariant of the solution. It has a very concrete **algebraic structure** which lends itself to a systematic attempt at classification.

I will explain the algebraic structure of Killing superalgebras, some of the recent results we have obtained from this point of view and some of the questions we are working on.

References

Results of an ongoing collaboration with **Andrea Santi** and **Paul de Medeiros**:

- **`arXiv:1511.08737 [hep-th]`** (CMP in press)
- **`arXiv:1511.09264 [hep-th]`** (JPhysA in press)
- **`arXiv:1605.00881 [hep-th]`** (JHEP in press)

and work in preparation.

Plan

- A geometric analogy
- Supergravity

  A geometric interlude
- Killing spinor equations
- Further results and outlook

A geometric analogy

where I introduce the basic ideas in the (hopefully) more familiar context of riemannian geometry

Back

Isometries

(M^n, g) (pseudo-) riemannian manifold

$G = \{\varphi : M \to M \mid \varphi^* g = g\}$ Lie group of **isometries**

$\mathfrak{g} = \{\xi \in TM \mid \mathcal{L}_\xi g = 0\}$ Lie algebra of **isometries**

What kind of Lie algebra is $\mathfrak{g}$?

The euclidean case

$M = \mathbb{R}^n$ $\quad g \quad$ euclidean inner product

$$G = \mathrm{O}(n) \ltimes \mathbb{R}^n \qquad \mathfrak{g} = \mathfrak{so}(n) \ltimes \mathbb{R}^n$$

isometries fixing the origin

translations

$$[A, B] = AB - BA$$
$$[A, v] = Av$$
$$[v, w] = 0 \qquad v, w \in \mathbb{R}^n$$
$$A, B \in \mathfrak{so}(n)$$

The flat model

(V, η) real vector space with symmetric inner product

$$\mathfrak{so}(V) = \{A : V \to V \mid \eta(Av, w) = -\eta(v, Aw)\}$$

$\mathfrak{e}(V) = \underset{0}{\mathfrak{so}(V)} \ltimes \underset{-2}{V}$ (pseudo-) euclidean Lie algebra

graded

$$[A, B] = AB - BA$$
$$[A, v] = Av \qquad v, w \in V$$
$$[v, w] = 0 \qquad A, B \in \mathfrak{so}(V)$$

The round sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

$$G = \mathrm{O}(n+1)$$
$$\mathfrak{g} = \mathfrak{so}(n+1)$$

Fix $x \in S^n$ $\quad V = T_x S^n$

stabiliser
$$H = \mathrm{O}(V)$$
$$\mathfrak{h} = \mathfrak{so}(V)$$

$$\mathfrak{g} \overset{\text{v.s.}}{\cong} \mathfrak{so}(V) \oplus V$$

$$[A, B] = AB - BA$$
$$[A, v] = Av$$
$$[v, w] = \rho(v, w)$$

$$\rho : \Lambda^2 V \to \mathfrak{so}(V)$$

curvature!

not graded but **filtered!**

Filtered Lie (super)algebras

$$\mathfrak{g}^\bullet : \qquad \cdots \supset \mathfrak{g}^{n-1} \supset \mathfrak{g}^n \supset \mathfrak{g}^{n+1} \supset \cdots$$

$$\bigcap_n \mathfrak{g}^n = 0 \qquad \bigcup_n \mathfrak{g}^n = \mathfrak{g} \qquad [\mathfrak{g}^n, \mathfrak{g}^m] \subset \mathfrak{g}^{n+m}$$

associated graded algebra

$$\mathfrak{g}_\bullet = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n \qquad \mathfrak{g}_n = \mathfrak{g}^n / \mathfrak{g}^{n+1}$$

$$[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$$

Filtered deformations

$$\mathfrak{a} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}_n \qquad [\mathfrak{a}_n, \mathfrak{a}_m] \subset \mathfrak{a}_{n+m} \qquad \mathbb{Z}\text{-graded}$$

A **filtered deformation** $\mathfrak{g}$ of $\mathfrak{a}$ is a filtered algebra whose associated graded algebra $\approx \mathfrak{a}$

The Lie brackets of $\mathfrak{g}$ are obtained by adding to those of $\mathfrak{a}$ terms with **positive** degree.

General result

- The Lie algebra $\mathfrak{g}$ of isometries of a (pseudo-)riemannian manifold is **filtered**
- Its associated graded Lie algebra is isomorphic to a Lie subalgebra of the (pseudo-) euclidean Lie algebra $\mathfrak{e}$

"$\mathfrak{g}$ is a **filtered subdeformation** of $\mathfrak{e}$"

i.e., a filtered deformation of a graded subalgebra of $\mathfrak{e}$

Killing transport

[Kostant (1955), Geroch (1969)]

$$\mathcal{E} = TM \oplus \mathfrak{so}(TM)$$

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} := \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A + R(\xi, X) \end{pmatrix}$$

$$\mathfrak{g} \cong \left\{ \begin{pmatrix} \xi \\ A \end{pmatrix} \in \mathcal{E} \;\middle|\; D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = 0 \right\}$$

$$\left[\begin{pmatrix} \xi \\ A \end{pmatrix}, \begin{pmatrix} \zeta \\ B \end{pmatrix} \right] = \begin{pmatrix} A(\zeta) - B(\xi) \\ [A, B] + R(\xi, \zeta) \end{pmatrix}$$

Localisation

$$p \in M \qquad V = T_pM$$

$$\left.\begin{aligned} \mathfrak{h} &:= \{\xi \in \mathfrak{g} \mid \xi(p) = 0\} < \mathfrak{g} \\ V' &:= \{\xi(p) \mid \xi \in \mathfrak{g}\} \subset V \end{aligned}\right\} \Longrightarrow \mathfrak{g} \overset{\text{v.s.}}{\cong} V' \oplus \mathfrak{h}$$

$$\begin{aligned} [A, B] &= AB - BA \\ [A, v] &= Av + \alpha(A, v) \\ [v, w] &= \tau(v, w) + \rho(v, w) \end{aligned} \qquad A, B \in \mathfrak{h} \quad v, w \in V'$$

$$\deg \alpha = \deg \tau = 2$$
$$\deg \rho = 4$$

filtered deformation of

$$\begin{aligned} [A, B] &= AB - BA \\ [A, v] &= Av \\ [v, w] &= 0 \end{aligned}$$

Supergravity

where I argue that every supergravity theory provides a "super-ization" of the previous construction and illustrate this with eleven-dimensional supergravity

Back

Geometric data

$(M, g, \ldots)$ lorentzian manifold with additional data

$$\begin{array}{c} \$ \\ \downarrow \\ M \end{array}$$ spinor bundle of modules over $C\ell(TM)$

bundle of Clifford algebras

$D = \nabla + \cdots$ connection on $\$$

(Possibly also additional endomorphisms of $\$$ depending on the additional data)

Example: d=11 SUGRA

$$(M, g, F) \qquad F \in \Omega^4(M) \qquad dF = 0$$

$\$$ real, rank 32, symplectic $\langle -, - \rangle$

$$D_X = \nabla_X - \tfrac{1}{24} X \cdot F + \tfrac{1}{8} F \cdot X$$

Killing spinors $= \{\varepsilon \in \$ \mid D\varepsilon = 0\}$

Why the name?

$\sqrt{\text{Killing vectors}}$

$\kappa : \$ \times \$ \to TM$ Dirac current

$$
\begin{aligned}
g(\kappa(\varepsilon_1, \varepsilon_2), X) &= \langle \varepsilon_1, X \cdot \varepsilon_2 \rangle \\
&= \langle \varepsilon_2, X \cdot \varepsilon_1 \rangle
\end{aligned}
$$

$\xi := \kappa(\varepsilon, \varepsilon)$ is either timelike or null

$$
D\varepsilon = 0 \implies \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0
$$

Killing superalgebra

[JMF+Meessen+Philip (2004)]

$$\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$$
$$\mathfrak{k}_{\bar{0}} = \{\xi \in TM \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\}$$
$$\mathfrak{k}_{\bar{1}} = \{\varepsilon \in \$ \mid D\varepsilon = 0\}$$

$[\xi, \varepsilon] := \mathcal{L}_\xi \varepsilon = \nabla_\xi \varepsilon + A_\xi \varepsilon$ spinorial Lie derivative

$\mathfrak{k}$ is a Lie superalgebra

Homogeneity theorem

[JMF+Hustler (2012)]

$\dim \mathfrak{k}_{\bar{1}} > \frac{1}{2} \operatorname{rank} \mathfrak{S}$ “background is >½-BPS”

(M, g, F) is locally homogeneous

This proves (a local version of) a conjecture of Patrick Meessen's (2004)

Quo vadimus?

- The Lie algebra of isometries of a riemannian manifold is a filtered subdeformation of the euclidean Lie algebra (i.e., Lie algebra of isometries of the flat model)
- In complete analogy, the Killing superalgebra of a supergravity background is a filtered subdeformation of the Poincaré superalgebra (i.e., Killing superalgebra of the flat model)

The flat model

$\mathbb{R}^{1,10}$ $\quad F = 0 \quad$ Minkowski spacetime (d=11)

$\mathfrak{k}$ is the (d=11) Poincaré superalgebra

$\mathfrak{k}_{\bar{0}} \cong \mathbb{R}^{1,10} \rtimes \mathfrak{so}(1,10)$ Poincaré algebra

$\mathfrak{k}_{\bar{1}} \cong S$ irreducible Clifford module of $C\ell(1,10)$

$$\mathfrak{k}_\bullet = \underbrace{\mathbb{R}^{1,10}}_{-2} \oplus \underbrace{S}_{-1} \oplus \underbrace{\mathfrak{so}(1,10)}_{0}$$

$\mathbb{Z}$-graded

Killing **super**-transport

$$\mathcal{E} = \mathcal{E}_{\bar{0}} \oplus \mathcal{E}_{\bar{1}}$$

$$\mathcal{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM) \qquad \mathcal{E}_{\bar{1}} = \$$$

$$D_X \begin{pmatrix} \xi \\ A \\ \varepsilon \end{pmatrix} := \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A + R(\xi, X) \\ \nabla_X \varepsilon - \frac{1}{24} X \cdot F \cdot \varepsilon + \frac{1}{8} F \cdot X \cdot \varepsilon \end{pmatrix}$$

$$\mathfrak{k}_{\bar{0}} \cong \left\{ \begin{pmatrix} \xi \\ A \end{pmatrix} \in \mathcal{E}_{\bar{0}} \;\middle|\; D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = 0, \quad \nabla_\xi F + AF = 0 \right\}$$

$$\mathfrak{k}_{\bar{1}} \cong \{ \varepsilon \in \mathcal{E}_{\bar{1}} \mid D_X \varepsilon = 0 \}$$

Localisation

$$p \in M \qquad V = T_pM \qquad S = \$_p$$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k}_{\bar{0}} \xrightarrow{\mathrm{ev_p}} V' \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{k}_{\bar{1}} \xrightarrow{\mathrm{ev_p}} S' \longrightarrow 0$$

$$\mathfrak{k} \overset{\text{v.s.}}{\cong} V' \oplus S' \oplus \mathfrak{h} \subset V \oplus S \oplus \mathfrak{so}(V) =: \mathfrak{p}(V)$$

Poincaré superalgebra

A filtered Lie superalgebra

Theorem [JMF+Santi (2016)]

1. $\mathfrak{k}$ is a filtered Lie superalgebra, and
2. its associated graded algebra is isomorphic to a Lie subalgebra of $\mathfrak{p}$.

"$\mathfrak{k}$ is a **filtered subdeformation** of $\mathfrak{p}$"

A geometric interlude

where I try to convince you that the “super-ization” is actually not all that exotic

Back

Geometric Killing spinors

$$(M,g) \qquad \begin{array}{c} \$ \\ \downarrow \\ M \end{array}$$

$\varepsilon \in \$$ (geometric) Killing spinor

$$\nabla_X \varepsilon = \lambda X \cdot \varepsilon \qquad \forall X \in TM$$

$$\mathrm{scal} \sim \lambda^2 \implies \lambda \in \mathbb{R} \sqcup i\mathbb{R}$$

Squaring geometric Killing spinors yields (conformal) Killing vectors.

The Lie derivative of a Killing spinor along a Killing vector is again a Killing spinor.

Killing ¾-Lie algebras

$$\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$$

$$\mathfrak{k}_{\bar{0}}(M, g) = \{\xi \in TM \mid \mathcal{L}_\xi g = 0\}$$

$$\mathfrak{k}_{\bar{1}}(M, g) = \{\varepsilon \in \$ \mid \nabla_X \varepsilon = \lambda X \cdot \varepsilon \quad \forall X \in TM\}$$

Brackets

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}]$ Lie bracket

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}]$ spinorial Lie derivative

$[\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$ Dirac current

Jacobi identities?

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}]$ ✓

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}]$ ✓

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$ ✓

$[\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$?

Some Killing Lie algebras

[JMF (2007)]

(octonion) Hopf fibration

$$\begin{array}{ccc} S^7 & \hookrightarrow & S^{15} \\ & & \downarrow \\ & & S^8 \end{array}$$

$$\mathfrak{k}(S^7) \cong \mathfrak{so}(9)$$

$$\mathfrak{k}(S^8) \cong \mathfrak{f}_4$$

$$\mathfrak{k}(S^{15}) \cong \mathfrak{e}_8$$

There are similar constructions for all simple Lie algebras and even some simple Lie superalgebras.

[de Medeiros (2014)]

Killing spinor equations

where I present a systematic approach to determining those geometries on which we can define rigidly supersymmetric theories

Back

Deformations

- **Deformations** of algebraic structures are typically governed by a **cohomology theory**.
- **Lie** (super)algebra deformations are governed by **Chevalley—Eilenberg** cohomology.
- **Filtered** deformations are governed by **generalised Spencer cohomology**: a bigraded refinement of Chevalley—Eilenberg cohomology.

(Generalised) Spencer cohomology

The terms of smallest positive degree in a filtered deformation $\mathfrak{g}$ of $\mathfrak{a}$ define a cocycle in bidegree (2,2) of a generalised Spencer cohomology theory associated to $\mathfrak{a}$.

In our applications, $\mathfrak{a}$ is a graded subalgebra of the Poincaré superalgebra $\mathfrak{p}$

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2} = \mathfrak{so}(V) \oplus S \oplus V$$

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} = \mathfrak{h} \oplus S' \oplus V'$$

The relevant cohomology group is

$$H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{a}_0}$$

$$\mathfrak{a}_- = \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} = S' \oplus V'$$

The first step in the calculation is to determine

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$$

and this yields the (differential) Killing spinor equations!

Killing spinor equations

One component of the Spencer cocycle gives a linear map

$$\beta : V \otimes S \to S$$

defining

$$\beta \in \Omega^1(M, \mathrm{End}\,\$)$$

and hence a spinor connection

$$D = \nabla - \beta$$

Killing superalgebras

And in many cases, the **Killing spinors**

$$D\varepsilon = 0$$

generate a Lie superalgebra.

(This is analogous to integrating an infinitesimal deformation.)

Examples

$\mathfrak{p}$ = 11-dimensional Poincaré superalgebra

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^4 V \qquad \text{(as } \mathfrak{so}(V) \text{ reps)}$$

$$F \in \Lambda^4 V \qquad \beta(v,s) = \tfrac{1}{24} v \cdot F \cdot s - \tfrac{1}{8} F \cdot v \cdot s$$

the 4-form! the gravitino connection!

(The existence of the Killing superalgebra (seems to) require that $dF=0$.)

$\mathfrak{p}$ = 4-dimensional N=1 Poincaré superalgebra

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$$

$$A \in \Lambda^0 V \quad B \in \Lambda^4 V \quad C \in \Lambda^1 V$$

$$\beta(v, s) = v \cdot (A + B) \cdot s - C \cdot v \cdot \mathrm{vol} \cdot s + \eta(C, v)\, \mathrm{vol} \cdot s$$

"old" off-shell formulation of d=4 N=1 supergravity!

(The Killing superalgebra exists without any differential constraints on A,B,C.)

Summary

- We may make the provocative claim to have derived eleven-dimensional supergravity from generalised Spencer cohomology!

- Okay, at least we recover the information necessary to define supersymmetric bosonic supergravity backgrounds:

  - Bosonic field equations are encoded in the Clifford trace of the gravitino connection

  - The gravitino connection defines the Killing spinors of the supergravity background

Further results and outlook

where I summarise some of our ongoing work in this area
and which questions we hope to address in the near future

Back

What is this good for?

- We can attack the classification of supersymmetric supergravity backgrounds by classifying their Killing superalgebras, whose algebraic structure is now very much under control

- We can determine the equations satisfied by Killing spinors in geometries admitting rigid supersymmetry by computing the relevant generalised Spencer cohomology groups

Maximal supersymmetry

The filtered deformations of graded subalgebras of the $d\!=\!11$ Poincaré superalgebra of the form

$$\mathfrak{a} = \mathfrak{h} \oplus S \oplus V$$

correspond *precisely* to the Killing superalgebras of the maximally supersymmetric backgrounds of $d\!=\!11$ supergravity:

$$\mathbb{R}^{1,10} \qquad \mathrm{AdS}_4 \times S^7 \qquad \mathrm{AdS}_7 \times S^4 \qquad \mathrm{KG}_{11}$$

4-dimensional rigid supersymmetry

The filtered deformations of graded subalgebras of the $N=1$ $d=4$ Poincaré superalgebra of the form

$$\mathfrak{a} = \mathfrak{h} \oplus S \oplus V$$

correspond to the Killing superalgebras of the following geometries:

$$\mathbb{R}^{1,3} \qquad \mathrm{AdS}_4 \qquad \mathrm{AdS}_3 \times S^1 \qquad \mathbb{R} \times S^3 \qquad \mathrm{NW}_4$$

Supersymmetry and field equations

The Jacobi identity of the Killing superalgebra of an 11-dimensional supergravity background is intimately related to the bosonic field equations.

$$\dim S' > \tfrac{1}{2}\dim S = 16$$

$$dF = 0$$

Einstein equation
&
Maxwell equation

(Known to fail for ½-BPS backgrounds)

Open questions

- Are all filtered Lie superalgebras which *look* like Killing superalgebras actually the Killing superalgebras of a supergravity background?
- How do algebraic Killing spinor equations coming from the variations of the dilatino, gaugino,… arise in this approach?
- What about the “odd” part of the Spencer cohomology?

In progress

- $d=11$ filtered deformations with $16 < \dim S' < 32$

- $d=10$ Type I filtered deformations: they seem to yield the (differential) Killing spinor equations associated to the $d=10$ conformal supermultiplet we studied recently [de Medeiros-JMF (2015)]

- $d=4$ with R-symmetry, in order to recover the “new” minimal off-shell formulation of $d=4$ $N=1$ supergravity

In utero

- Other dimensions: 5, 6, 10
- Extended supersymmetries
- Superconformal symmetries
- A possible supergeometrical approach

Recap

- The Killing superalgebra continues to be useful.

- It is gratifying to see **old friends** (supersymmetric supergravity backgrounds) in a **new guise** (filtered subdeformations of the Poincaré superalgebra).

- Especially when this suggests **new approaches** to **old problems** (classification of supersymmetric supergravity backgrounds).

- It is even more gratifying when the **new techniques** (Spencer cohomology) shed light on a **different problem**: determination of geometries admitting rigid supersymmetry.