Homogeneous M2 duals

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MOTIVATION

AdS
/
CFT

Underlying symmetry argument for AdS/CFT:

Symmetry superalgebra
of near horizon limit
of $(d\text{-}1)$-brane

$$\cong$$

Conformal superalgebra
of d-dimensional
CFT

Underlying symmetry argument for AdS/CFT:

Symmetry superalgebra of near horizon limit of $M2$-brane $\cong$ Conformal superalgebra of 3-dimensional CFT

In this talk: $d = 3$

The superalgebra: $\mathfrak{osp}(N|4)$

$\mathfrak{g} = \mathfrak{osp}(N|4)$ $\qquad$ $\mathfrak{g}_0 = \mathfrak{o}(N) \oplus \mathfrak{sp}(4;\mathbb{R})$

$$\mathfrak{sp}(4;\mathbb{R}) \cong \mathfrak{o}(3,2)$$

Lie algebra of isometries of AdS_4 $\cong$ Lie algebra of conformal isometries of $Mink_3$

This suggests dual geometries of the form

$$\mathrm{AdS}_4 \times X^7$$

For Freund-Rubin backgrounds ($F \propto \mathrm{dvol}_{\mathrm{AdS}_4}$)

$$N \geq 4 \quad \Longrightarrow \quad X^7 = S^7/\Gamma \qquad \exists\, \Gamma \subset \mathrm{SO}(8)$$

Such backgrounds have been classified:

Morrison+Plesser (1999)
de Medeiros+JMF+Ghadia+Méndez-Escobar (2009)
de Medeiros+JMF (2010)

The group Γ is a fibred product of ADE subgroups of $Sp(1)$. (cf. my talk here in April)

There are also examples of backgrounds with geometry

$$\mathrm{AdS}_4 \times X^7$$

but with nonvanishing internal flux.

They will be reviewed later: they are constructed from a Freund-Rubin background by adding a “torsion” term to the 3-form potential.

Natural Question

¿ Are there M-theory backgrounds with symmetry superalgebra $\mathfrak{osp}(N|4)$ which are **not** Freund-Rubin $AdS_4 \times X$?

NB: Any such background would be

$$\frac{N}{8} - \text{ supersymmetric.}$$

Homogeneity Conjecture

> “ M-theory backgrounds preserving $>\frac{1}{2}$ of the supersymmetry are (locally) homogeneous. ”

Strong form: the Killing vectors generated by the Killing spinors already acts locally transitively.

Known to hold for type I and heterotic SUGRAs, and for $>\frac{3}{4}$ -BPS M-theory and IIB backgrounds.

JMF+Meessen+Philip (2004)
JMF+Hackett-Jones+Moutsopoulos (2007)

Therefore...

Assuming (the strong form of) the homogeneity conjecture, an M-theory background with symmetry superalgebra $\mathfrak{osp}(N|4)$ and $N>4$ must be (locally) homogeneous under the action of $\mathrm{SO}(N)\times\mathrm{SO}(3,2)$.

Programme

(Work in progress with Mara Ungureanu.)

Classify (up to local isometry) 11-dimensional homogeneous lorentzian manifolds with an isometric transitive action of $\mathrm{SO}(N)\times\mathrm{SO}(3,2)$ for $N>4$.

Identify those which give rise to eleven-dimensional supergravity backgrounds.

Determine how much supersymmetry is preserved by the backgrounds.

HOMOGENEOUS LORENTZIAN MANIFOLDS

A (locally) homogeneous lorentzian manifold is (locally) isometric to $\mathrm{G/H}$. We are interested in $\mathrm{G{=}SO}(N)\times\mathrm{SO(3,2)}$ for $N{>}4$ and H a closed subgroup of G. Since $\mathrm{dim(G/H){=}11}$,

$$\mathrm{dimH{+}1{=}dimSO(N)}.$$

These are characterised by the following algebraic data:

$$\mathfrak{h} \subset \mathfrak{g} = \mathfrak{so}(N) \oplus \mathfrak{so}(3,2) \qquad \dim \mathfrak{h} = \binom{N}{2} - 1$$

fixing a lorentzian inner product on $\mathfrak{g}/\mathfrak{h}$.

(Lie algebraic) **Goursat Lemma**:

Every $\mathfrak{h} \subset \mathfrak{so}(N) \oplus \mathfrak{so}(3,2)$ is a fibred product of two subalgebras $\mathfrak{h}_1 \subset \mathfrak{so}(N)$ and $\mathfrak{h}_2 \subset \mathfrak{so}(3,2)$ over a common quotient.

$$\begin{tikzcd}
\mathfrak{h} \arrow[r] \arrow[d] & \mathfrak{h}_1 \arrow[d] \\
\mathfrak{h}_2 \arrow[r] & \mathfrak{q}
\end{tikzcd}$$

Let's meet the Lie subalgebras of $\mathfrak{so}(3,2)$!

Patera+Sharp+Winternitz+Zassenhaus (1977)

TABLE IV. Subalgebras of the similitude algebra.

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{7,1}$	$F;K_1,K_2,L_3,P_0,P_1,P_2$	LSim(2,1)	self	$(K_1P_2-K_2P_1-L_3P_0)^2/$ $(P_0^2-P_1^2-P_2^2)$	$a_{6,1},a_{6,2},a_{5,7},$ $a_{4,4}$
$a_{6,1}$	$;K_1,K_2,L_3,$ P_0,P_1,P_2	LE(2,1)	$a_{7,1}$	$P_0^2-P_1^2-P_2^2,$ $K_1P_2-K_2P_1-L_3P_0$	$a_{5,1},a_{4,3},a_{3,24}$
$a_{6,2}$	$F,K_2;$ K_1-L_3,P_0,P_1,P_2	$F\square A_{5,30}^0$	self	none	$a_{5,1},a_{5,2},a_{5,3},$ $a_{5,4}^a,a_{5,5},a_{5,6}\;a_{5,8}$
$a_{5,1}$	$K_2;L_3-K_1,$ P_0,P_1,P_2	$A_{5,30}^0$	$a_{6,2}$	$P_0^2-P_1^2-P_2^2$	$a_{4,1},a_{4,7},a_{4,14}$
$a_{5,2}$	$F-K_2;$ $-K_1+L_3,P_0,\;P_1,P_2$	$A_{5,30}^{-1}$	$a_{6,2}$	P_0-P_2	$a_{4,2},a_{4,7},a_{4,13}$
$a_{5,3}$	$F+K_2;$ $-K_1+L_3,P_0,P_1,P_2$	$A_{5,30}^1$	$a_{6,2}$	$(P_0^2-P_1^2-P_2^2)/$ (P_0-P_2)	$a_{4,2},a_{4,7},a_{4,12}^\epsilon,$ $\bar{a}_{4,18}$
$a_{5,4}^a$ $a\neq 0,\pm 1$	$F+aK_2;$ $-K_1+L_3,P_0,P_1,P_2$	$A_{5,30}^{1/a}$	$a_{6,2}$	$(P_0-P_2)^{2/(1+a)}/$ $(P_0^2-P_1^2-P_2^2)$	$a_{4,7},a_{4,10}^{-a},a_{4,15}^a,a_{4,16}^\epsilon\,(a=\frac{1}{2})$
$a_{5,5}$	$F,L_3-K_1;$ P_0,P_1,P_2	$A_{5,32}^0$	$a_{6,2}$	$(P_0^2-P_1^2-P_2^2)/$ $(P_0-P_2)^2$	$a_{4,7},a_{4,8}^\epsilon,a_{4,9},\tilde{a}_{4,14}$
$a_{5,6}$	$F,K_2;P_0,$ P_1,P_2	$A_{5,33}^{1/2,1/2}$	self	$(P_0^2-P_1^2-P_2^2)/\,P_1^2$	$a_{4,1},a_{4,2},a_{4,5},$ $a_{4,6},a_{4,9},\;a_{4,10}^b$
$a_{5,7}$	$F,L_3;P_0,P_1,\;P_2$	$A_{5,35}^{0,1}$	self	$(P_0^2-P_1^2-P_2^2)/\,P_0^2$	$a_{4,3},a_{4,9},a_{4,11},$ $a_{4,17},a_{3,5}$
$a_{5,8}$	$F,K_2;L_3-K_1,$ P_0-P_2,P_1	$A_{5,36}$	self	$[(L_3-K_1)P_1/$ $(P_0-P_2)]+K_2-F$	$a_{4,5},\tilde{a}_{4,5},a_{4,13},a_{4,14},$ $\tilde{a}_{4,14},a_{4,15}^b,\;\bar{a}_{4,18}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{4,1}$	$P_1 \oplus$ $\{K_2; P_0, P_2\}$	$4_1 \oplus A_{3,4}$	$a_{5,6}$	$P_1, P_0^2 - P_2^2$	$a_{3,1}, a_{3,2}, a_{3,13}, a_{3,14}^{\epsilon}$
$a_{4,2}$	$P_0 - P_2 \oplus$ $\{F - K_2; P_0 + P_2, P_1\}$	$A_1 \oplus A_{3,5}^{1/2}$	$a_{5,6}$	$P_0 - P_2,$ $(P_0 + P_2)/P_1^2$	$a_{3,1}, a_{3,6}, a_{3,7},$ $a_{3,16}, a_{3,17}^{\epsilon}$
$a_{4,3}$	$P_0 \oplus \{L_3; P_1, P_2\}$	$A_1 \oplus A_{3,6}$	$a_{5,7}$	$P_0, P_1^2 + P_2^2$	$a_{3,1}, a_{3,21}, a_{3,22}^{\epsilon}, a_{2,3}$
$a_{4,4}$	$F \oplus \{; K_1, K_2, L_3\}$	$A_1 \oplus A_{3,8}$	self	$F, K_1^2 + K_2^2 - L_3^2$	$a_{3,3}, a_{3,24}, a_{2,7}$
$a_{4,5}$	$\{K_2; P_0 - P_2\} \oplus$ $\{F - K_2; P_1\}$	$A_2 \oplus A_2$	self	none	$a_{3,2}, \tilde{a}_{3,3}, a_{3,4}, a_{3,7},$ $\tilde{a}_{3,10}, a_{3,15}, a_{3,16}, a_{3,19}^{\epsilon}$
$\tilde{a}_{4,5}$	$\{F; P_0 - P_2\} \oplus \{F$ $- K_2, L_3 - K_1\}$	$A_2 \oplus A_2$	self	none	$\tilde{a}_{3,2}, a_{3,3}, a_{3,4}, \tilde{a}_{3,7}, a_{3,10},$ $\tilde{a}_{3,15}, \tilde{a}_{3,16}, \tilde{a}_{3,19}^{\epsilon}$
$a_{4,6}$	$\{F + K_2; P_0 - P_2\} \oplus$ $\{F - K_2; P_0 + P_2\}$	$A_2 \oplus A_2$	self	none	$a_{3,4}, a_{3,6}, a_{3,12},$ $a_{3,13}, a_{3,20}^{\epsilon}$
$a_{4,7}$	$L_3 - K_1, P_0 + P_2;$ $P_0 - P_2, P_1$	$A_{4,1}$	$a_{6,2}$	$P_0 - P_2, P_0^2$ $- P_1^2 - P_2^2$	$a_{3,1}, a_{3,8}^{\epsilon}, \bar{a}_{3,25}$
$a_{4,8}^{\epsilon}$ $\epsilon = 1$ $[\epsilon = \pm 1]$	$F + \epsilon(L_3 - K_1);$ P_0, P_1, P_2	$A_{4,4}$	$a_{5,5}$	$(P_0 - P_2)^2/(P_0^2 - P_1^2 - P_2^2),$ $(P_0 - P_2)^{\epsilon} \exp[P_1/(P_2 - P_0)]$	$a_{3,1}, \tilde{a}_{3,9}^{\epsilon}$
$a_{4,9}$	$F; P_0, P_1, P_2$	$A_{4,5}^{1,1}$	$a_{7,1}$	$P_1/P_0,\ P_2/P_0$	$a_{3,1}, \tilde{a}_{3,10},$ $a_{3,11}, a_{3,12}$
$a_{4,10}^{b}$ $b > 0, \neq 1$	$F - bK_2;$ P_0, P_1, P_2	$A_{4,5}^{1/(1-b),(1+b)/(1-b)}$	$a_{5,6}$	$P_1^2/(P_0^2 - P_1^2 - P_2^2),$ $P_1^{2b}(P_0 - P_2)/(P_0 + P_2)$	$a_{3,1}, a_{3,15}(b = 2)$ $a_{3,19}^{b}, a_{3,20}^{-b}$
$a_{4,11}^{b}$ $b > 0[b \neq 0]$	$F + bL_3;$ P_0, P_1, P_2	$A_{4,6}^{1/b,1/b}$	$a_{5,7}$	$P_0^2/(P_1^2 + P_2^2), (P_1^2 + P_2^2)^b$ $(P_1 + iP_2)^i (P_1 - iP_2)^{-i}$	$a_{3,1}, a_{3,23}^{b},$ $a_{2,17}^{b}$
$a_{4,12}^{\epsilon}$ $\epsilon = 1^*$ $[\epsilon = \pm 1]$	$F + K_2 + \epsilon(P_0 + P_2);$ $-K_1 + L_3, P_0 - P_2, P_1$	$A_{4,7}$	$a_{5,3}$	none	$a_{3,17}^{\epsilon}, \bar{a}_{3,25}$
$a_{4,13}$	$F - K_2;$ $P_0 - P_2, -K_1 + L_3, P_1$	$A_{4,8}$	$a_{5,8}$	$P_0 - P_2, (P_0 - P_2)(F - K_2)$ $- P_1(-K_1 + L_3)$	$a_{3,7}, \tilde{a}_{3,7}, \bar{a}_{3,25}$
$a_{4,14}$	$K_2, P_1;$ $-K_1 + L_3, P_0 - P_2$	$A_{4,9}^{0}$	$a_{5,8}$	none	$a_{3,2}, a_{3,9}^{\epsilon}, a_{3,10},$ $\bar{a}_{3,25}$
$\tilde{a}_{4,14}$	$F, -K_1 + L_3;$ $P_1, P_0 - P_2$	$A_{4,9}^{0}$	$a_{5,8}$	none	$\tilde{a}_{3,2}, \tilde{a}_{3,9}^{\epsilon}, \tilde{a}_{3,10}, \bar{a}_{3,25}$
$a_{4,15}^{b}$ $0 < \lvert b \rvert < 1$ $[b \neq 0, \pm 1]$	$F + bK_2; -K_1 + L_3,$ $P_0 - P_2, P_1$	$A_{4,9}^{h}$ $h = b, \lvert b \rvert < 1$ $h = b^{-1}, \lvert b \rvert > 1$	$a_{5,8}$	none	$a_{3,15}(b = -2), \tilde{a}_{3,15}(b = -\frac{1}{2}),$ $a_{3,19}^{b}, \tilde{a}_{3,19}^{b}, \bar{a}_{3,25}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a^{\epsilon}_{4,16}$ $\epsilon = 1 * [\epsilon = \pm 1]$	$F + \frac{1}{2}K_2; -K_1 + L_3 + \epsilon(P_0 + P_2), P_0 - P_2, P_1$	$A^{1/2}_{4,9}$	$a^{1/2}_{5,4}$	none	$a^{\epsilon}_{3,8}, a^{\epsilon}_{3,18}, a^{1/2}_{3,19}$
$a_{4,17}$	$F, L_3; P_1, P_2$	$A_{4,12}$	self	none	$a_{3,11}, a_{3,21}, a^{c}_{3,23}, a_{2,7}$
$\bar{a}_{4,18} = b_{4,5}$	$F + K_2; -K_1 + L_3, P_0 - P_2, P_1$	$A^{1}_{4,9}$	$b_{7,1}$	none	$a_{3,16}, \tilde{a}_{3,16}, \bar{a}_{3,25}$
$a_{3,1}$	P_0, P_1, P_2	$3A_1$	$a_{7,1}$	P_0, P_1, P_2	$a_{2,2}, a_{2,4}, a_{2,5}$
$a_{3,2}$	$P_1 \oplus \{K_2; P_0 - P_2\}$	$A_1 \oplus A_2$	$a_{4,5}$	P_1	$\tilde{a}_{2,1}, a_{2,2}, a_{2,11}, a^{\epsilon}_{2,12}$
$\tilde{a}_{3,2}$	$-K_1 + L_3 \oplus \{F; P_0 - P_2\}$	$A_1 \oplus A_2$	$\tilde{a}_{4,5}$	$-K_1 + L_3$	$a_{2,1}, \tilde{a}_{2,2}, \tilde{a}_{2,11}, \tilde{a}^{\epsilon}_{2,12}$
$a_{3,3}$	$F \oplus \{K_2; -K_1 + L_3\}$	$A_1 \oplus A_2$	self	F	$a_{2,1}, a_{2,6}, a_{2,10}, a^{d}_{2,14}, \tilde{a}^{\epsilon}_{2,15}$
$\tilde{a}_{3,3}$	$K_2 \oplus \{F; P_1\}$	$A_1 \oplus A_2$	self	K_2	$\tilde{a}_{2,1}, a_{2,6}, \tilde{a}_{2,10}, \tilde{a}^{d}_{2,14}, a_{2,15}$
$a_{3,4}$	$F - K_2 \oplus \{F + K_2; P_0 - P_2\}$	$A_1 \oplus A_2$	self	$F - K_2$	$a_{2,6}, a_{2,8}, a_{2,11}, \tilde{a}_{2,11}, a^{d}_{2,18}, \bar{a}_{2,21}$
$a_{3,5}$	$L_3 \oplus \{F; P_0\}$	$A_1 \oplus A_2$	self	L_3	$a_{2,3}, a_{2,7}, a_{2,13}, a^{d}_{2,17}$
$a_{3,6}$	$P_0 - P_2 \oplus \{F - K_2; P_0 + P_2\}$	$A_1 \oplus A_2$	$a_{4,6}$	$P_0 - P_2$	$a_{2,5}, a_{2,8}, a^{\epsilon}_{2,19}, \bar{a}_{2,21}$
$a_{3,7}$	$P_0 - P_2 \oplus \{F - K_2; P_1\}$	$A_1 \oplus A_2$	$a_{4,5}$	$P_0 - P_2$	$a_{2,2}, a_{2,8}, a_{2,15}, a^{\epsilon}_{2,16}$
$\tilde{a}_{3,7}$	$P_0 - P_2 \oplus \{F - K_2; -K_1 + L_3\}$	$A_1 \oplus A_2$	$\tilde{a}_{4,5}$	$P_0 - P_2$	$\tilde{a}_{2,2}, a_{2,8}, \tilde{a}^{1}_{2,15}, \tilde{a}^{\epsilon}_{2,16}$
$a^{\epsilon}{}_{3,8}$ $\epsilon = 1 * [\epsilon = \pm 1]$	$-K_1 + L_3 + \epsilon(P_0 + P_2), P_1; P_0 - P_2$	$A_{3,1}$	$a^{1/2}_{5,4}$	$P_0 - P_2$	$a_{2,2}, a^{\epsilon}_{2,9}$
$a^{\epsilon}_{3,9}$ $\epsilon = 1 \ [\epsilon = \pm 1]$	$K_2 - \epsilon P_1; -K_1 + L_3, P_0 - P_2$	$A_{3,2}$	$a_{4,14}$	$(P_0 - P_2)^{-\epsilon} \exp\{(-K_1 + L_3)/(P_0 - P_2)\}$	$\tilde{a}_{2,2}, a^{\epsilon}_{2,12}$
$\tilde{a}^{\epsilon}_{3,9}$ $\epsilon = 1 \ [\epsilon = \pm 1]$	$F + \epsilon(-K_1 + L_3); P_1, P_0 - P_2$	$A_{3,2}$	$\tilde{a}_{4,14}$	$(P_0 - P_2)^{-\epsilon} \exp\{P_1/(P_0 - P_2)\}$	$a_{2,2}, \tilde{a}^{\epsilon}_{2,12}$
$a_{3,10}$	$K_2; -K_1 + L_3, P_0 - P_2$	$A_{3,3}$	$a_{5,8}$	$(-K_1 + L_3)/(P_0 - P_2)$	$\tilde{a}_{2,2}, a_{2,10}, a_{2,11}$
$\tilde{a}_{3,10}$	$F; P_1, P_0 - P_2$	$A_{3,3}$	$a_{5,8}$	$P_1/(P_0 - P_2)$	$a_{2,2}, \tilde{a}_{2,10}, \tilde{a}_{2,11}$
$a_{3,11}$	$F; P_1, P_2$	$A_{3,3}$	$a_{4,17}$	P_1/P_2	$a_{2,4}, \tilde{a}_{2,10}$
$a_{3,12}$	$F; P_0, P_2$	$A_{3,3}$	$a_{4,6}$	P_0/P_2	$a_{2,5}, \tilde{a}_{2,10}, \tilde{a}_{2,11}, a_{2,13}$
$a_{3,13}$	$K_2; P_0, P_2$	$A_{3,4}$	$a_{5,6}$	$P_0^2 - P_2^2$	$a_{2,5}, a_{2,11}$
$a^{\epsilon}_{3,14}$ $\epsilon = 1 \ [\epsilon = \pm 1]$	$K_2 - \epsilon P_1; P_0, P_2$	$A_{3,4}$	$a_{4,1}$	$P_0^2 - P_2^2$	$a_{2,5}, a^{\epsilon}_{2,12}$
$a_{3,15}$	$F - 2K_2; P_1, P_0 - P_2$	$A_{3,4}$	$a_{4,5}$	$P_1(P_0 - P_2)$	$a_{2,2}, \tilde{a}^{2}_{2,14}, a^{-2}_{2,18}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\tilde{a}_{3,15}$	$F-\frac{1}{2}K_2;$ $-K_1+L_3, P_0-P_2$	$A_{3,4}$	$\tilde{a}_{4,5}$	$(-K_1+L_3)(P_0-P_2)$	$\tilde{a}_{2,2}, a^{1/2}_{2,14}, a^{-1/2}_{2,18}$
$a_{3,16}$	$F+K_2;P_1,$ P_0-P_2	$A^{1/2}_{3,5}$	$a_{5,6}$	$P_1^2/(P_0-P_2)$	$a_{2,2}, a_{2,15}, \bar{a}_{2,21}$
$\tilde{a}_{3,16}$	$F+K_2; -K_1+L_3, P_0-P_2$	$A^{1/2}_{3,5}$	$\sim a_{5,6}$	$(-K_1+L_3)^2/$ (P_0-P_2)	$\tilde{a}_{2,2}, \tilde{a}^{-1}_{2,15}, \bar{a}_{2,21}$
$a^{\epsilon}_{3,17}$ $\epsilon=1*$ $[\epsilon=\pm 1]$	$F+K_2+\epsilon(P_0+P_2);$ P_0-P_2, P_1	$A^{1/2}_{3,5}$	$a_{4,2}$	$P_1^2/(P_0-P_2)$	$a_{2,2}a^{\epsilon}_{2,16}, a^{\epsilon}_{2,19}$
$a^{\epsilon}_{3,18}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$F+\frac{1}{2}K_2; -K_1+L_3$ $+\epsilon(P_0+P_2), P_0-P_2$	$A^{1/3}_{3,5}$	self	$(P_0-P_2)/[-K_1+L_3$ $+\epsilon(P_0+P_2)]^3$	$a^{\epsilon}_{2,9}, a^{1/2}_{2,18}, a^{\epsilon}_{2,20}$
$a^{c}_{3,19}$ $c\neq 0, \pm 1, -2$	$F+cK_2;P_1,$ P_0-P_2	$A^{h}_{3,5}$ $h=1+c, -2<c<0$ $h=(1+c)^{-1}, c<-2$ or $c>0$	$a_{4,5}$	$(P_0-P_2)P_1^{-1-c}$	$a_{2,2}, \tilde{a}^{-c}_{2,14}, a^{c}_{2,18}$
$\tilde{a}^{c}_{3,19}$ $c\neq 0, -\frac{1}{2}, \pm 1$	$F+cK_2;$ $-K_1+L_3, P_0-P_2$	$A^{h}_{3,5}$ $h=\frac{c}{c+1}, c>-\frac{1}{2}$ $h=\frac{c+1}{c}, c<-\frac{1}{2}$	$\tilde{a}_{4,5}$	(P_0-P_2) $(L_3-K_1)^{-(1+c)/c}$	$\tilde{a}_{2,2}, a^{-c}_{2,14}, a^{c}_{2,18}$
$a^{c}_{3,20}$ $c>0, c\neq 1$	$F+cK_2;P_0, P_2$	$A^{(1-c)/(1+c)}_{3,5}$	$a_{4,6}$	(P_0+P_2) $(P_0-P_2)^{(-1+c)/(1+c)}$	$a_{2,5}, a^{c}_{2,18}$
$a_{3,21}$	$L_3;P_1, P_2$	$A_{3,6}$	$a_{5,7}$	$P_1^2+P_2^2$	$a_{2,4}, \bar{a}_{1,10}$
$a^{\epsilon}_{3,22}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$L_3+\epsilon P_0;P_1, P_2$	$A_{3,6}$	$a_{4,3}$	$P_1^2+P_2^2$	$a_{2,4}, a^{\epsilon}_{1,6}$
$a^{c}_{3,23}$ $c>0$ $[c\neq 0]$	$F+cL_3;P_1, P_2$	$A^{1/c}_{3,7}$	$a_{4,17}$	$(P_1^2+P_2^2)^c$ $(P_1+iP_2)^i(P_1-iP_2)^{-i}$	$a_{2,4}, a^{c}_{1,}$[illegible]
$a_{3,24}$	$;K_1, K_2, L_3$	$A_{3,8}$	$a_{4,4}$	$K_1^2+K_2^2-L_3^2$	$a_{2,10}, \bar{a}_{1,10}$
$\bar{a}_{3,25}=b_{3,2}$	$-K_1+L_3, P_1:$ P_0-P_2	$A_{3,1}$	$b_{7,1}$	P_0-P_2	$a_{2,2}, \tilde{a}_{2,2}$
$a_{2,1}$	$F, -K_1+L_3$	$2A_1$	$a_{3,3}$	$F, -K_1+L_3$	$a_{1,1}, \tilde{a}_{1,2}, a^{\epsilon}_{1,4}$
$\tilde{a}_{2,1}$	K_2, P_1	$2A_1$	$\tilde{a}_{3,3}$	K_2, P_1	$\tilde{a}_{1,1}, a_{1,2}, \tilde{a}^{\epsilon}_{1,4}$
$a_{2,2}$	P_1, P_0-P_2	$2A_1$	$a_{6,2}$	P_1, P_0-P_2	$a_{1,2}, \bar{a}_{1,11}$
$\tilde{a}_{2,2}$	$-K_1+L_3,$ P_0-P_2	$2A_1$	$b_{6,2}$	$-K_1+L_3,$ P_0-P_2	$\tilde{a}_{1,2}, \bar{a}_{1,11}$
$a_{2,3}$	L_3, P_0	$2A_1$	$a_{3,5}$	L_3, P_0	$a_{1,3}, a^{\epsilon}_{1,6}, \bar{a}_{1,10}$
$a_{2,4}$	P_1, P_2	$2A_1$	$a_{5,7}$	P_1, P_2	$a_{1,2}$
$a_{2,5}$	P_0, P_2	$2A_1$	$a_{5,6}$	P_0, P_2	$a_{1,2}, a_{1,3}, \bar{a}_{1,11}$
$a_{2,6}$	F, K_2	$2A_1$	self	F, K_2	$a_{1,1}, \tilde{a}_{1,1}, a^{\epsilon}_{1,8},$ $\bar{a}_{1,12}$
$a_{2,7}$	F, L_3	$2A_1$	self	F, L_3	$a_{1,1}, a^{\epsilon}_{1,7}, \bar{a}_{1,10}$
$a_{2,8}$	$F-K_2,$ P_0-P_2	$2A_1$	$a_{3,4}$	$F-K_2, P_0-P_2$	$a^{\epsilon}_{1,9}, \bar{a}_{1,11}, \bar{a}_{1,12}$
$a^{\epsilon}_{2,9}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$L_3-K_1+\epsilon(P_0+P_2),$ P_0-P_2	$2A_1$	$a^{\epsilon}_{3,18}$	$L_3-K_1+\epsilon(P_0+P_2),$ P_0-P_2	$a^{\epsilon}_{1,5}, \bar{a}_{1,11}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{2,10}$	$K_2; -K_1+L_3$	A_2	$a_{3,3}$	none	$\tilde{a}_{1,1}, \tilde{a}_{1,2}$
$\tilde{a}_{2,10}$	$F;P_1$	A_2	$\tilde{a}_{3,3}$	none	$a_{1,1}, a_{1,2}$
$a_{2,11}$	$K_2;P_0-P_2$	A_2	$a_{4,5}$	none	$\tilde{a}_{1,1}, \bar{a}_{1,11}$
$\tilde{a}_{2,11}$	$F;P_0-P_2$	A_2	$\tilde{a}_{4,5}$	none	$a_{1,1}, \bar{a}_{1,11}$
$a^{\epsilon}_{2,12}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$K_2-\epsilon P_1;P_0-P_2$	A_2	$a_{3,2}$	none	$\tilde{a}^{\epsilon}_{1,4}, \bar{a}_{1,11}$
$\tilde{a}^{\epsilon}_{2,12}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$F+\epsilon(-K_1+L_3);$ P_0-P_2	A_2	$\tilde{a}_{3,2}$	none	$a^{\epsilon}_{1,4}, \bar{a}_{1,11}$
$a_{2,13}$	$F;P_0$	A_2	$a_{3,5}$	none	$a_{1,1}, a_{1,3}$
$a^{d}_{2,14}$ $d>0, d\neq 1$ $[d\neq 0,\pm 1]$	$F-dK_2;$ $-K_1+L_3$	A_2	$a_{3,3}$	none	$\tilde{a}_{1,2}, a^{d}_{1,8}$
$\tilde{a}^{d}_{2,14}$ $d>0, d\neq 1$ $[d\neq 0,\pm 1]$	$F-dK_2;P_1$	A_2	$\tilde{a}_{3,3}$	none	$a_{1,2}, a^{d}_{1,8}$
$a_{2,15}$	$F-K_2;P_1$	A_2	$a_{4,5}$	none	$a_{1,2}, a_{1,12}$
$\tilde{a}^{\epsilon}_{2,15}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$F-\epsilon K_2;$ $-K_1+L_3$	A_2	$\tilde{a}_{4,5}$	none	$\tilde{a}_{1,2}, \bar{a}_{1,12}$
$a^{\epsilon}_{2,16}$ $\epsilon=1*$ $[\epsilon=\pm 1]$	$F-K_2+\epsilon(P_0-P_2);P_1$	A_2	$a_{3,7}$	none	$a_{1,2}, a^{\epsilon}_{1,9}$
$\tilde{a}^{\epsilon}_{2,16}$ $\epsilon=1*$ $[\epsilon=\pm 1]$	$F-K_2+\epsilon(P_0-P_2); -K_1+L_3$	A_2	$\tilde{a}_{3,7}$	none	$\tilde{a}_{1,2}, a^{\epsilon}_{1,9}$
$a^{d}_{2,17}$ $d>0$ $[d\neq 0]$	$F+dL_3;P_0$	A_2	$a_{3,5}$	none	$a_{1,3}, a^{d}_{1,7}$
$a^{d}_{2.18}$ $0<\lvert d\rvert<1$ $[d\neq 0,\pm 1]$	$F+dK_2;P_0$ $-P_2$	A_2	$a_{3,4}$	none	$a^{d}_{1,8}, \bar{a}_{1,11}$
$a^{\epsilon}_{2,19}$ $\epsilon=1*$ $[\epsilon=\pm 1]$	$F-K_2+\epsilon(P_0-P_2);$ $;P_0+P_2$	A_2	$a_{3,6}$	none	$a^{\epsilon}_{1,9}, \bar{a}_{1,11}$
$a^{\epsilon}_{2,20}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$F+\frac{1}{2}K_2; -K_1$ $+L_3+\epsilon(P_0+P_2)$	A_2	self	none	$a^{\epsilon}_{1,5}, a^{1/2}_{1,8}$
$\bar{a}_{2,21}=b_{2,4}$	$F+K_2;P_0-P_2$	A_2	$b_{5,1}$	none	$\bar{a}_{1,11}, \bar{a}_{1,12}$
$a_{1,1}$	F	A_1	$a_{4,4}$	F	
$\tilde{a}_{1,1}$	K_2	A_1	$\sim a_{4,4}$	K_2	
$a_{1,2}$	P_1	A_1	$a_{5,6}$	P_1	
$\tilde{a}_{1,2}$	$-K_1+L_3$	A_1	$\sim a_{5,6}$	$-K_1+L_3$	
$a_{1,3}$	P_0	A_1	$a_{5,7}$	P_0	
$a^{\epsilon}_{1,4}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$F+\epsilon(-K_1+L_3)$	A_1	$a_{2,1}$	$F+\epsilon(-K_1+L_3)$	
$\tilde{a}^{\epsilon}_{1,4}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$K_2-\epsilon P_1$	A_1	$\tilde{a}_{2,1}$	$K_2-\epsilon P_1$	
$a^{\epsilon}_{1,5}$ $\epsilon=1*$ $[\epsilon=\pm 1]$	$-K_1+L_3$ $+\epsilon(P_0+P_2)$	A_1	$a^{\epsilon}_{3,18}$	$-K_1+L_3$ $+\epsilon(P_0+P_2)$	
$a^{\epsilon}_{1,6}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$L_3+\epsilon P_0$	A_1	$a_{2,3}$	$L_3+\epsilon P_0$	
$a^{e}_{1,7}$ $e>0$ $[e\neq 0]$	$F+eL_3$	A_1	$a_{2,7}$	$F+eL_3$	
$a^{e}_{1,8}$ $0<e<1$ $[e>0]$	$F+eK_2$	A_1	$a_{2,6}$	$F+eK_2$	
$a^{\epsilon}_{1,9}$ $\epsilon=1$ $[\epsilon=\pm 1]$	$F-K_2$ $+\epsilon(P_0-P_2)$	A_1	$a_{2,8}$	$F-K_2+\epsilon(P_0-P_2)$	
$\bar{a}_{1,10}=d_{1,1}$	L_3	A_1	$d_{4,1}$	L_3	
$\bar{a}_{1,11}=b_{1,4}$	P_0-P_2	A_1	$b_{7,1}$	P_0-P_2	
$\bar{a}_{1,12}=b_{1,5}$	$F+K_2$	A_1	$b_{4,2}$	$F+K_2$	

TABLE V. Subalgebras of the optical algebra LOpt(2,1).

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$b_{7,1}$	$W;K_1,K_2,L_3,$ M,Q,N	LOpt(2,1)	self	$\frac{1}{N}[MQK_1+\frac12(K_2-L_3)M^2$ $-\frac12(K_2+L_3)Q^2]_{\text{sym}}+K_1^2+K_2^2-L_3^2$	$b_{6,1},\bar{b}_{6,2},b_{5,1}$ $b_{5,2}$
$b_{6,1}$	$;K_1,K_2,L_3,$ M,Q,N	LSch$_1$	$b_{7,1}$	$[MQK_1+\frac12(K_2-L_3)M^2$ $-\frac12(K_2+L_3)Q^2]_{\text{sym}}+N(K_1^2+K_2^2-L_3^2),N$	$\bar{b}_{5,4},b_{4,1},b_{4,3}$
$\bar{b}_{6,2}=a_{6,2}$	$W,K_1;K_2+L_3,$ M,Q,N		self	none	$\bar{b}_{5,3},\bar{b}_{5,4},\bar{b}_{5,5},$ $\bar{b}^a_{5,6},\bar{b}_{5,7},\bar{b}_{5,8},\bar{b}_{5,9}$
$b_{5,1}$	$\{;K_1,K_2,L_3\}$ $\oplus\{W;N\}$	$A_{3,8}\oplus A_2$	self	$K_1^2+K_2^2-L_3^2$	$b_{4,1},b_{4,2},\bar{b}_{4,9},b_{3,1}$
$b_{5,2}$	$W,L_3;M,Q,N$	$A_{5,37}$	self	$(Q^2+M^2+4NL_3)/N$	$b_{4,3},b^b_{4,4},b_{4,5},b_{3,1}$
$\bar{b}_{5,3}=a_{5,1}$	$W+K_1;K_2+L_3,$ M,Q,N	$A^0_{5,30}$	$\bar{b}_{6,2}$	$M^2+2N(K_2+L_3)$	$\bar{b}_{4,6},\bar{b}_{4,10},\bar{b}_{4,16}$
$\bar{b}_{5,4}=a_{5,2}$	$K_1;K_2+L_3,$ M,Q,N	$A^{-1}_{5,30}$	$\bar{b}_{6,2}$	N	$\bar{b}_{4,7},\bar{b}_{4,10},\bar{b}_{4,15}$
$\bar{b}_{5,5}=a_{5,3}$	$W,K_2+L_3;$ M,Q,N	$A^1_{5,30}$	$\bar{b}_{6,2}$	$2(K_2+L_3)+M^2/N$	$b_{4,5},\tilde{\bar{b}}_{4,7},\bar{b}_{4,10},\bar{b}^{\epsilon}_{4,14}$
$\bar{b}^a_{5,6}=a^{(1+a)/(1-a)}_{5,4}$ $a\neq 0,+1$	$W+aK_1;K_2+L_3,$ M,Q,N	$A^{(1-a)/(1+a)}_{5,30}$	$\bar{b}_{6,2}$	$[M^2+2N(K_2+L_3)]N^{a-1}$	$\bar{b}_{4,10},\bar{b}^{-a}_{4,13},\bar{b}^{-a}_{4,17},$ $\bar{b}_{4,18}(a=-\frac13)$
$\bar{b}_{5,7}=a_{5,5}$	$W-K_1;K_2+L_3,$ M,Q,N	$A^0_{5,32}$	$\bar{b}_{6,2}$	$[M^2+2N(K_2+L_3)]/N^2$	$\bar{b}_{4,10},\bar{b}_{4,11},\bar{b}_{4,12},\bar{b}_{4,16}$
$\bar{b}_{5,8}=a_{5,6}$	$W,K_1;K_2+L_3,M,N$	$A^{1/2,1/2}_{5,33}$	self	$(K_2+L_3)N/M^2$	$\bar{b}_{4,6},\bar{b}_{4,7},\overset{\simeq}{b}_{4,7},\bar{b}_{4,8},$ $\overset{\simeq}{b}_{4,8},\bar{b}_{4,9},\bar{b}_{4,12},\bar{b}_{4,13}$
$\bar{b}_{5,9}=a_{5,8}$	$W,K_1;M,Q,N$	$A_{5,36}$	self	$[QM+MQ-2K_1(W+2K_1)]/N$	$b_{4,5},\bar{b}_{4,8},\bar{b}_{4,15},$ $\bar{b}_{4,16},\bar{b}^b_{4,17}$
$b_{4,1}$	$N\oplus\{;K_1,K_2,L_3\}$	$A_1\oplus A_{3,8}$	$b_{5,1}$	$N,K_1^2+K_2^2-L_3^2$	$\bar{b}_{3,3},\bar{b}_{3,8},b_{2,1}$
$b_{4,2}$	$W\oplus\{;K_1,K_2,L_3\}$	$A_1\oplus A_{3,8}$	self	$W,K_1^2+K_2^2-L_3^2$	$\bar{b}_{3,3},\bar{b}_{3,7},b_{2,2}$
$b_{4,3}$	$L_3;Q,M,N$	$A_{4,10}$	$b_{5,2}$	$N,M^2+Q^2+4L_3N$	$b_{3,2},b_{2,1}$
$b^b_{4,4}$ $b>0*[b\neq 0]$	$W+bL_3;Q,M,N$	$A^{1/b}_{4,11}$	$b_{5,2}$	none	$b_{3,2},b^b_{2,3}$
$b_{4,5}=a_{4,18}$	$W;Q,M,N$	$A^1_{4,9}$	$b_{7,1}$	none	$b_{3,2},\bar{b}_{3,19}$
$\bar{b}_{4,6}=a_{4,1}$	$M\oplus\{W+K_1;K_2$ $+L_3,N\}$	$A_1\oplus A_{3,4}$	$\bar{b}_{5,8}$	$M,N(K_2+L_3)$	$\bar{b}_{3,4},\bar{b}_{3,5},\overset{\simeq}{b}_{3,5},$ $\bar{b}_{3,16},\bar{b}_{3,17}$
$\bar{b}_{4,7}=a_{4,2}$	$N\oplus\{K_1;K_2+L_3,M\}$	$A_1\oplus A^{1/2}_{3,5}$	$\bar{b}_{5,8}$	$N,(K_2+L_3)/M^2$	$\bar{b}_{3,4},\bar{b}_{3,8},\bar{b}_{3,9},$ $\overset{\simeq}{b}_{3,19},\overset{\simeq}{b}^{\epsilon}_{3,20}$
$\overset{\simeq}{b}_{4,7}\sim a_{4,2}$	$K_2+L_3\oplus\{W;M,N\}$	$A_1\oplus A^{1/2}_{3,5}$	$\bar{b}_{5,8}$	$K_2+L_3,M^2/N$	$\bar{b}_{3,4},\overset{\simeq}{b}_{3,8},\overset{\simeq}{b}_{3,9},$ $\bar{b}_{3,19},\bar{b}^{\epsilon}_{3,20}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{b}_{4,8} = a_{4,5}$	$\{W+K_1;N\}$ $\oplus \{K_1;M\}$	$A_2 \oplus A_2$	self	none	$\bar{b}_{3,5}, \bar{b}_{3,6}, \tilde{\bar{b}}_{3,7},$ $\bar{b}_{3,9}, \bar{b}_{3,12},$ $\bar{b}_{3,18}, \bar{b}_{3,19}, \tilde{\bar{b}}^c_{3,22}$
$\tilde{\bar{b}}_{4,8} \sim a_{4,5}$	$\{W+K_1;K_2+L_3\}$ $\oplus \{W;M\}$	$A_2 \oplus A_2$	self	none	$\tilde{\bar{b}}_{3,5}, \bar{b}_{3,6}, \bar{b}_{3,7}, \tilde{\bar{b}}_{3,9}, \tilde{\bar{b}}_{3,12},$ $\tilde{\bar{b}}_{3,18}, \tilde{\bar{b}}_{3,19}, \bar{b}^c_{3,22}$
$\bar{b}_{4,9} = a_{4,6}$	$\{K_1;K_2+L_3\}$ $\oplus \{W;N\}$	$A_2 \oplus A_2$	self	none	$\bar{b}_{3,7}, \tilde{\bar{b}}_{3,7}, \bar{b}_{3,8},$ $\tilde{\bar{b}}_{3,8}, \bar{b}_{3,15}, \bar{b}_{3,16}, \bar{b}^c_{3,22}$
$\bar{b}_{4,10} = a_{4,7}$	$K_2+L_3, Q;M, N$	$A_{4,1}$	$\bar{b}_{6,2}$	$N, M^2+2N(K_2+L_3)$	$b_{3,2}, \bar{b}_{3,4}, \bar{b}_{3,10}$
$\bar{b}_{4,11} \sim a^{-1}_{4,8}$	$W-K_1+Q;$ K_2+L_3, M, N	$A_{4,4}$	$\bar{b}_{5,7}$	$N\exp(-M/N),$ $(2N(K_2+L_3)+M^2)/N^2$	$\bar{b}_{3,4}, \bar{b}_{3,11}$
$\bar{b}_{4,12} = a_{4,9}$	$W-K_1:$ K_2+L_3, M, N	$A^{1,1}_{4,5}$	$a_{7,1}$	$M/N, (K_2+L_3)/N$	$\bar{b}_{3,4}, \bar{b}_{3,12}, \tilde{\bar{b}}_{3,12},$ $\bar{b}_{3,13}, \bar{b}_{3,14}, \bar{b}_{3,15}$
$\bar{b}^b_{4,13} = a^{(b-1)/(b+1)}_{4,10}$ $0 < \lvert b \rvert < 1$ $[b \neq 0, \pm 1]$	$W-bK_1;$ K_2+L_3, M, N	$A^{b,(b+1)/2}_{4,5}$	$b_{5,8}$	$N^{(1+b)/2}/M, N^b/(K_2+L_3)$	$\bar{b}_{3,4}, \bar{b}_{3,18}(b=-3), \bar{b}^b_{3,23}$ $\tilde{\bar{b}}_{3,18}(b=-\frac{1}{3}), \bar{b}^{-b}_{3,22}, \tilde{\bar{b}}^b_{3,22},$
$\bar{b}^\epsilon_{4,14} = a^\epsilon_{4,12}$ $\epsilon = 1 * [\epsilon = \pm 1]$	$W+\epsilon(K_2+L_3);$ M, Q, N	$A_{4,7}$	$\bar{b}_{5,3}$	none	$b_{3,2}, \bar{b}^\epsilon_{3,20}$
$\bar{b}_{4,15} = a_{4,13}$	$K_1;M, Q, N$	$A_{4,8}$	$\bar{b}_{5,9}$	$N, QM+MQ+4NK_1$	$b_{3,2}, \bar{b}_{3,9}$
$\bar{b}_{4,16} = a_{4,14}$	$W+K_1, M;Q, N$	$A^0_{4,9}$	$\bar{b}_{5,9}$	none	$b_{3,2}, \bar{b}_{3,5}, \bar{b}_{3,11},$ $\bar{b}_{3,12}$
$\bar{b}^b_{4,17} = a^{(1-b)/(1+b)}_{4,15}$ $b > 0, b \neq 1$	$W-bK_1;M, Q, N$	$A^{(1-b)/(1+b)}_{4,9}$	$\bar{b}_{5,9}$	none	$b_{3,2}, \bar{b}_{3,18}\ (b=3),\ \tilde{\bar{b}}^b_{3,22}$
$\bar{b}_{4,18} \sim a^{-1}_{4,16}$	$W-\frac{1}{3}K_1;$ K_2+L_3+Q, M, N	$A^{1/2}_{4,9}$	$\bar{b}^{-1/3}_{5,6}$	none	$\bar{b}_{3,10}, \bar{b}_{3,21}, \tilde{\bar{b}}^{1/3}_{3,22}$
$b_{3,1}$	$L_3 \oplus \{W;N\}$	$A_1 \oplus A_2$	self	L_3	$b_{2,1}, b_{2,2}, b^d_{2,3}, b_{2,4}$
$b_{3,2} = \bar{a}_{3,25}$	$Q, M;N$	$A_{3,1}$	$b_{7,1}$	N	$\tilde{\bar{b}}_{2,6}$
$\bar{b}_{3,3} = e_{3,1}$	$;K_1, K_2, L_3$	$A_{3,8}$	$e_{6,1}$	$K_1^2+K_2^2-L_3^2$	$\tilde{b}_{2,4}, \bar{b}_{1,6}$
$\bar{b}_{3,4} = a_{3,1}$	K_2+L_3, M, N	$3A_1$	$a_{7,1}$	K_2+L_3, M, N	$\bar{b}_{2,6}, \tilde{\bar{b}}_{2,6}, \bar{b}_{2,7},$ $\bar{b}_{2,8}, \tilde{\bar{b}}_{2,8}$
$\bar{b}_{3,5} = a_{3,2}$	$M \oplus \{W+K_1;N\}$	$A_1 \oplus A_2$	$\bar{b}_{4,8}$	M	$\bar{b}_{2,5}, \tilde{\bar{b}}_{2,6}, \bar{b}_{2,13}, \tilde{\bar{b}}_{2,14}$
$\tilde{\bar{b}}_{3,5} \sim a_{3,2}$	$M \oplus \{W+K_1;$ $K_2+L_3\}$	$A_1 \oplus A_2$	$\tilde{\bar{b}}_{4,8}$	M	$\bar{b}_{2,5}, \bar{b}_{2,6}, \overset{\approx}{\bar{b}}_{2,13}, \bar{b}_{2,14}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{b}_{3,6}\sim a_{3,3}$	$W+K_1\oplus$ $\{W-K_1;M\}$	$A_1\oplus A_2$	self	$W+K_1$	$\bar{b}_{2,5},\bar{b}_{2,9},\bar{b}_{2,12},$ $\bar{b}^{d}_{2,16},\bar{b}_{2,17},$ $\overset{\simeq}{b}_{2,17}$
$\bar{b}_{3,7}\sim a_{3,4}$	$W\oplus\{K_1;K_2+L_3\}$	$A_1\oplus A_2$	self	W	$\tilde{b}_{2,4},\bar{b}_{2,9},\overset{\simeq}{\bar{b}}_{2,10},$ $\overset{\simeq}{b}_{2,13},\overset{\approx}{\bar{b}}_{2,13},$ $\overset{\simeq}{b}{}^{d}_{2,19}$
$\overset{\simeq}{\bar{b}}_{3,7}=a_{3,4}$	$K_1\oplus\{W;N\}$	$A_1\oplus A_2$	self	K_1	$b_{2,4},\bar{b}_{2,9},\bar{b}_{2,10},$ $\bar{b}_{2,13},\bar{b}^{d}_{2,19}$
$\bar{b}_{3,8}=a_{3,6}$	$N\oplus\{K_1;K_2+L_3\}$	$A_1\oplus A_2$	$\bar{b}_{4,9}$	N	$\overset{\simeq}{\bar{b}}_{2,4},\overset{\simeq}{\bar{b}}_{2,8},\bar{b}_{2,10},$ $\bar{b}^{\epsilon}_{2,20}$
$\overset{\simeq}{\bar{b}}_{3,8}\sim a_{3,6}$	$K_2+L_3\oplus\{W;N\}$	$A_1\oplus A_2$	$\bar{b}_{4,9}$	K_2+L_3	$b_{2,4},\overset{\simeq}{\bar{b}}_{2,8},\overset{\simeq}{\bar{b}}_{2,10},$ $\overset{\simeq}{\bar{b}}{}^{\epsilon}_{2,20}$
$\bar{b}_{3,9}=a_{3,7}$	$N\oplus\{K_1;M\}$	$A_1\oplus A_2$	$\bar{b}_{4,8}$	N	$\overset{\simeq}{\bar{b}}_{2,6},\bar{b}_{2,10},\bar{b}_{2,17},$ $\bar{b}^{\epsilon}_{2,18}$
$\overset{\simeq}{\bar{b}}_{3,9}\sim a_{3,7}$	$K_2+L_3\oplus\{W;M\}$	$A_1\oplus A_2$	$\overset{\simeq}{\bar{b}}_{4,8}$	K_2+L_3	$\bar{b}_{2,6},\overset{\simeq}{\bar{b}}_{2,10},\overset{\simeq}{\bar{b}}_{2,17},$ $\overset{\simeq}{\bar{b}}{}^{\epsilon}_{2,18}$
$\bar{b}_{3,10}\sim a^{-1}_{3,8}$	$K_2+L_3+Q,M;N$	$A_{3,1}$	$\bar{b}^{-1/3}_{5,6}$	N	$\overset{\simeq}{\bar{b}}_{2,6},\bar{b}_{2,11}$
$\bar{b}_{3,11}\sim a^{-1}_{3,9}$	$W+K_1+M;Q,N$	$A_{3,2}$	$\bar{b}_{4,17}$	$N\exp(Q/N)$	$\overset{\simeq}{\bar{b}}_{2,6},\overset{\simeq}{\bar{b}}_{2,14}$
$\bar{b}_{3,12}=\tilde{a}_{3,10}$	$W-K_1;M,N$	$A_{3,3}$	$\bar{b}_{5,9}$	M/N	$\overset{\simeq}{\bar{b}}_{2,6},\bar{b}_{2,12},\bar{b}_{2,13}$
$\overset{\simeq}{\bar{b}}_{3,12}\sim\tilde{a}_{3,10}$	$W-K_1;K_2+L_3,M$	$A_{3,3}$	$\sim\bar{b}_{5,9}$	$(K_2+L_3)/M$	$\bar{b}_{2,6},\bar{b}_{2,12},\overset{\simeq}{\bar{b}}_{2,12},$ $\overset{\simeq}{b}_{2,13}$
$\bar{b}_{3,13}=a_{3,11}$	$W-K_1;K_2+L_3+N,M$	$A_{3,3}$	$a_{4,17}$	$(K_2+L_3+N)/M$	$\bar{b}_{2,7},\bar{b}_{2,12},\overset{\simeq}{\bar{b}}_{2,12}$
$\bar{b}_{3,14}\sim a_{3,12}$	$W-K_1,K_2+L_3-N,M$	$A_{3,3}$	$\sim a_{4,6}$	$(K_2+L_3-N)/M$	$\bar{b}_{2,8},\bar{b}_{2,12},\overset{\simeq}{\bar{b}}_{2,12},$ $\overset{\simeq}{\bar{b}}_{2,13},\bar{b}_{2,15}$
$\bar{b}_{3,15}=a_{3,12}$	$W-K_1;K_2+L_3,N$	$A_{3,3}$	$\bar{b}_{4,9}$	$(K_2+L_3)/N$	$\overset{\simeq}{b}_{2,8},\overset{\simeq}{\bar{b}}_{2,12},\bar{b}_{2,13},$ $\overset{\simeq}{b}_{2,13},\bar{b}_{2,15}$
$\bar{b}_{3,16}=a_{3,13}$	$W+K_1;K_2+L_3,N$	$A_{3,4}$	$\bar{b}_{5,8}$	$N(K_2+L_3)$	$\overset{\simeq}{b}_{2,8},\bar{b}_{2,13},\overset{\approx}{\bar{b}}_{2,13}$
$\bar{b}_{3,17}\sim a^{-1}_{3,14}$	$W+K_1+M;K_2+L_3,N$	$A_{3,4}$	$\bar{b}_{4,6}$	$N(K_2+L_3)$	$\overset{\simeq}{b}_{2,8},\bar{b}_{2,14},\overset{\simeq}{\bar{b}}_{2,14}$
$\bar{b}_{3,18}=a_{3,15}$	$W+3K_1;M,N$	$A_{3,4}$	$\bar{b}_{4,8}$	NM	$\overset{\simeq}{b}_{2,6},\bar{b}^{-3}_{2,16},\bar{b}^{3}_{2,19}$
$\overset{\simeq}{b}_{3,18}\sim a_{3,15}$	$W+\frac{1}{3}K_1;K_2+L_3,M$	$A_{3,4}$	$\overset{\simeq}{\bar{b}}_{4,8}$	$M(K_2+L_3)$	$\bar{b}_{2,6},\bar{b}^{-1/3}_{2,16},\overset{\simeq}{\bar{b}}{}^{-1/3}_{2,19}$
$\bar{b}_{3,19}=a_{3,16}$	$W;M,N$	$A^{1/2}_{3,5}$	$\bar{b}_{5,8}$	M^2/N	$b_{2,4},\overset{\simeq}{\bar{b}}_{2,6},\overset{\simeq}{\bar{b}}_{2,17}$
$\overset{\simeq}{b}_{3,19}\sim a_{3,16}$	$K_1;K_2+L_3,M$	$A^{1/2}_{3,5}$	$\bar{b}_{5,8}$	$M^2/(K_2+L_3)$	$\bar{b}_{2\,4},\bar{b}_{2,6},\bar{b}_{2,17}$
$\bar{b}^{\epsilon}_{3,20}=a^{\epsilon}_{3,17}$ $\epsilon=1*[\epsilon=\pm1]$	$W+\epsilon(K_2+L_3);$ M,N	$A^{1/2}_{3,5}$	$\overset{\simeq}{b}_{4,7}$	M^2/N	$\overset{\simeq}{\bar{b}}_{2,6},\overset{\simeq}{\bar{b}}{}^{-\epsilon}_{2,18},\overset{\simeq}{\bar{b}}{}^{\epsilon}_{2,20}$
$\overset{\simeq}{\bar{b}}{}^{\epsilon}_{3,20}\sim a^{\epsilon}_{3,17}$ $\epsilon=1*[\epsilon=\pm1]$	$K_1+\epsilon N;K_2+L_3,M$	$A^{1/2}_{3,5}$	$\bar{b}_{4,7}$	$M^2/(K_2+L_3)$	$\bar{b}_{2,6},\bar{b}^{\epsilon}_{2,18},\bar{b}^{\epsilon}_{2,20}$
$\bar{b}_{3,21}\sim a^{-1}_{3,18}$	$W-\frac{1}{3}K_1;K_2+L_3+Q,N$	$A^{1/3}_{3,5}$	self	$(K_2+L_3+Q)^3/N$	$\bar{b}_{2,11},\bar{b}^{1/3}_{2,19}\ \bar{b}_{2,21}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{b}^c_{3,22} \sim a^{(c+1)/(c-1)}_{3,19}$ $c \neq 0, \frac{1}{3}, \pm 1$	$W+cK_1;K_2+L_3,M$	$A^h_{3,5}\ h=\frac{2c}{c-1}, -1<c<\frac{1}{3}$ $h=\frac{c-1}{2c}, c>\frac{1}{3}$ or $c<-1$	$\overset{\approx}{b}_{4,8}$	$(K_2+L_3)M^{2c/(1-c)}$	$\bar{b}_{2,6}, \bar{b}^{-c}_{2,16}, \overset{\approx}{b}^{-c}_{2,19}$
$\overset{\approx}{b}^c_{3,22} = a^{(1-c)/(1+c)}_{3,19}$ $c \neq 0, \pm 1, -3$	$W-cK_1;M,N$	$A^h_{3,5}\ h=\frac{c+1}{2}, -3<c<1$ $h=\frac{2}{c+1}, c>1$ or $c<-3$	$\bar{b}_{4,8}$	$MN^{-(c+1)/2}$	$\overset{\approx}{b}_{2,6}, \bar{b}^c_{2,16}, \bar{b}^c_{2,19}$
$\bar{b}^c_{3,23} = a^{(1-c)/(1+c)}_{3,20}$ $0<\lvert c \rvert<1\ [c \neq 0, \pm 1]$	$W-cK_1;K_2+L_3,N$	$A^c_{3,5}$	$\bar{b}_{4,9}$	$(K_2+L_3)N^{-c}$	$\overset{\approx}{b}_{2,8}, \bar{b}^c_{2,19}, \overset{\approx}{b}^c_{2,19}$
$b_{2,1}$	L_3,N	$2A_1$	$b_{3,1}$	L_3,N	$b_{1,1}, b_{1,2}, b_{1,4}, \bar{b}_{1,6}$
$b_{2,2}$	W,L_3	$2A_1$	self	W,L_3	$b^e_{1,3}, b_{1,5}, \bar{b}_{1,6}$
$b^d_{2,3}$ $d>0\ [d \neq 0]$	$W+dL_3;N$	A_2	$b_{3,1}$	none	$b^d_{1,3}, b_{1,4}$
$b_{2,4} = \bar{a}_{2,21}$	$W;N$	A_2	$b_{5,1}$	none	$b_{1,4}, b_{1,5}$
$\tilde{b}_{2,4} \sim \bar{a}_{2,21}$	$K_1;K_2+L_3$	A_2	$\sim b_{5,1}$	none	$\tilde{b}_{1,4}, \tilde{b}_{1,5}$
$\bar{b}_{2,5} = \tilde{a}_{2,1}$	$W+K_1,M$	$2A_1$	$\bar{b}_{3,6}$	$W+K_1,M$	$\bar{b}_{1,7}, \bar{b}_{1,8}, \bar{b}_{1,10}$
$\bar{b}_{2,6} \sim a_{2,2}$	K_2+L_3,M	$2A_1$	$\sim \bar{b}_{6,2}$	K_2+L_3,M	$\tilde{b}_{1,4}, \bar{b}_{1,8}, \overset{\approx}{b}_{1,8}$
$\overset{\approx}{b}_{2,6} = a_{2,2}$	M,N	$2A_1$	$\bar{b}_{6,2}$	M,N	$b_{1,4}, \bar{b}_{1,8}$
$\bar{b}_{2,7} = a_{2,4}$	K_2+L_3+N,M	$2A_1$	$a_{5,7}$	K_2+L_3+N,M	$\bar{b}_{1,8}, \overset{\approx}{b}_{1,8}$
$\bar{b}_{2,8} \sim a_{2,5}$	K_2+L_3-N,M	$2A_1$	$\sim \bar{b}_{5,8}$	K_2+L_3-N,M	$\tilde{b}_{1,4}, \bar{b}_{1,8}, \overset{\approx}{b}_{1,8}, \bar{b}_{1,9}$
$\overset{\approx}{b}_{2,8} = a_{2,5}$	K_2+L_3,N	$2A_1$	$\bar{b}_{5,8}$	K_2+L_3,N	$b_{1,4}, \tilde{b}_{1,4}, \overset{\approx}{b}_{1,8}, \bar{b}_{1,9}$
$\bar{b}_{2,9} = a_{2,6}$	W,K_1	$2A_1$	self	W,K_1	$b_{1,5}, \tilde{b}_{1,5}, \bar{b}_{1,7}, \bar{b}^e_{1,12}$
$\bar{b}_{2,10} = a_{2,8}$	K_1,N	$2A_1$	$\overset{\approx}{b}_{3,7}$	K_1,N	$b_{1,4}, \tilde{b}_{1,5}, \bar{b}_{1,13}$
$\overset{\approx}{b}_{2,10} \sim a_{2,8}$	W,K_2+L_3	$2A_1$	$\bar{b}_{3,7}$	W,K_2+L_3	$\tilde{b}_{1,4}, b_{1,5}, \overset{\approx}{b}_{1,13}$
$\bar{b}_{2,11} \sim a^{-1}_{2,9}$	K_2+L_3+Q,N	$2A_1$	$\bar{b}_{3,20}$	K_2+L_3+Q,N	$b_{1,4}, \bar{b}_{1,11}$
$\bar{b}_{2,12} = \tilde{a}_{2,10}$	$W-K_1;M$	A_2	$\bar{b}_{3,6}$	none	$\bar{b}_{1,7}, \bar{b}_{1,8}$
$\overset{\approx}{b}_{2,12} \sim \tilde{a}_{2,10}$	$W-K_1;K_2+L_3+N$	A_2	$\sim \bar{b}_{3,6}$	none	$\bar{b}_{1,7}, \overset{\approx}{b}_{1,8}$
$\bar{b}_{2,13} = a_{2,11}$	$W+K_1;N$	A_2	$\bar{b}_{4,8}$	none	$b_{1,4}, \bar{b}_{1,7}$
$\overset{\approx}{b}_{2,13} \sim \tilde{a}_{2,11}$	$W-K_1;K_2+L_3$	A_2	$\overset{\approx}{b}_{4,8}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,7}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\tilde{\bar{b}}_{2,13} \sim a_{2,11}$	$W+K_1;K_2+L_3$	A_2	$\tilde{\bar{b}}_{4,8}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,7}$
$\bar{b}_{2,14} \sim a^{1}_{2,12}$	$W+K_1+M;$ K_2+L_3	A_2	$\tilde{\bar{b}}_{3,5}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,10}$
$\tilde{\bar{b}}_{2,14} \sim a^{-1}_{2,12}$	$W+K_1+M;N$	A_2	$\bar{b}_{3,5}$	none	$b_{1,4}, \bar{b}_{1,10}$
$\bar{b}_{2,15} = a_{2,13}$	$W-K_1,$ K_2+L_3-N	A_2	$\sim a_{3,5}$	none	$\bar{b}_{1,7}, \bar{b}_{1,9}$
$\bar{b}^{d}_{2,16} = \tilde{a}^{(-1+d)/(1+d)}_{2,14}$ $0<\|d\|<1$ $[d \neq 0, \pm 1]$	$W-dK_1;M$	A_2	$\bar{b}_{3,6}$	none	$\bar{b}_{1,8}, \bar{b}^{d}_{1,12}$
$\bar{b}_{2,17} = a_{2,15}$	$K_1;M$	A_2	$\bar{b}_{4,8}$	none	$\tilde{b}_{1,5}, \bar{b}_{1,8}$
$\tilde{\bar{b}}_{2,17} \sim a_{2,15}$	$W;M$	A_2	$\bar{b}_{4,8}$	none	$b_{1,5}, \bar{b}_{1,8}$
$\bar{b}^{\epsilon}_{2,18} = a^{\epsilon}_{2,16}$ $\epsilon = 1*$ $[\epsilon = \pm 1]$	$K_1+\epsilon N;M$	A_2	$\bar{b}_{3,9}$	none	$\bar{b}_{1,8}, \bar{b}_{1,13}$
$\tilde{\bar{b}}^{\epsilon}_{2,18} \sim \bar{a}^{\epsilon}_{2,16}$ $\epsilon = 1*$ $[\epsilon = \pm 1]$	$W-\epsilon(K_2+L_3);M$	A_2	$\tilde{\bar{b}}_{3,9}$	none	$\bar{b}_{1,8}, \tilde{\bar{b}}^{\epsilon}_{1,13}$
$\bar{b}_{2,19} \sim a^{(1-d)/(1+d)}_{2,18}$ $d>0, \neq 1$	$W-dK_1;N$	A_2	$\tilde{\bar{b}}_{3,7}$	none	$b_{1,4}, \bar{b}^{d}_{1,12}$
$\tilde{\bar{b}}_{2,19} \sim a^{(1-d)/(1+d)}_{2,18}$ $d>0, \neq 1$ $[d \neq 0, \pm 1]$	$W-dK_1;K_2+L_3$	A_2	$\bar{b}_{3,7}$	none	$\tilde{b}_{1,4}, \bar{b}^{d}_{1,12}$
$\bar{b}^{\epsilon}_{2,20} = a^{\epsilon}_{2,19}$ $\epsilon = 1*$ $[\epsilon = \pm 1]$	$K_1+\epsilon N;K_2+L_3$	A_2	$\bar{b}_{3,8}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,13}$
$\tilde{\bar{b}}^{\epsilon}_{2,20} \sim a^{\epsilon}_{2,19}$ $\epsilon = 1*$ $[\epsilon = \pm 1]$	$W+\epsilon(K_2+L_3);N$	A_2	$\tilde{\bar{b}}_{3,8}$	none	$b_{1,4}, \tilde{\bar{b}}^{\epsilon}_{1,13}$
$\bar{b}_{2,21} = a^{-1}_{2,20}$	$W-\frac{1}{3}K_1;$ K_2+L_3+Q	A_2	self	none	$\bar{b}_{1,11}, \bar{b}^{1/3}_{1,12}$
$b_{1,1}$	L_3+N	A_1	$b_{2,1}$	L_3+N	
$b_{1,2}$	L_3-N	A_1	$b_{2,1}$	L_3-N	
$b^{e}_{1,3}$, $e>0[e \neq 0]$	$W+eL_3$	A_1	$b_{2,2}$	$W+eL_3$	
$b_{1,4} = \bar{a}_{1,11}$	N	A_1	$b_{7,1}$	N	
$\tilde{b}_{1,4} \sim \bar{a}_{1,11}$	K_2+L_3	A_1	$\sim b_{7,1}$	K_2+L_3	
$b_{1,5} = \bar{a}_{1,12}$	W	A_1	$b_{4,2}$	W	
$\tilde{b}_{1,5} \sim \bar{a}_{1,12}$	K_1	A_1	$\sim b_{4,2}$	K_1	
$\bar{b}_{1,6} = e_{1,1}$	L_3	A_1	$e_{4,1}$	L_3	
$\bar{b}_{1,7} = a_{1,1}$	$W-K_1$	A_1	$a_{4,4}$	$W-K_1$	
$\bar{b}_{1,8} = a_{1,2}$	M	A_1	$\bar{b}_{5,8}$	M	
$\tilde{\bar{b}}_{1,8} \sim a_{1,2}$	K_2+L_3+N	A_1	$\sim \bar{b}_{5,8}$	K_2+L_3+N	
$\bar{b}_{1,9} = a_{1,3}$	K_2+L_3-N	A_1	$a_{5,7}$	K_2+L_3-N	
$\bar{b}_{1,10} \sim \tilde{a}^{-1}_{1,4}$	$W+K_1+M$	A_1	$\bar{b}_{2,5}$	$W+K_1+M$	
$\bar{b}_{1,11} \sim a^{-1}_{1,5}$	K_2+L_3+Q	A_1	$\bar{b}_{3,20}$	K_2+L_3+Q	
$\bar{b}^{e}_{1,12} = a^{(1-e)/(1+e)}_{1,8}$ $0<e<1$ $[e>0, \neq 1]$	$W-eK_1$	A_1	$\bar{b}_{2,9}$	$W-eK_1$	
$\bar{b}_{1,13} = a^{1}_{1,9}$	K_1+N	A_1	$\bar{b}_{2,10}$	K_1+N	
$\tilde{\bar{b}}^{\epsilon}_{1,13} = a^{\epsilon}_{1,9}$ $\epsilon = 1$ $[\epsilon = \pm 1]$	$W+\epsilon(K_2+L_3)$	A_1	$\tilde{\bar{b}}_{2,10}$	$W+\epsilon(K_2+L_3)$	

TABLE VI. Subalgebras of LO(3) ⊕ LO(2).

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$c_{4,1}$	$D \oplus \{;A, B, C\}$	$A_1 \oplus A_{3.9}$	self	$D,\ A^2 + B^2 + C^2$	$c_{3.1}, c_{2,1}$
$c_{3,1}$	$;A, B, C$	$A_{3,9}$	$c_{4,1}$	$A^2 + B^2 + C^2$	$\bar{c}_{1,3}$
$c_{2,1}$	D, C	$2A_1$	self	D, C	$c_{1,1}, c^e_{1,2}, \bar{c}_{1,3}, \bar{c}_{1,4}$
$c_{1,1}$	D	A_1	$c_{4,1}$	D	
$c^e_{1,2}$ $e > 0, \neq 1$	$D + eC$	A_1	$c_{2,1}$	$D + eC$	
$\bar{c}_{1,3} = d_{1,1}$	C	A_1	$d_{4,1}$	C	
$\bar{c}_{1,4} = e_{1,1}$	$A + D$	A_1	$e_{4,1}$	$A + D$	

TABLE VII. Subalgebras of LO(2) ⊕ LO(2,1).

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$d_{4,1}$	$C \oplus \{;D, E, F\}$	$A_1 \oplus A_{3,8}$	self	$C, D^2 - E^2 - F^2$	$d_{3,1}, \bar{d}_{3,2}, \bar{d}_{2,3}$
$d_{3,1}$	$;D, E, F$	$A_{3,8}$	$d_{4,1}$	$D^2 - E^2 - F^2$	$\bar{d}_{2,4}, \bar{d}_{1,2}$
$\bar{d}_{3,2} = a_{3,5}$	$C \oplus \{F; D + E\}$	$A_1 \oplus A_2$	self	C	$\bar{d}_{2,1}, \bar{d}_{2,2}, \bar{d}_{2,4}, \bar{d}^d_{2,5}$
$\bar{d}_{2,1} = a_{2,7}$	C, F	$2A_1$	self	C, F	$d_{1,1}, \bar{d}_{1,4}, \bar{d}^e_{1,6}$
$\bar{d}_{2,2} = a_{2,3}$	$C, D + E$	$2A_1$	$\bar{d}_{3,2}$	$C, D + E$	$d_{1,1}, \bar{d}_{1,5}, \bar{d}^\epsilon_{1,7}$
$\bar{d}_{2,3} = c_{2,1}$	C, D	$2A_1$	self	C, D	$d_{1,1}, \bar{d}_{1,2}, \bar{d}^e_{1,3}, \bar{d}^\epsilon_{1,8}$
$\bar{d}_{2,4} = a_{2,13}$	$F; D + E$	A_2	$\bar{d}_{3,2}$	none	$\bar{d}_{1,4}, \bar{d}_{1,5}$
$\bar{d}^d_{2,5} = a^d_{2,17}$ $d > 0\ [d \neq 0]$	$F + dC; D + E$	A_2	$\bar{d}_{3,2}$	none	$\bar{d}_{1,5}, \bar{d}^d_{1,6}$
$d_{1,1}$	C	A_1	$d_{4,1}$	C	
$\bar{d}_{1,2} = c_{1,1}$	D	A_1	$c_{4,1}$	D	
$\bar{d}^e_{1,3} = c^e_{1,2}$ $e > 0, \neq 1\ [e \neq 0, \pm 1]$	$D + eC$	A_1	$\bar{d}_{2,3}$	$D + eC$	
$\bar{d}_{1,4} = a_{1,1}$	F	A_1	$a_{4,4}$	F	
$\bar{d}_{1,5} = a_{1,3}$	$D + E$	A_1	$a_{5,7}$	$D + E$	
$\bar{d}^e_{1,6} = a^e_{1,7}, \quad e > 0$	$F + eC$	A_1	$a_{2,7}$	$F + eC$	
$\bar{d}^\epsilon_{1,7} = a^\epsilon_{1,6}$ $\epsilon = 1\ [\epsilon = \pm 1]$	$C + \epsilon(D + E)$	A_1	$a_{2,3}$	$C + \epsilon(D + E)$	
$\bar{d}^\epsilon_{1,8} \sim e_{1,1}$ $\epsilon = 1\ [\epsilon = \pm 1]$	$D + \epsilon C$	A_1	$\sim e_{4,1}$	$D + \epsilon C$	

TABLE VIII. Subalgebras of LO(2,2).

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$e_{6,1}$	$\{;A_1, A_2, A_3\}$ $\oplus \{;B_1, B_2, B_3\}$	$A_{3,8} \oplus A_{3,8}$	self	$A_1^2 + A_2^2 - A_3^2$, $B_1^2 + B_2^2 - B_3^2$	$\bar{e}_{5,1}, \overset{\simeq}{e}_{5,1}, e_{4,1}, \tilde{e}_{4,1}, \bar{e}_{3,8}$ $\bar{e}_{3,9}$
$\bar{e}_{5,1} = b_{5,1}$	$\{A_2; A_1 - A_3\}$ $\oplus \{;B_1, B_2, B_3\}$	$A_2 \oplus A_{3,8}$	self	$B_1^2 + B_2^2 - B_3^2$	$\bar{e}_{4,2}, \bar{e}_{4,3}, \bar{e}_{4,4}, \bar{e}_{3,4}$
$\overset{\simeq}{e}_{5,1}$	$\{;A_1, A_2, A_3\}$ $\oplus \{B_2; B_1 - B_3\}$	$A_{3,8} \oplus A_2$	self	$A_1^2 + A_2^2 - A_3^2$	$\bar{e}_{4,2}, \overset{\simeq}{e}_{4,3}, \overset{\simeq}{e}_{4,4}, \overset{\simeq}{e}_{3,4}$
$e_{4\;1}$	$A_3 \oplus \{;B_1, B_2, B_3\}$	$A_1 \oplus A_{3,8}$	self	$A_3, B_1^2 + B_2^2 - B_3^2$	$e_{3,1}, \overset{\simeq}{e}_{3,4}, \bar{e}_{2,6}$
$\tilde{e}_{4,1}$	$\{;A_1, A_2, A_3\} \oplus B_3$	$A_{3,8} \oplus A_1$	self	$A_1^2 + A_2^2 - A_3^2, B_3$	$\tilde{e}_{3,1}, \bar{e}_{3,4}, \bar{e}_{2,6}$

TABLE VIII. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{e}_{4,2} = a_{4,6}$	$\{A_2;A_1 - A_3\}$ $\oplus \{B_2;B_1 - B_3\}$	$A_2 \oplus A_2$	self	none	$\bar{e}_{3,2}, \overset{\simeq}{e}_{3,2}, \bar{e}_{3,3}, \overset{\simeq}{e}_{3,3},$ $\bar{e}_{3,5}, \bar{e}_{3,6}, \bar{e}^c_{3,7}$
$\bar{e}_{4,3} = b_{4,1}$	$(A_1 - A_3) \oplus$ $(;B_1, B_2, B_3\}$	$A_1 \oplus A_{3,8}$	$\bar{e}_{5,1}$	$A_1 - A_3, B_1^2 + B_2^2 - B_3^2$	$e_{3,1}, \bar{e}_{3,3}, \bar{e}_{2,4}$
$\overset{\simeq}{e}_{4,3}$	$\{;A_1, A_2, A_3\}$ $\oplus (B_1 - B_3)$	$A_{3,8} \oplus A_1$	$\sim \bar{e}_{5,1}$	$A_1^2 + A_2^2 - A_3^2, B_1 - B_3$	$\overset{\simeq}{e}_{3,1}, \overset{\simeq}{e}_{3,3}, \overset{\simeq}{e}_{2,4}$
$\bar{e}_{4,4} = b_{4,2}$	$A_2 \oplus \{;B_1, B_2, B_3\}$	$A_1 \oplus A_{3,8}$	self	$A_2, B_1^2 + B_2^2 - B_3^2$	$e_{3,1}, \overset{\simeq}{e}_{3,2}, \bar{e}_{2,5}$
$\overset{\simeq}{e}_{4,4}$	$\{;A_1, A_2, A_3\} \oplus B_2$	$A_{3,8} \oplus A_1$	self	$A_1^2 + A_2^2 - A_3^2, B_2$	$\tilde{e}_{3,1}, \bar{e}_{3,2}, \overset{\simeq}{e}_{2,5}$
$e_{3,1} = \bar{b}_{3,3}$	$;B_1, B_2, B_3$	$A_{3,8}$	$e_{6,1}$	$B_1^2 + B_2^2 - B_3^2$	$\bar{e}_{2,7}, \tilde{e}_{1,1}$
$\tilde{e}_{3,1}$	$;A_1, A_2, A_3$	$A_{3,8}$	$e_{6,1}$	$A_1^2 + A_2^2 - A_3^2$	$\bar{e}_{2,7}, e_{1,1}$
$\bar{e}_{3,2} = a_{3,4}$	$B_2 \oplus \{A_2;A_1 - A_3\}$	$A_1 \oplus A_2$	self	B_2	$\bar{e}_{2,2}, \bar{e}_{2,3}, \bar{e}_{2,7}, \bar{e}_{2,10},$ $\bar{e}^d_{2,11}$
$\overset{\simeq}{e}_{3,2}$	$A_2 \oplus \{B_2;B_1 - B_3\}$	$A_1 \oplus A_2$	self	A_2	$\bar{e}_{2,2}, \overset{\simeq}{e}_{2,3}, \overset{\simeq}{e}_{2,7}, \overset{\simeq}{e}_{2,10},$ $\overset{\simeq}{e}^d_{2,11}$
$\bar{e}_{3,3} = a_{3,6}$	$(A_1 - A_3) \oplus$ $\{B_2;B_1 - B_3\}$	$A_1 \oplus A_2$	$\bar{e}_{4,2}$	$A_1 - A_3$	$\bar{e}_{2,1}, \bar{e}_{2,3}, \overset{\simeq}{e}_{2,7}, \bar{e}^e_{2,13}$
$\overset{\simeq}{e}_{3,3}$	$(B_1 - B_3) \oplus$ $\{A_2; A_1 - A_3\}$	$A_1 \oplus A_2$	$\bar{e}_{4,2}$	$B_1 - B_3$	$\bar{e}_{2,1}, \overset{\simeq}{e}_{2,3}, \overset{\simeq}{e}_{2,7}, \overset{\simeq}{e}^e_{2,13}$
$\bar{e}_{3,4} = b_{3,1}$	$B_3 \oplus \{A_2;A_1 - A_3\}$	$A_1 \oplus A_2$	self	B_3	$\bar{e}_{2,4}, \bar{e}_{2,5}, \bar{e}_{2,7}, \bar{e}^d_{2,12}$
$\overset{\simeq}{e}_{3,4}$	$A_3 \oplus \{B_2;B_1 - B_3\}$	$A_1 \oplus A_2$	self	A_3	$\overset{\simeq}{e}_{2,4}, \overset{\simeq}{e}_{2,5}, \overset{\simeq}{e}_{2,7}, \overset{\simeq}{e}^d_{2,12}$
$\bar{e}_{3,5} = a_{3,12}$	$A_2 + B_2;A_1 - A_3,$ $B_1 - B_3$	$A_{3,3}$	$\bar{e}_{4,2}$	$(A_1 - A_3)/(B_1 - B_3)$	$\bar{e}_{2,1}, \bar{e}_{2,8}, \bar{e}_{2,9}, \bar{e}_{2,10},$ $\overset{\simeq}{e}_{2,10}$
$\bar{e}_{3,6} = a_{3,13}$	$A_2 - B_2;A_1 - A_3,$ $B_1 - B_3$	$A_{3,4}$	$a_{5,6}$	$(A_1 - A_3)(B_1 - B_3)$	$\bar{e}_{2,1}, \bar{e}_{2,10}, \overset{\simeq}{e}_{2,10}$
$\bar{e}^c_{3,7} = a^{(1-c)/(1+c)}_{3,20}$ $0 < \lvert c \rvert < 1$ $[c \neq 0, \pm 1]$	$A_2 + cB_2;$ $A_1 - A_3, B_1 - B_3$	$A^h_{3,5}$ $h = c, \lvert c \rvert < 1$ $h = 1/c, \lvert c \rvert > 1$	$\bar{e}_{4,2}$	$(A_1 - A_3)^{-c}(B_1 - B_3)$	$\bar{e}_{2,1}, \bar{e}^c_{2,11}, \overset{\simeq}{e}^c_{2,11}$
$\bar{e}_{3,8} = d_{3,1}$	$;A_1 - B_1,$ $A_2 + B_2, A_3 - B_3$	$A_{3,8}$	$d_{4,1}$	$(A_1 - B_1)^2 + (A_2 + B_2)^2$ $- (A_3 - B_3)^2$	$\bar{e}_{2,9}, \bar{e}_{1,13}$
$\bar{e}_{3,9} \sim a_{3,24}$	$;A_1 + B_1, A_2 + B_2,$ $A_3 + B_3$	$A_{3,8}$	$\sim a_{4,4}$	$(A_1 + B_1)^2 + (A_2 + B_2)^2$ $- (A_3 + B_3)^2$	$\bar{e}_{2,8}, \bar{e}_{1,14}$
$\bar{e}_{2,1} = a_{2,5}$	$A_1 - A_3, B_1 - B_3$	$2A_1$	$a_{5,6}$	$A_1 - A_3, B_1 - B_3$	$\bar{e}_{1,3}, \bar{e}_{1,4}, \bar{e}_{1,10}, \overset{\simeq}{e}_{1,10}$
$\bar{e}_{2,2} = a_{2,6}$	A_2, B_2	$2A_1$	self	A_2, B_2	$\bar{e}_{1,2}, \bar{e}^e_{1,5}, \bar{e}_{1,11}, \overset{\simeq}{e}_{1,11}$
$\bar{e}_{2,3} = a_{2,8}$	$B_2, A_1 - A_3$	$2A_1$	$\bar{e}_{3,2}$	$B_2, A_1 - A_3$	$\bar{e}_{1,6}, \bar{e}_{1,10}, \overset{\simeq}{e}_{1,11}$
$\overset{\simeq}{e}_{2,3}$	$A_2, B_1 - B_3$	$2A_1$	$\overset{\simeq}{e}_{3,2}$	$A_2, B_1 - B_3$	$\overset{\simeq}{e}_{1,6}, \overset{\simeq}{e}_{1,10}, \bar{e}_{1,11}$
$\bar{e}_{2,4} = b_{2,1}$	$A_1 - A_3, B_3$	$2A_1$	$\bar{e}_{3,4}$	$A_1 - A_3, B_3$	$\tilde{e}_{1,1}, \bar{e}_{1,7}, \bar{e}_{1,8}, \bar{e}_{1,10}$
$\overset{\simeq}{e}_{2,4}$	$A_3, B_1 - B_3$	$2A_1$	$\overset{\simeq}{e}_{3,4}$	$A_3, B_1 - B_3$	$e_{1,1}, \overset{\simeq}{e}_{1,7}, \overset{\simeq}{e}_{1,8}, \overset{\simeq}{e}_{1,10}$
$\bar{e}_{2,5} = b_{2,2}$	A_2, B_3	$2A_1$	self	A_2, B_3	$\tilde{e}_{1,1}, \bar{e}^e_{1,9}, \bar{e}_{1,11}$
$\overset{\simeq}{e}_{2,5}$	A_3, B_2	$2A_1$	self	A_3, B_2	$e_{1,1}, \overset{\simeq}{e}^e_{1,9}, \overset{\simeq}{e}_{1,11}$
$\bar{e}_{2,6} \sim c_{2,1}$	A_3, B_3	$2A_1$	self	A_3, B_3	$e_{1,1}, \tilde{e}_{1,1}, \bar{e}^e_{1,12}, \bar{e}_{1,13},$ $\bar{e}_{1,14}$
$\bar{e}_{2,7} = b_{2,4}$	$A_2;A_1 - A_3$	A_2	$\bar{e}_{5,1}$	none	$\bar{e}_{1,10}, \bar{e}_{1,11}$
$\overset{\simeq}{e}_{2,7} = b_{2,4}$	$B_2;B_1 - B_3$	A_2	$\overset{\simeq}{e}_{5,1}$	none	$\overset{\simeq}{e}_{1,10}, \overset{\simeq}{e}_{1,11}$
$\bar{e}_{2,8} \sim \tilde{a}_{2,10}$	$A_2 + B_2;$ $-A_1 + A_3 - B_1 + B_3$	A_2	$\sim \tilde{a}_{3,3}$	none	$\bar{e}_{1,2}, \bar{e}_{1,3}$

TABLE VIII. (continued)

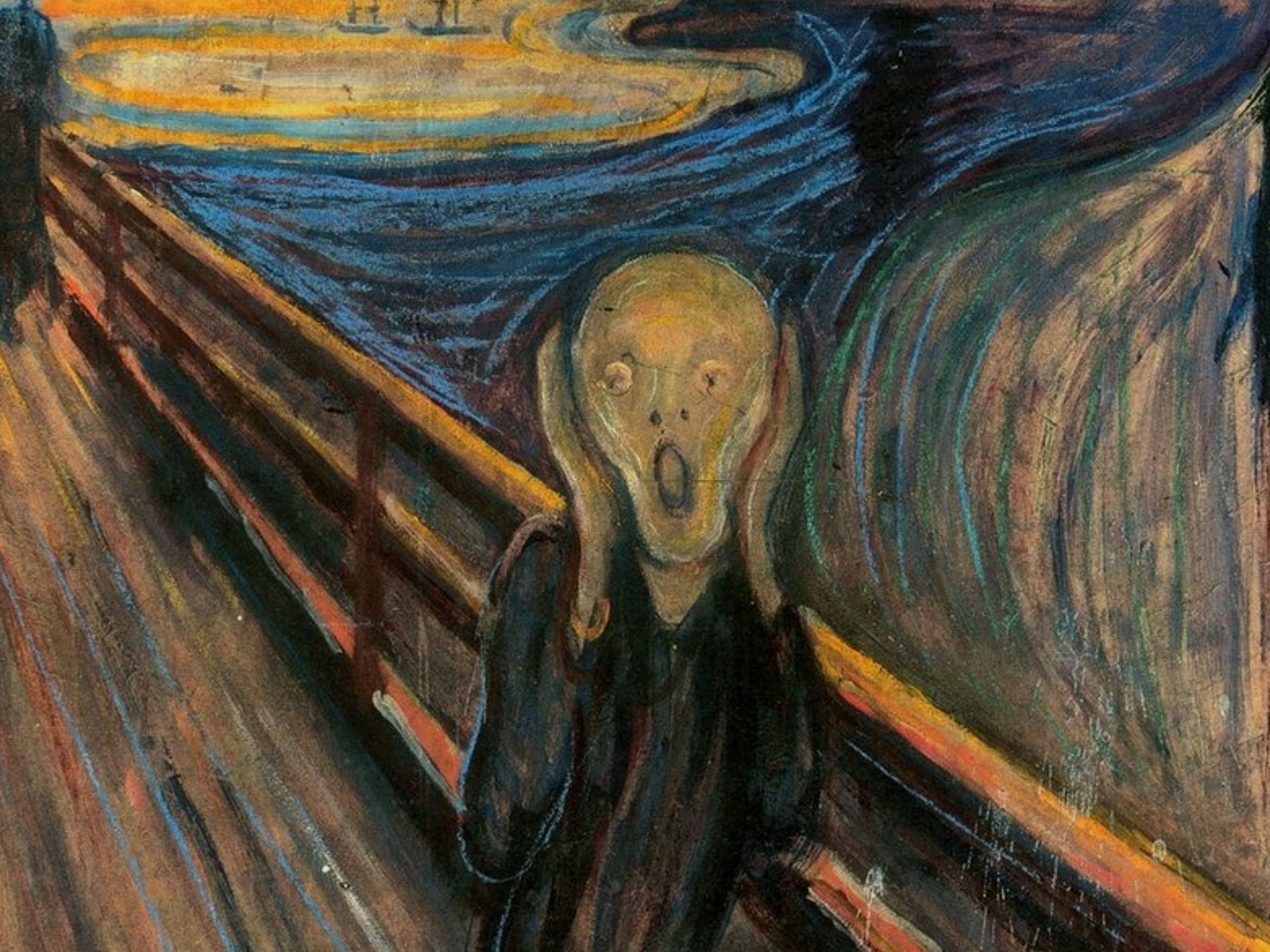
Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{e}_{2,9}=a_{2,13}$	$A_2+B_2;\ -A_1+A_3$ $+B_1-B_3$	A_2	$a_{3,5}$	none	$\bar{e}_{1,2},\bar{e}_{1,4}$
$\bar{e}_{2,10}=\tilde{a}_{2,11}$	$A_2+B_2;\ -A_1+A_3$	A_2	$\tilde{a}_{4,5}$	none	$\bar{e}_{1,2},\bar{e}_{1,10}$
$\overset{\simeq}{e}_{2,10}$	$A_2+B_2;\ -B_1+B_3$	A_2	$\sim\tilde{a}_{4,5}$	none	$\bar{e}_{1,2},\overset{\simeq}{e}_{1,10}$
$\bar{e}^d_{2,11}=a^{(1-d)/(1+d)}_{2,18}$ $d>0,\neq 1$	$A_2+dB_2;\ -A_1+A_3$	A_2	$\bar{e}_{3,2}$	none	$\bar{e}^d_{1,5},\bar{e}_{1,10}$
$\overset{\simeq}{e}^d_{2,11}$ $d>0,\neq 1$	$A_2+dB_2;\ -B_1+B_3$	A_2	$\overset{\simeq}{e}_{3,2}$	none	$\bar{e}^d_{1,5},\overset{\simeq}{e}_{1,10}$
$\bar{e}^d_{2,12}=b^d_{2,3}$ $d>0\ [d\neq 0]$	$A_2-dB_3;\ -A_1+A_3$	A_2	$e_{3,4}$	none	$\bar{e}^d_{1,9},\bar{e}_{1,10}$
$\overset{\simeq}{e}^d_{2,12}$ $d>0\ [d\neq 0]$	$B_2-dA_3;\ -B_1+B_3$	A_2	$\overset{\simeq}{e}_{3,4}$	none	$\overset{\simeq}{e}^d_{1,9},\overset{\simeq}{e}_{1,10}$
$\bar{e}^\epsilon_{2,13}=a^\epsilon_{2,19}$ $\epsilon=1\ [\epsilon=\pm 1]$	$B_2+\epsilon(A_3-A_1);$ B_1-B_3	A_2	$\bar{e}_{3,3}$	none	$\bar{e}_{1,6},\overset{\simeq}{e}_{1,10}$
$\overset{\simeq}{e}^\epsilon_{2,13}$ $\epsilon=1\ [\epsilon=\pm 1]$	$A_2+\epsilon(B_3-B_1);$ A_1-A_3	A_2	$\overset{\simeq}{e}_{3,3}$	none	$\overset{\simeq}{e}_{1,6},\bar{e}_{1,10}$
$e_{1,1}$	A_3	A_1	$e_{4,1}$	A_3	
$\tilde{e}_{1,1}$	B_3	A_1	$\tilde{e}_{4,1}$	B_3	
$\bar{e}_{1,2}=a_{1,1}$	A_2+B_2	A_1	$a_{4,4}$	A_2+B_2	
$\bar{e}_{1,3}\sim a_{1,2}$	$-A_1+A_3-B_1+B_3$	A_1	$\sim a_{5,6}$	$-A_1+A_3-B_1+B_3$	
$\bar{e}_{1,4}=a_{1,3}$	$-A_1+A_3+B_1-B_3$	A_1	$a_{5,7}$	$-A_1+A_3+B_1-B_3$	
$\bar{e}^e_{1,5}=a^{(1-e)/(1+e)}_{1,8},0<e<1$ $[e>0,\neq 1]$	A_2+eB_2	A_1	$\bar{e}_{2,2}$	A_2+eB_2	
$\bar{e}_{1,6}=a^1_{1,9}$	$B_2-A_1+A_3$	A_1	$\bar{e}_{2,3}$	$B_2-A_1+A_3$	
$\overset{\simeq}{e}_{1,6}$	$A_2-B_1+B_3$	A_1	$\overset{\simeq}{e}_{2,3}$	$A_2-B_1+B_3$	
$\bar{e}_{1,7}=b_{1,1}$	$-A_1+A_3+B_3$	A_1	$\bar{e}_{2,4}$	$-A_1+A_3+B_3$	
$\overset{\simeq}{e}_{1,7}$	$-B_1+B_3+A_3$	A_1	$\overset{\simeq}{e}_{2,4}$	$-B_1+B_3+A_3$	
$\bar{e}_{1,8}=b_{1,2}$	$-A_1+A_3-B_3$	A_1	$\bar{e}_{2,4}$	$-A_1+A_3-B_3$	
$\overset{\simeq}{e}_{1,8}$	$-B_1+B_3-A_3$	A_1	$\overset{\simeq}{e}_{2,4}$	$-B_1+B_3-A_3$	
$\bar{e}^e_{1,9}=b_{1,3},\quad e>0$	A_2-eB_3	A_1	$\bar{e}_{2,5}$	A_2-eB_3	
$\overset{\simeq}{e}^e_{1,9},e>0$	B_2-eA_3	A_1	$\overset{\simeq}{e}_{2,5}$	B_2-eA_3	
$\bar{e}_{1,10}=b_{1,4}$	$-A_1+A_3$	A_1	$b_{7,1}$	$-A_1+A_3$	
$\overset{\simeq}{e}_{1,10}=\tilde{b}_{1,4}$	$-B_1+B_3$	A_1	$\sim b_{7,1}$	$-B_1+B_3$	
$\bar{e}_{1,11}=b_{1,5}$	A_2	A_1	$\bar{e}_{4,4}$	A_2	
$\overset{\simeq}{e}_{1,11}=\tilde{b}_{1,5}$	B_2	A_1	$\overset{\simeq}{e}_{4,4}$	B_2	
$\bar{e}^e_{1,12}\sim c^{(1+e)/(1-e)}_{1,2},0<\|e\|<1$ $[e\neq 0,\pm 1]$	A_3+eB_3	A_1	$\bar{e}_{2,6}$	A_3+eB_3	
$\bar{e}_{1,13}=c_{1,1}$	A_3-B_3	A_1	$c_{4,1}$	A_3-B_3	
$\bar{e}_{1,14}\sim d_{1,1}$	A_3+B_3	A_1	$d_{4,1}$	A_3+B_3	

TABLE IX. Subalgebras of LO(3,1).

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$f_{6,1}$	$;L_1, L_2, L_3, K_1, K_2, K_3$	LO(3,1)	self	$\mathbf{L}^2 - \mathbf{K}^2, (\mathbf{L}, \mathbf{K})$	$\bar{f}_{4,1}, \bar{f}_{3,4}, \bar{f}_{3,5}$
$\bar{f}_{4,1} = a_{4,17}$	$K_1, L_1; L_2 - K_3, L_3 + K_2$	$A_{4,12}$	self	none	$\bar{f}_{3,1}, \bar{f}_{3,2}, \bar{f}^c_{3,3}, \bar{f}_{2,2}$
$\bar{f}_{3,1} = a_{3,11}$	$K_1; L_2 - K_3, L_3 + K_2$	$A_{3,3}$	$f_{4,1}$	$(L_2 - K_3)/(L_3 + K_2)$	$\bar{f}_{2,1}, \bar{f}_{2,3}$
$\bar{f}_{3,2} = a_{3,21}$	$L_1; L_2 - K_3, L_3 + K_2$	$A_{3,6}$	$a_{5,7}$	$(L_2 - K_3)^2 + (L_3 + K_2)^2$	$\bar{f}_{2,1}, \bar{f}_{1,1}$
$\bar{f}^c_{3,3} = a^c_{3,23}$ $c > 0$ $[c \neq 0]$	$K_1 - cL_1; L_2 - K_3, L_3 + K_2$	$A^{1/c}_{3,7}$	$\bar{f}_{4,1}$	$[(L_2 - K_3)^2 + (L_3 + K_2)^2]^c \times \left(\frac{L_2 - K_3 - i(L_3 + K_2)}{L_2 - K_3 + i(L_3 + K_2)}\right)^i$	$\bar{f}_{2,1}, \bar{f}^c_{1,4}$
$\bar{f}_{3,4} \sim a_{3,24}$	$;K_2, K_3, L_1$	$A_{3,8}$	$a_{4,4}$	$L_1^2 - K_2^2 - K_3^2$	$\bar{f}_{2,3}, \bar{f}_{1,1}$
$\bar{f}_{3,5} = c_{3,1}$	$;L_1, L_2, L_3$	$A_{3,9}$	$c_{4,1}$	$L_1^2 + L_2^2 + L_3^2$	$\bar{f}_{1,1}$
$\bar{f}_{2,1} = a_{2,4}$	$L_2 - K_3, L_3 + K_2$	$2A_1$	$a_{5,7}$	$L_2 - K_3, L_3 + K_2$	$\bar{f}_{1,3}$
$\bar{f}_{2,2} = a_{2,7}$	L_1, K_1	$2A_1$	self	L_1, K_1	$\bar{f}_{1,1}, \bar{f}_{1,2}, \bar{f}^e_{1,4}$
$\bar{f}_{2,3} = \tilde{a}_{2,10}$	$K_1; L_2 - K_3$	A_2	$\tilde{a}_{3,3}$	none	$\bar{f}_{1,2}, \bar{f}_{1,3}$
$\bar{f}_{1,1} = d_{1,1}$	L_1	A_1	$d_{4,1}$	L_1	
$\bar{f}_{1,2} = a_{1,1}$	K_1	A_1	$a_{4,4}$	K_1	
$\bar{f}_{1,3} = a_{1,2}$	$L_2 - K_3$	A_1	$a_{5,6}$	$L_2 - K_3$	
$\bar{f}^e_{1,4} = a^e_{1,7}$, $e > 0$ $[e \neq 0]$	$K_1 - eL_1$	A_1	$\bar{f}_{2,2}$	$K_1 - eL_1$	

TABLE X. Subalgebras of the irreducibly embedded LO(2,1).

Name	Generators	Isomorphism class	Normalizer	Invariants
$g_{3,1}$	$;K_1, K_2, L_3$	$A_{3,8}$	self	$K_1^2 + K_2^2 - L_3^2$
$\bar{g}_{2,1} \sim a^1_{2,20}$	$K_2; K_1 - L_3$	A_2	self	none
$\bar{g}_{1,1} \sim c^2_{1,2}$	L_3	A_1	$\sim c_{2,1}$	L_3
$\bar{g}_{1,2} \sim a^1_{1,5}$	$K_1 - L_3$	A_1	$\sim a^1_{3,18}$	$K_1 - L_3$
$\bar{g}_{1,3} \sim a^{1/2}_{1,8}$	K_2	A_1	$\sim a_{2,6}$	K_2



HOMOGENEOUS LORENTZIAN MANIFOLDS

(the smart way)

We observe that $\mathrm{SO}(N)\times\mathrm{SO}(3,2)$ is **semisimple** for $N>2$.

Theorem

Let M be a lorentzian homogeneous space of a semisimple Lie group G. Then **either**

(1) the action of G is proper; **or**
(2) M is the product of (anti) de Sitter space with a riemannian homogeneous manifold.

Kowalsky (1996)
Deffaf+Melnick+Zeghib (2008)

✓ There are no dS backgrounds.

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✓ We already classified Freund-Rubin AdS backgrounds, but still need to check AdS backgrounds which are not Freund-Rubin.

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✓ If the action of $\mathbb{G}$ is proper, then $\mathbb{H}$ is compact.

- ✓ There are no dS backgrounds.

- ✓ We already classified Freund-Rubin AdS backgrounds, but still need to check AdS backgrounds which are not Freund-Rubin.

- ✓ If the action of G is proper, then H is compact.

- ✓ In all cases, G/H is reductive.

The original problem branches into two:

1. Classify homogeneous AdS backgrounds with internal flux ➻ classify 7-dimensional homogeneous riemannian manifolds G/H for $G{=}SO(N)$ and $N{>}4$.

2. Classify other homogeneous backgrounds ➻ classify homogeneous lorentzian manifolds G/H for $G{=}SO(3{,}2)\times SO(N)$ with $N{>}4$ and H compact.

$\mathbf{d=11}$ supergravity backgrounds

(M, g, F)

(M^{11}, g) lorentzian spin manifold

$$F \in \Omega^4(M) \qquad dF = 0$$

$$d \star F = \tfrac{1}{2} F \wedge F$$

$$\mathrm{Ric}(X,Y) = \tfrac{1}{2} \langle \iota_X F, \iota_Y F \rangle - \tfrac{1}{6} |F|^2 g(X,Y)$$

Not unlike Einstein-Maxwell theory

Homogeneous backgrounds

$M = G/H$ $\qquad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ $\qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$

g ↔ H-invariant inner product on $\mathfrak{m}$

F ↔ H-invariant 4-form on $\mathfrak{m}$

$(\boldsymbol{e}_a)$ basis for $\mathfrak{m}$ $\qquad (\boldsymbol{\theta}^a)$ canonical dual basis

$$g = \gamma_{ab}\boldsymbol{\theta}^a\boldsymbol{\theta}^b \qquad F = \tfrac{1}{4!}\varphi_{abcd}\boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c \wedge \boldsymbol{\theta}^d$$

H-invariance ➡ linear equations for the constants γ_{ab} and φ_{abcd}, resulting in a finite number of free parameters.

Supergravity equations become algebraic equations in the free parameters.

We can eliminate some parameters using the action of the normaliser of $\mathfrak{h}$ in $\mathfrak{g}$.

We can eliminate one further parameter using the homothety invariance of the field equations:

$$g \mapsto t^2 g \qquad F \mapsto t^3 F$$

which holds in general and not just in the homogeneous case.

HOMOGENEOUS
ADS
BACKGROUNDS

AdS backgrounds were much studied in the early 1980s in the context of Kaluza-Klein supergravity.

Freund-Rubin

This is the original AdS background:

$\mathrm{AdS}_4 \times S^7$ curvatures in ratio $2{:}1$

$F \propto \mathrm{dvol}_{\mathrm{AdS}_4}$

$\mathrm{SO(3,2)}\times\mathrm{SO(8)}$ symmetry

Maximal $\mathrm{N}=8$ supersymmetry.

Freund+Rubin (1980)

Englert

$\mathrm{AdS}_4 \times S^7$ curvatures not in $2{:}1$ratio

$\mathbf{F}$ a linear combination of $\mathrm{dvol}_{\mathrm{AdS}_4}$

and unique Spin(7)-invariant form on

$$S^7 = \mathrm{Spin}(7)/G_2$$

The Englert solution is **not** supersymmetric.

Englert (1982)

Pope-Warner

$$\begin{tikzcd}
S^1 \arrow[r] & S^7 \arrow[d] \\
 & \mathbb{CP}^3
\end{tikzcd}$$

and stretch the circle fibre.

The resulting metric on $\mathbf{S^7}$ has $\mathbf{SO(6)\times SO(2)}$ invariance which the flux breaks further to $\mathbf{SO(6)}$. The resulting solution is **not** supersymmetric.

Pope+Warner (1985) ×2

Squashed 7-sphere

$$\begin{tikzcd} S^3 \arrow[r] & S^7 \arrow[d] \\ & S^4 \end{tikzcd}$$

and now “squash” the $\mathbf{S}^3$ fibre to get an Einstein metric.

$$F \propto \mathrm{dvol}_{\mathrm{AdS}_4}$$

The solution has $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ symmetry and $\mathrm{N}{=}1$ supersymmetry.

Awada+Duff+Pope (1983)

Squashed Englert

Perform the "canonical variation" on Englert's solution. The resulting solution has $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ symmetry and **no** supersymmetry.

Englert+Rooman+Spindel (1983)
Duff+Nilsson+Pope (1983)
Bais+Nicolai+van Nieuwenhuizen (1983)

General FR & Englert

$\mathrm{AdS}_4 \times (G/H)$ $\mathbf{G/H}$ Einstein

If $\mathbf{G/H}$ admits a Killing spinor ε, then one can adding to F a term proportional to $\mathbf{dA}$ with

$$A_{abc} = \overline{\varepsilon}\Gamma_{abc}\varepsilon$$

There are solutions with $\mathbf{G{=}SO(5)}$.

Castellani+Romans+Warner (1984)

We recovered these solutions and **in addition** some new (to us) $\mathrm{SO}(5)$-invariant backgrounds.

They do **not** correspond to solutions of gauged $\mathrm{N{=}8}$ supergravity.

The $\mathrm{SO}(5)$-invariant solutions of gauged $\mathrm{N{=}8}$ supergravity were classified by Romans who only found the Englert and Pope-Warner solutions.

Romans (1983)

A 7-dimensional homogeneous space of $\mathrm{SO}(5)$ must have a 3-dimensional isotropy subgroup.

There are (up to conjugation) three 3-dimensional subalgebras of $\mathfrak{so}(5)$ and they are all isomorphic to $\mathfrak{so}(3)$. They are distinguished by their action on the vector representation:

$\mathfrak{so}(3)_{\mathrm{irr}}$	$\mathbf{5} \to [2]$
$\mathfrak{so}(3)$	$\mathbf{5} \to 2[0] \oplus [1]$
$\mathfrak{so}(3)_{+}$	$\mathbf{5} \to [0] \oplus [\tfrac{3}{2}]$

$\mathfrak{so}(3)_{\mathrm{irr}}$

$\mathfrak{m} \cong [3]$ ➡ round S^7

And we recover the known Freund–Rubin and Englert solutions.

$\mathfrak{so}(3)$

$\mathfrak{m} \cong [0] \oplus 2[1]$ $\quad$ L_{ia}, L_{45} $\quad$ $a = 4, 5$

$\langle L_{ia}, L_{jb} \rangle = \delta_{ij} m_{ab}$ $\quad$ $m_{ab} = m_{ba}$ $\quad$ $\langle L_{45}, L_{45} \rangle = M > 0$

Can take $\mathfrak{m}_{ab}$ diagonal.

There is a 6-dimensional space of closed invariant 4-forms.

We find, in addition to recovering the Freund–Rubin, Pope–Warner and Castellani–Romans–Warner solutions, two additional backgrounds, one of which we can only approximate numerically.

Full isometry group?

Supersymmetry?

$\mathfrak{so}(3)_+$

We have not finished analysing this case due to its computational complexity. So far we have not found any new backgrounds.

OTHER HOMOGENEOUS BACKGROUNDS

We need to **classify** compact Lie subalgebras

$$\mathfrak{h} \subset \mathfrak{so}(N) \oplus \mathfrak{so}(3,2) \qquad \dim \mathfrak{h} = \binom{N}{2} - 1$$

for $\mathrm{N}{\geq}5$, leaving invariant a lorentzian inner product on $\mathfrak{g}/\mathfrak{h}$.

In particular, $\mathfrak{h} \subset \mathfrak{so}(N) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)$

Because $\mathfrak{h}$ is compact, it leaves invariant a positive-definite inner product β on $\mathfrak{m}$, and every $\mathfrak{h}$-invariant lorentzian inner product on $\mathfrak{m}$ is of the form

$$\beta - \alpha\otimes\alpha$$

for some $\mathfrak{h}$-invariant α in $\mathfrak{m}$.

See, e.g., Alekseevsky (2011)

In particular, unless there are nonzero $\mathfrak{h}$-invariants in $\mathfrak{m}$, there is no $\mathfrak{h}$-invariant lorentzian inner product either.

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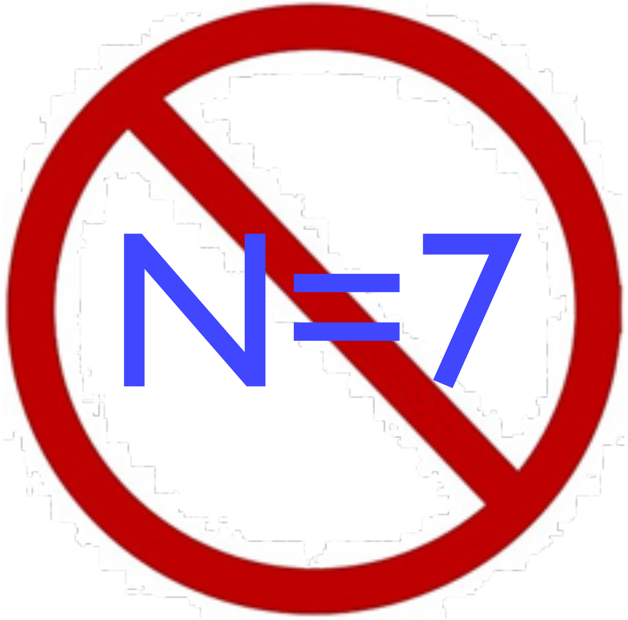
✓ Classify fibred products of the right dimension yielding all possible $\mathfrak{h}$.

✓ Determine $\mathfrak{h}$-invariant lorentzian inner products and $\mathfrak{h}$-invariant 4-forms.

N=7

There are no compact subalgebras of the right dimension, assuming that $\mathfrak{so}(3,2)$ acts effectively.

Otherwise $\mathfrak{so}(3,2)$ acts trivially, in which case we can take SO(7)/SO(5), which does have an invariant lorentzian metric and even invariant fluxes, but they do **not** satisfy the supergravity field equations.



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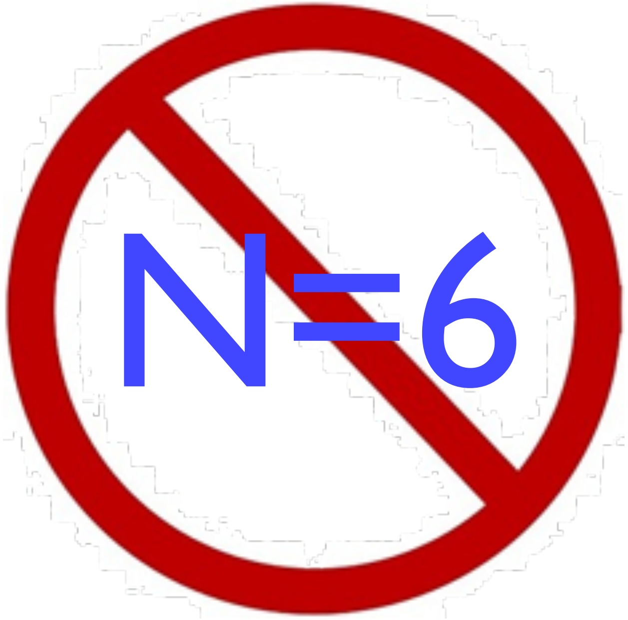
N=6

If $\mathfrak{so}(3,2)$ acts effectively, then

$$\mathfrak{h} = \mathfrak{so}(5)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(2),$$

but there is no lorentzian invariant metric on such a homogeneous space.

If $\mathfrak{so}(3,2)$ acts trivially, $\mathfrak{h} \cong \mathfrak{so}(3)\oplus \mathfrak{so}(2)$. There are 5 inequivalent $\mathfrak{so}(3)\oplus\mathfrak{so}(2)$ subalgebras of $\mathfrak{so}(6)$, 3 of which have invariant lorentzian metrics and invariant fluxes. (Still must check the field equations.)



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N=5

Here $\mathfrak{so}(3,2)$ has to act effectively, by dimension.

There is precisely one possible subalgebra of the right dimension:

$$\mathfrak{h} = \mathfrak{so}(4)\oplus\mathfrak{so}(3)$$

and there are invariant lorentzian inner products.

The resulting geometry is

$$S^4 \times X^7 \qquad\qquad X^7 \cong \mathrm{SO}(3,2)/\mathrm{SO}(3)$$

The backgrounds in question are all of the form

$$S^4 \times X^7 \qquad X^7 \cong \mathrm{SO}(3,2)/\mathrm{SO}(3)$$

$$\mathfrak{so}(3,2) = \mathfrak{so}(3) \oplus \mathfrak{m} \qquad \mathfrak{m} \cong 2[0] \oplus [1]$$

The most general lorentzian metric has four parameters: $\gamma_1, \underbrace{\gamma_2, \gamma_3}_{V_1 \oplus V_1}, \gamma_0$

$$\gamma_0 < 0$$

radius of S^4

$$\gamma_{1,2,3} > 0$$

V_0

Used normaliser of $\mathfrak{so}(3)$ in $\mathfrak{so}(3,2)$ to diagonalise inner product on $V_1 \oplus V_1$

There is a 6-dimensional space of invariant 4-forms:

$$\left(\Lambda^4\mathfrak{m}\right)^{\mathrm{SO}(3)} = \mathbb{R}\omega^2 \oplus S^3\mathbb{R}^2 \otimes \left(\Lambda^3 V_1\right)^{\mathrm{SO}(3)} \otimes V_0$$

and the volume form on S^4

ω = SO(3)-invariant symplectic form on $V_1 \oplus V_1$

Finally we use the homothety symmetry

$$g \mapsto t^2 g \qquad F \mapsto t^3 F$$

to set $\gamma_0 = -1$

In summary, we have a number of (almost polynomial) equations on 9 real parameters:

$$\gamma_1, \gamma_2, \gamma_3, \varphi_1, \ldots, \varphi_6 \qquad \gamma_{1,2,3} > 0$$

The Maxwell equations:

$$\varphi_4 = \frac{\gamma_2^{1/2}\varphi_1\varphi_3}{3\gamma_1^2\gamma_3^{1/2}} \qquad \frac{\gamma_2\varphi_3}{\gamma_3} = \left(\frac{2\gamma_3}{\gamma_2} + \frac{\gamma_3^{1/2}\varphi_1}{\gamma_1^2\gamma_2^{1/2}}\right)\varphi_4$$

$$\varphi_5 = \frac{\gamma_3^{1/2}\varphi_1\varphi_2}{3\gamma_1^2\gamma_2^{1/2}} \qquad \frac{\gamma_3\varphi_2}{\gamma_2} = \left(\frac{2\gamma_2}{\gamma_3} + \frac{\gamma_2^{1/2}\varphi_1}{\gamma_1^2\gamma_3^{1/2}}\right)\varphi_5$$

$$0 = \left(\frac{\varphi_1}{\gamma_1^2} + \frac{2}{\gamma_2^{1/2}\gamma_3^{1/2}}\right)\varphi_6 \,.$$

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$$0 = \left(\frac{\varphi_1}{\gamma_1^2} + \frac{2}{\gamma_2^{1/2}\gamma_3^{1/2}}\right)\varphi_6 \, .$$

⇨ $$\varphi_3 = \frac{1}{3}\left(\frac{2\gamma_3}{\gamma_2} + \frac{\gamma_3^{1/2}\varphi_1}{\gamma_1^2\gamma_2^{1/2}}\right)\frac{\gamma_3^{1/2}\varphi_1}{\gamma_1^2\gamma_2^{1/2}}\varphi_3 \quad \text{and} \quad \varphi_2 = \frac{1}{3}\left(\frac{2\gamma_2}{\gamma_3} + \frac{\gamma_2^{1/2}\varphi_1}{\gamma_1^2\gamma_3^{1/2}}\right)\frac{\gamma_2^{1/2}\varphi_1}{\gamma_1^2\gamma_3^{1/2}}\varphi_2$$

The Einstein equations:

$$
\begin{aligned}
0 &= \varphi_1\varphi_2\varphi_3\left(3\gamma_1^2(\gamma_2+\gamma_3)+2\sqrt{\gamma_2\gamma_3}\varphi_1\right)\\
0 &= 3\gamma_1^4\varphi_6^2+\gamma_2^2\gamma_3^2(12\gamma_1^3-\varphi_1^2)+3\gamma_1^4\gamma_2\gamma_3\left((\gamma_2-\gamma_3)^2-1\right)\\
0 &= \left(3\gamma_2\gamma_1^4+\gamma_3\varphi_1^2\right)\gamma_2^2\varphi_3^2+\left(3\gamma_3\gamma_1^4+\gamma_2\varphi_1^2\right)\gamma_3^2\varphi_2^2-9\gamma_1^4\gamma_2\gamma_3\varphi_6^2+6\gamma_2^3\gamma_3^3\varphi_1^2-54\gamma_1^3\gamma_2^3\gamma_3^3\\
0 &= 6\gamma_1^4\gamma_2^3\varphi_3^2+\left(\gamma_2\varphi_1^2-3\gamma_1^4\gamma_3\right)\gamma_3^2\varphi_2^2-9\gamma_1^4\gamma_2\gamma_3\varphi_6^2+3\gamma_2^3\gamma_3^3\varphi_1^2+9\gamma_1^4\gamma_2^2\gamma_3^2\left(\gamma_2^2-\gamma_3^2-6\gamma_2+1\right)\\
0 &= 6\gamma_1^4\gamma_3^3\varphi_2^2+\left(\gamma_3\varphi_1^2-3\gamma_1^4\gamma_2\right)\gamma_2^2\varphi_3^2-9\gamma_1^4\gamma_2\gamma_3\varphi_6^2+3\gamma_2^3\gamma_3^3\varphi_1^2+9\gamma_1^4\gamma_2^2\gamma_3^2\left(\gamma_3^2-\gamma_2^2-6\gamma_3+1\right)
\end{aligned}
$$

(some time passes)

When the dust settles we find **three** backgrounds, and no others!

Probably related by “double Wick rotation” to the $\mathrm{AdS}_4\times \mathrm{SO}(5)/\mathrm{SO}(3)$ backgrounds.

An approximate background

$$\begin{aligned}\gamma_1 &= 0.22776420155467458\\ \gamma_2 &= 0.4670546272324634\\ \gamma_3 &= 0.12728016028858763\\ \varphi_1 &= 0.14715771499261474\\ \varphi_3 &= \pm 0.27380714065085027\end{aligned}$$

We know little else about this solution, except that it has no additional symmetries.

Supersymmetry?

A circle's worth of backgrounds

$$\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$$

$$\varphi_1 = -\frac{1}{3}, \quad \varphi_2 = \frac{1}{\sqrt{3}} \cos\alpha, \quad \varphi_3 = \frac{1}{\sqrt{3}} \sin\alpha$$

The 7-dimensional lorentzian geometry is **Sasakian**, and the background has an extra Killing vector: the Sasaki vector field.

The timelike cone is an 8-dimensional **pseudo-Kähler** manifold; that is, of holonomy $U(3,1) \subset SO(6,2)$.

Supersymmetry?

A susy Freund-Rubin background

$$\gamma_2 = \gamma_3 = \frac{2}{3} \quad \gamma_1 = \frac{4}{9} \quad \varphi_1 = \pm\frac{8}{9}$$

The 7-dimensional lorentzian geometry is **Sasaki-Einstein**. $\therefore$ it has an additional Killing vector and the (time-like) cone is **pseudo-Calabi-Yau**; i.e., of holonomy SU(3,1) ⊂ SO(6,2).

Freund-Rubin ⇒ The background is **supersymmetric**!

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Freund-Rubin → The background is **supersymmetric**!

JMF+Leitner+Simón (201?)

However it is only **N=2** supersymmetric.

Killing spinors

$$\varepsilon^{Ia} = \zeta^{Ia} \otimes \psi \qquad I = 1, \ldots, 4 \qquad a = 1, 2$$

(symplectic Majorana)

geometric Killing spinors on S^4

geometric Killing spinor on LES

Sasaki vector field

Killing superalgebra: $[\varepsilon^{Ia}, \varepsilon^{Jb}] = \Omega^{IJ} \epsilon^{ab} \chi$

SO(3,2)×SO(5) is an "accidental" symmetry.

A LOOK AHEAD

Still left to do...

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- For completeness: **explore** the homogeneous $\mathrm{SO}(4)$-invariant backgrounds and **classify** those cases where $\mathrm{SO}(3,2)$ acts trivially

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ご清聴ありがとうございました。