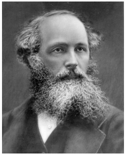
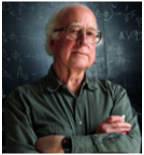
Homogeneous Lorentzian Manifolds in Supergravity

José Miguel Figueroa O'Farrill





HelgaFest

Greifswald, 18 March 2014


Baumschulenweg

¡Feliz cumpleaños!

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Liebe Helga,

Alles Gute zum Geburtstag!

Outline

1. Supergravity

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2. A geometrical interlude
3. Homogeneity in Supergravity

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  - a connection $D = \nabla + \cdots$ on the spinor (actually Clifford) bundle S

Poincaré supergravities

	32			24		20	16		12	8	4
11	M										
10	IIA	IIB					I				
9	$\mathrm{N}=2$						$\mathrm{N}=1$				
8	$\mathrm{N}=2$						$\mathrm{N}=1$				
7	$\mathrm{N}=4$						$\mathrm{N}=2$				
6	$(2,2)$	$(3,1)$	$(4,0)$	$(2,1)$	$(3,0)$		$(1,1)$	$(2,0)$		$(1,0)$	
5	$\mathrm{N}=8$			$\mathrm{N}=6$			$\mathrm{N}=4$			$\mathrm{N}=2$	
4	$\mathrm{N}=8$			$\mathrm{N}=6$		$\mathrm{N}=5$	$\mathrm{N}=4$		$\mathrm{N}=3$	$\mathrm{N}=2$	$\mathrm{N}=1$

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- Explicitly,

$$d \star F = \frac{1}{2} F \wedge F$$
$$\mathrm{Ric}(X, Y) = \frac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \frac{1}{6} g(X, Y)|F|^2$$

together with $dF = 0$

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- It is convenient to organise this information according to how much “supersymmetry” the background preserves.

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- such spinor fields are called **Killing spinors**

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- a background is said to be ν-**BPS** if $n = 32\nu$

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- those in the 2nd row are now known to be **homogeneous**!

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- It behaves as expected: it deforms under geometric limits (e.g., Penrose) and it embeds under asymptotic limits.

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JMF+MEESSEN+PHILIP (2004)

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BAUM+FRIEDRICH+GRUNEWALD+KATH (1991)

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- $\lambda \in \mathbb{R}$ case reduces to a holonomy problem: *which metric cones admit parallel spinors?*

  BÄR (1993), ALEKSEEVESKY+CORTÉS+GALAEV+LEISTNER (2007)

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- This is the obstacle to a classification of supersymmetric Freund–Rubin backgrounds

  JMF+LEITNER+SIMÓN (20??)

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DE MEDEIROS+HOLLANDS (2013)

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- There is hope of constructing non-homogeneous examples via deformations

  MOROIANU+SEMMELMANN+NAGY (2006)

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- This is the “right” working notion in supergravity

The homogeneity theorem

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

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The existence of non-homogeneous $\nu = \frac{1}{2}$ backgrounds shows that the theorem is sharp. But, **why** $\frac{1}{2}$?

Generalisations

Theorem

Every ν-BPS background of type IIB supergravity with $\nu > \frac{1}{2}$ is locally homogeneous.

Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is locally homogeneous.

JMF+HACKETT-JONES+MOUTSOPOULOS (2007)

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Every ν-BPS background of six-dimensional $(1,0)$ and $(2,0)$ supergravities with $\nu > \frac{1}{2}$ is locally homogeneous.

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The theorems actually prove the strong version of the conjecture: that the symmetries which are generated from the supersymmetries already act (locally) transitively.

Poincaré supergravities again

	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N = 2$			$N = 1$			
8	$N = 2$			$N = 1$			
7	$N = 4$			$N = 2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N = 8$	$N = 6$		$N = 4$		$N = 2$	
4	$N = 8$	$N = 6$	$N = 5$	$N = 4$	$N = 3$	$N = 2$	$N = 1$

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This actually only shows local homogeneity.

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This is **good** because

- the supergravity field equations for homogeneous backgrounds are algebraic and hence simpler to solve than PDEs
- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

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subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

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4. Solve the equations!

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Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper (i.e., inverse image of compact is compact). In particular, proper actions have compact stabilisers.

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If a semisimple Lie group acts transitively and non-properly on a lorentzian manifold (M, g)*, then* (M, g) *is locally isometric to the product of (anti) de Sitter spacetime and a riemannian homogeneous space.*

This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

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- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$
  JMF+UNGUREANU (IN PREPARATION)

Outlook

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- Checking supersymmetry is an additional problem, but there is an efficient algorithm which has already discarded many of the symmetric eleven-dimensional backgrounds.

  LISCHEWSKI (2014), HUSTLER+LISCHEWSKI (IN PROGRESS)