# (The algebraic structure of) Killing superalgebras

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# Happy Birthday!

# References

Results of an ongoing collaboration with **Andrea Santi** and **Paul de Medeiros**:

- **arXiv:1511.08737 [hep-th]** (CMP in press)
- **arXiv:1511.09264 [hep-th]** (JPhysA)
- **arXiv:1605.00881 [hep-th]** (JHEP in press)

and work in preparation.

# Plan

- A geometric analogy
- Supergravity

  A geometric interlude

- Killing spinor equations from cohomology

# A geometric analogy

where I introduce the basic ideas in the (hopefully) more familiar context of riemannian geometry



Back

# Isometries

$(M^n, g)$ (pseudo-) riemannian manifold

$G = \{\varphi : M \to M \mid \varphi^* g = g\}$ Lie group of **isometries**

$\mathfrak{g} = \{\xi \in TM \mid \mathcal{L}_\xi g = 0\}$ Lie algebra of **isometries**

$$\Updownarrow$$

$$\nabla \xi \in \mathfrak{so}(TM)$$

**Question:** What kind of Lie algebra is $\mathfrak{g}$?

# The euclidean case

$M = \mathbb{R}^n$ $\quad g \quad$ euclidean inner product

$$G = \mathrm{O}(n) \ltimes \mathbb{R}^n \qquad \mathfrak{g} = \mathfrak{so}(n) \ltimes \mathbb{R}^n$$

$\mathrm{O}(n)$: isometries fixing the origin

$\mathbb{R}^n$: translations

$$[A, B] = AB - BA$$
$$[A, v] = Av$$
$$[v, w] = 0$$

$v, w \in \mathbb{R}^n$

$A, B \in \mathfrak{so}(n)$

# The flat model

$(V, \eta)$ real vector space with symmetric inner product

$$\mathfrak{so}(V) = \{A : V \to V \mid \eta(Av, w) = -\eta(v, Aw)\}$$

$\mathfrak{e}(V) = \mathfrak{so}(V) \ltimes V$ (pseudo-) euclidean Lie algebra

0 -2

graded

$$[A, B] = AB - BA$$

$$[A, v] = Av \qquad v, w \in V$$

$$[v, w] = 0 \qquad A, B \in \mathfrak{so}(V)$$

# The round sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

$$G = \mathrm{O}(n+1)$$
$$\mathfrak{g} = \mathfrak{so}(n+1)$$

Fix $x \in S^n$ $\quad V = T_x S^n$

stabiliser $\quad H = \mathrm{O}(V)$
$$\mathfrak{h} = \mathfrak{so}(V)$$

$$\mathfrak{g} \overset{\text{v.s.}}{\cong} \mathfrak{so}(V) \oplus V$$

$$[A, B] = AB - BA$$
$$[A, v] = Av$$
$$[v, w] = \rho(v, w)$$

$$\rho : \Lambda^2 V \to \mathfrak{so}(V)$$

curvature

not graded but filtered!

# Filtered Lie (super)algebras

$$\mathfrak{g}^\bullet : \quad \cdots \supset \mathfrak{g}^{n-1} \supset \mathfrak{g}^{n} \supset \mathfrak{g}^{n+1} \supset \cdots$$

"degree at least $n$"

$$\bigcap_n \mathfrak{g}^n = 0 \qquad \bigcup_n \mathfrak{g}^n = \mathfrak{g} \qquad [\mathfrak{g}^n, \mathfrak{g}^m] \subset \mathfrak{g}^{n+m}$$

associated graded algebra

$$\mathfrak{g}_\bullet = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n \qquad \mathfrak{g}_n = \mathfrak{g}^n / \mathfrak{g}^{n+1}$$

$$[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$$

# Filtered deformations

$$\mathfrak{a} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}_n \qquad [\mathfrak{a}_n, \mathfrak{a}_m] \subset \mathfrak{a}_{n+m} \qquad \mathbb{Z}\text{-graded}$$

A filtered deformation $\mathfrak{g}$ of $\mathfrak{a}$ is a filtered algebra whose associated graded algebra ≈ $\mathfrak{a}$

The Lie brackets of $\mathfrak{g}$ are obtained by adding to those of $\mathfrak{a}$ terms with *positive* degree.

# General result

- The Lie algebra $\mathfrak{g}$ of isometries of a (pseudo-)riemannian manifold is **filtered**
- Its associated graded Lie algebra is isomorphic to a Lie subalgebra of the (pseudo-) euclidean Lie algebra $\mathfrak{e}$

"$\mathfrak{g}$ is a filtered subdeformation of $\mathfrak{e}$"

i.e., a filtered deformation of a graded subalgebra of $\mathfrak{e}$

# Killing transport

[Kostant (1955), Geroch (1969)]

$$\mathcal{E} = TM \oplus \mathfrak{so}(TM)$$

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} := \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A + R(\xi, X) \end{pmatrix}$$

$$\mathfrak{g} \cong \left\{ \begin{pmatrix} \xi \\ A \end{pmatrix} \in \mathcal{E} \;\middle|\; D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = 0 \right\}$$

← Killing's equation

← Killing's identity

$$\left[ \begin{pmatrix} \xi \\ A \end{pmatrix}, \begin{pmatrix} \zeta \\ B \end{pmatrix} \right] = \begin{pmatrix} A(\zeta) - B(\xi) \\ [A, B] + R(\xi, \zeta) \end{pmatrix}$$

↑ obstruction to $\mathfrak{g} < \mathfrak{e}$

# Localisation

$$p \in M \qquad V = T_pM$$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{\mathrm{ev_p}} V' \longrightarrow 0$$

$$[A, B] = AB - BA$$

$$[A, v] = Av + \alpha(A, v) \qquad A, B \in \mathfrak{h} \quad v, w \in V'$$

$$[v, w] = \tau(v, w) + \rho(v, w)$$

$$\deg \alpha = \deg \tau = 2$$

$$\deg \rho = 4$$

filtered deformation of

$$[A, B] = AB - BA$$

$$[A, v] = Av$$

$$[v, w] = 0$$

# Homogeneous case

$G \curvearrowright (M, g)$ transitive $\Rightarrow$ $V' = V$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{\mathrm{ev_p}} V \longrightarrow 0$$

$$[A, B] = AB - BA$$
$$[A, v] = Av + \alpha(A, v)$$
$$[v, w] = \tau(v, w) + \rho(v, w)$$

$\alpha = 0$ $\Rightarrow$ reductive

$\tau = 0$ $\Rightarrow$ symmetric

# Supergravity

where I argue that every supergravity theory provides a "super-ization" of the previous construction and illustrate this with eleven-dimensional supergravity

# (super)Gravity

"If Grossmann had met Grassmann,
Einstein would have discovered supergravity."
— Peter van Nieuwenhuizen

- Supersymmetric theory of gravity $\Longleftrightarrow$ a gauge theory of supersymmetry
- Dates from the mid 1970s
- There are many supergravities, in dimensions $\leq 11$ (lorentzian)
- They rank among the crown jewels of (late) 20th century mathematical physics
- Their construction is typically a technical tour de force
- … but they are dictated by the representation theory of the Poincaré or conformal superalgebras

# Geometric data

$(M, g, \ldots)$ lorentzian manifold with additional data

$$\begin{array}{c} \$ \\ \downarrow \\ M \end{array}$$

spinor bundle of modules over $C\ell(TM)$ ← bundle of Clifford algebras

$D = \nabla + \cdots$ connection on $\$$, where $\cdots \in \Omega^1(M, \operatorname{End} \$)$

(Possibly also additional endomorphisms of $\$$ depending on the additional data)

# Example: d=11 SUGRA

$(M, g, F) \qquad F \in \Omega^4(M) \qquad dF = 0$

$\$$ real, rank 32, symplectic $\langle -, - \rangle$

$$D_X = \nabla_X - \frac{1}{24} X \cdot F + \frac{1}{8} F \cdot X$$

Killing spinors $= \{\varepsilon \in \$ \mid D\varepsilon = 0\}$

Why the name?

# $\sqrt{\text{Killing vectors}}$

Clifford action $TM \times \$ \to \$$

Transpose $\kappa : \$ \times \$ \to TM$ Dirac current

$\xi := \kappa(\varepsilon, \varepsilon)$ is either timelike or null

$$D\varepsilon = 0 \implies \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$$

# Spinorial Lie derivative

[Kosmann(-Schwarzbach) (1972)]

$$\xi \in \mathfrak{g} \quad \varepsilon \in \$ \qquad\qquad \mathcal{L}_\xi \varepsilon := \nabla_\xi \varepsilon + A_\xi \varepsilon$$

cf. $\eta \in TM \qquad \nabla_\xi \eta + A_\xi \eta = \nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$

$$\xi \in \mathfrak{g} \quad \mathcal{L}_\xi F = 0 \implies [\mathcal{L}_\xi, D] = 0$$

$$D\varepsilon = 0 \implies D\mathcal{L}_\xi \varepsilon = 0$$

# Killing superalgebra

[JMF+Meessen+Philip (2004)]

$$\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$$

$$\mathfrak{k}_{\bar{0}} = \{\xi \in TM \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\}$$

$$\mathfrak{k}_{\bar{1}} = \{\varepsilon \in \$ \mid D\varepsilon = 0\}$$

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}]$ Lie bracket

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}]$ spinorial Lie derivative

$[\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$ Dirac current

$\mathfrak{k}$ is a Lie superalgebra

# Homogeneity theorem

[JMF+Hustler (2012)]

$\dim \mathfrak{k}_{\bar{1}} > \frac{1}{2} \operatorname{rank} \mathfrak{S}$ “background is >½-BPS”

$$\Downarrow$$

$(M, g, F)$ is locally homogeneous

**(Old) problem:** classify >½-BPS backgrounds

This proves (a local version of) a conjecture of Patrick Meessen's (2004)

# Quo vadimus?

- The Lie algebra of isometries of a riemannian manifold is a filtered subdeformation of the euclidean Lie algebra (i.e., Lie algebra of isometries of the flat model)

- In complete analogy, the Killing superalgebra of a supergravity background is a filtered subdeformation of the Killing superalgebra of the flat model

# The flat model

$\mathbb{R}^{1,10}$ $F = 0$ Minkowski spacetime ($d$=$11$)

$\mathfrak{k}$ is the ($d$=$11$) Poincaré superalgebra

$\mathfrak{k}_{\bar{0}} \cong \mathbb{R}^{1,10} \rtimes \mathfrak{so}(1,10)$ Poincaré algebra

$\mathfrak{k}_{\bar{1}} \cong S$ irreducible Clifford module of $C\ell(1,10)$

$\mathfrak{k}_\bullet = \mathbb{R}^{1,10} \oplus S \oplus \mathfrak{so}(1,10)$ $\mathbb{Z}$-graded

-2 -1 0

# Killing **super**-transport

$$\mathcal{E} = \mathcal{E}_{\bar{0}} \oplus \mathcal{E}_{\bar{1}}$$

$$\mathcal{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM) \qquad \mathcal{E}_{\bar{1}} = \$$$

$$D_X \begin{pmatrix} \xi \\ A \\ \varepsilon \end{pmatrix} := \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A + R(\xi, X) \\ \nabla_X \varepsilon - \frac{1}{24} X \cdot F \cdot \varepsilon + \frac{1}{8} F \cdot X \cdot \varepsilon \end{pmatrix}$$

$$\mathfrak{k}_{\bar{0}} \cong \left\{ \begin{pmatrix} \xi \\ A \end{pmatrix} \in \mathcal{E}_{\bar{0}} \,\middle|\, D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = 0, \quad \nabla_\xi F + AF = 0 \right\}$$

$$\mathfrak{k}_{\bar{1}} \cong \{ \varepsilon \in \mathcal{E}_{\bar{1}} \mid D_X \varepsilon = 0 \}$$

# Localisation

$$p \in M \qquad V = T_pM \qquad S = \$_p$$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k}_{\bar{0}} \xrightarrow{\mathrm{ev_p}} V' \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{k}_{\bar{1}} \xrightarrow{\mathrm{ev_p}} S' \longrightarrow 0$$

$$\mathfrak{k} \overset{\text{v.s.}}{\cong} V' \oplus S' \oplus \mathfrak{h} \subset V \oplus S \oplus \mathfrak{so}(V) =: \mathfrak{p}$$

Poincaré superalgebra

# A filtered Lie superalgebra

**Theorem** [JMF+Santi (2016)]

1. $\mathfrak{k}$ is a filtered Lie superalgebra, and
2. its associated graded algebra is isomorphic to a Lie subalgebra of $\mathfrak{p}$.

"$\mathfrak{k}$ is a **filtered subdeformation** of $\mathfrak{p}$"

# What is this good for?

We can attack the classification of supersymmetric supergravity backgrounds by classifying their Killing superalgebras, whose algebraic structure is now very much under control.

We have already recovered the classification of maximally supersymmetric backgrounds from this approach.

Question: Is every filtered subdeformations of the Poincaré superalgebra which *could* be a Killing superalgebra *actually* the Killing superalgebra of a supersymmetric background?

There is potential counterexample.

# A geometric interlude

where I try to convince you that the “super-ization” is actually not all that exotic



Back

# Geometric Killing spinors

$$(M,g) \qquad \begin{array}{c} \$ \\ \downarrow \\ M \end{array} \qquad \varepsilon \in \$$$

(geometric) Killing spinor

$$\nabla_X \varepsilon = \lambda X \cdot \varepsilon \qquad \forall X \in TM$$

$$\text{scal} \sim \lambda^2 \implies \lambda \in \mathbb{R} \sqcup i\mathbb{R}$$

Squaring geometric Killing spinors yields (conformal) Killing vectors.

The Lie derivative of a Killing spinor along a Killing vector is again a Killing spinor.

# Killing ¾-Lie algebras

$$\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$$

$$\mathfrak{k}_{\bar{0}}(M,g) = \{\xi \in TM \mid \mathcal{L}_\xi g = 0\}$$

$$\mathfrak{k}_{\bar{1}}(M,g) = \{\varepsilon \in \$ \mid \nabla_X \varepsilon = \lambda X \cdot \varepsilon \quad \forall X \in TM\}$$

Brackets

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}]$ Lie bracket

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}]$ spinorial Lie derivative

$[\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$ Dirac current

Jacobi identities?

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}]$ ✓

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}]$ ✓

$[\mathfrak{k}_{\bar{0}}\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$ ✓

$[\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}\mathfrak{k}_{\bar{1}}]$ ?

# Some Killing Lie algebras

[JMF (2007)]

(octonion) Hopf fibration

$$\begin{array}{ccc} S^7 & \hookrightarrow & S^{15} \\ & & \downarrow \\ & & S^8 \end{array}$$

$$\mathfrak{k}(S^7) \cong \mathfrak{so}(9)$$

$$\mathfrak{k}(S^8) \cong \mathfrak{f}_4$$

$$\mathfrak{k}(S^{15}) \cong \mathfrak{e}_8$$

There are similar constructions for all simple Lie algebras and even some simple Lie superalgebras.

[de Medeiros (2014)]

# Killing spinor equations from cohomology

where I present a systematic approach to determining which spinor equations give rise to a Lie superalgebra — in the context of Poincaré supersymmetry



Back

# Deformations

- **Deformations** of algebraic structures are typically governed by a **cohomology theory**.

- **Lie** (super)algebra deformations are governed by **Chevalley—Eilenberg** cohomology.

- **Filtered** deformations are governed by **generalised Spencer cohomology**: a bigraded refinement of Chevalley—Eilenberg cohomology.

# (Generalised) Spencer cohomology

The terms of smallest positive degree in a filtered deformation $\mathfrak{g}$ of $\mathfrak{a}$ define a cocycle in bidegree (2,2) of a generalised Spencer cohomology theory associated to $\mathfrak{a}$.

In our applications, $\mathfrak{a}$ is a graded subalgebra of the Poincaré superalgebra $\mathfrak{p}$

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2} = \mathfrak{so}(V) \oplus S \oplus V$$

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} = \mathfrak{h} \oplus S' \oplus V'$$

The relevant cohomology group is

$$H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{a}_0}$$

$$\mathfrak{a}_- = \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} = S' \oplus V'$$

A first step in the calculation is to determine

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$$

and this yields the (differential) Killing spinor equations!

# Killing spinor equations

One component of the Spencer cocycle gives a linear map

$$\beta : V \otimes S \to S$$

defining

$$\beta \in \Omega^1(M, \mathrm{End}\,\$)$$

and hence a spinor connection

$$D = \nabla - \beta$$

# Killing superalgebras

In many cases, the **Killing spinors**

$$D\varepsilon = 0$$

generate a Lie superalgebra.

(This is analogous to integrating an infinitesimal deformation.)

# Examples

$\mathfrak{p}$ = 11-dimensional Poincaré superalgebra

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^4 V \qquad \text{(as } \mathfrak{so}(V) \text{ reps)}$$

$$F \in \Lambda^4 V \qquad \beta(v,s) = \tfrac{1}{24} v \cdot F \cdot s - \tfrac{1}{8} F \cdot v \cdot s$$

the 4-form!

the gravitino connection!

(The existence of the Killing superalgebra (seems to) require that $dF=0$.)

$\mathfrak{p}$ = 4-dimensional *N=1* Poincaré superalgebra

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V$$

$$A \in \Lambda^0 V \quad B \in \Lambda^4 V \quad C \in \Lambda^1 V$$

$$\beta(v, s) = v \cdot (A + B) \cdot s - C \cdot v \cdot \mathrm{vol} \cdot s + \eta(C, v)\, \mathrm{vol} \cdot s$$

“old” off-shell formulation of *d=4 N=1* supergravity!

(The Killing superalgebra exists without any differential constraints on $A$,$B$,$C$.)

# Summary

- We may make the provocative claim to have derived eleven-dimensional supergravity from generalised Spencer cohomology!
- In the same way we claim to have derived the (old, minimal) off-shell formulation of four-dimensional supergravity.
- Other supergravity theories in other dimensions are work in progress.