On supersymmetric space forms

José Figueroa-O'Farrill

Edinburgh Mathematical Physics Group

School of Mathematics

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Space forms

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Killing's identity $\Longrightarrow$ a Killing vector ξ is uniquely determined by:

- its value $\xi(p)$ at a point p; and
- the value $\nabla_\mu \xi_\nu(p) = -\nabla_\nu \xi_\mu(p)$ of its derivative at p.

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The flat and spherical cases are solved (culminating in the work of Wolf in the 1970s), but the hyperbolic and lorentzian cases remain largely open despite many partial results.

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In this talk I will report on progress towards answering this question.

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- George Papadopoulos (King's College, London)
  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170`

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  - ⋆ `math.AG/0211170`
- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
  - ⋆ `in preparation`

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N=2$			$N=1$			
8	$N=2$			$N=1$			
7	$N=4$			$N=2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
4	$N=8$	$N=6$	$N=5$	$N=4$	$N=3$	$N=2$	$N=1$

[Van Proeyen, hep-th/0301005]

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defined by the supersymmetric variation of the gravitino:

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where I is an index labeling the following elements

$$\Gamma_a \quad \Gamma_{ab} \quad \Gamma_{abc} \quad \Gamma_{abcd} \quad \Gamma_{abcde}$$

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- if the plane is null, we can choose a pseudo-orthonormal frame in which the only nonzero component of F is F_{-123}

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- F null: a one parameter $\mu \in \mathbb{R}$ family of *symmetric plane waves*:

$$g = 2dx^+dx^- - \tfrac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2 \right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

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All vacua embed isometrically in $\mathbb{E}^{2,11}$ as the intersections of two quadrics.

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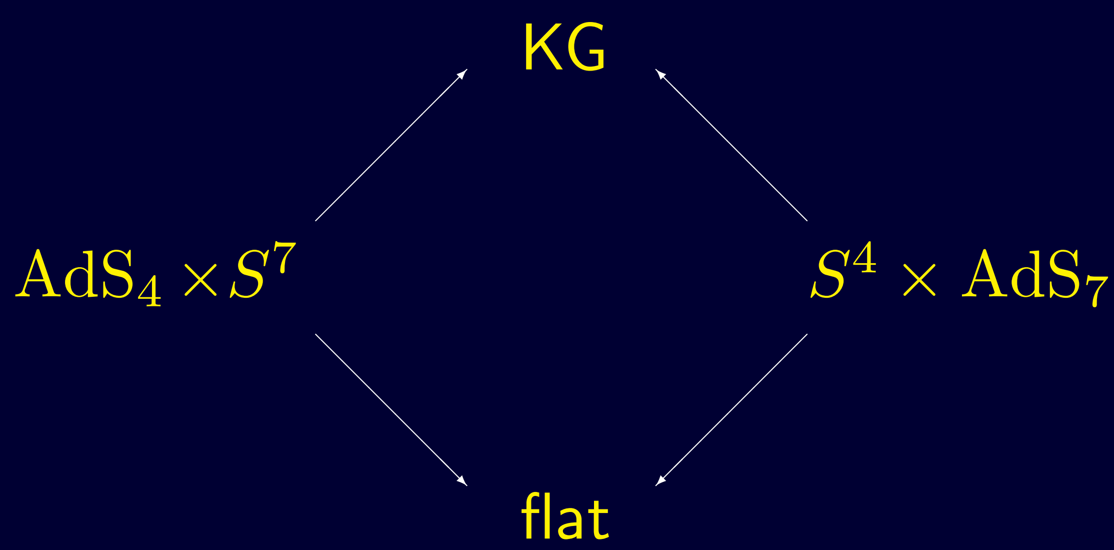

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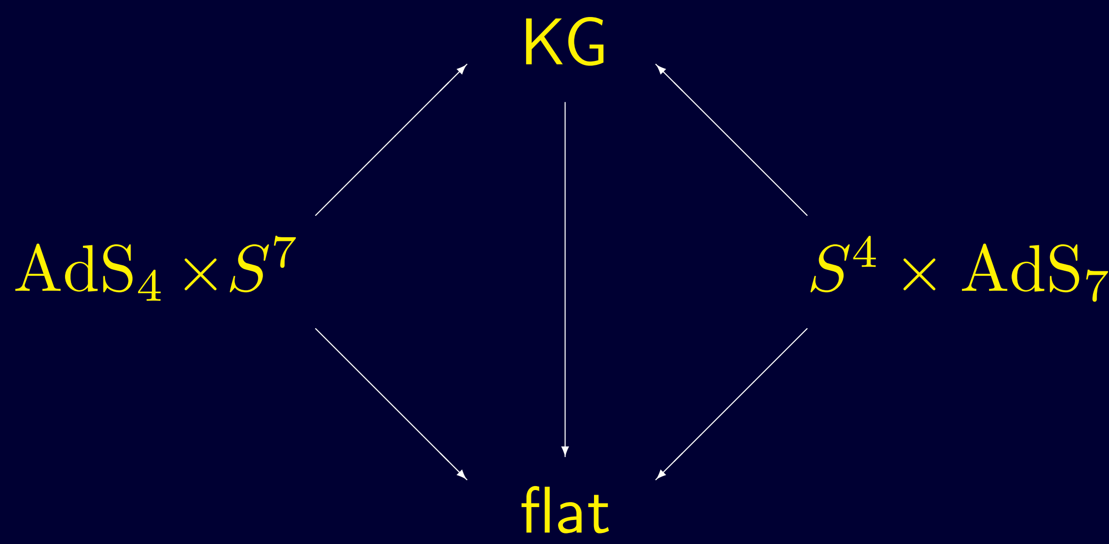

[Back]

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The solution to this problem is known.

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But there is a more general construction.

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- since $\mathfrak{h}$ preserves the metric on $\mathfrak{g}$, there is a linear map

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- $\mathfrak{h}$ acts on $\mathfrak{g} \times_\omega \mathfrak{h}^*$ preserving the Lie bracket, so we can form the *double extension*

$$\mathfrak{d}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{h} \ltimes (\mathfrak{g} \times_\omega \mathfrak{h}^*)$$

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This construction is due to Medina and Revoy who proved an important structure theorem.

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(Any semisimple factors in $\mathfrak{a}$ factor out of the double extension. [FO–Stanciu `hep-th/9402035`])

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The third case is a six-dimensional version of the Nappi-Witten spacetime, NW_6, discovered by Meessen. [Meessen hep-th/0111031]

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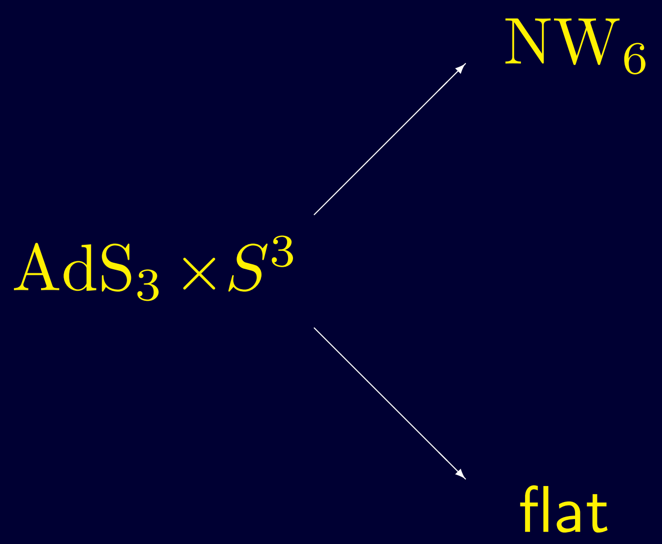

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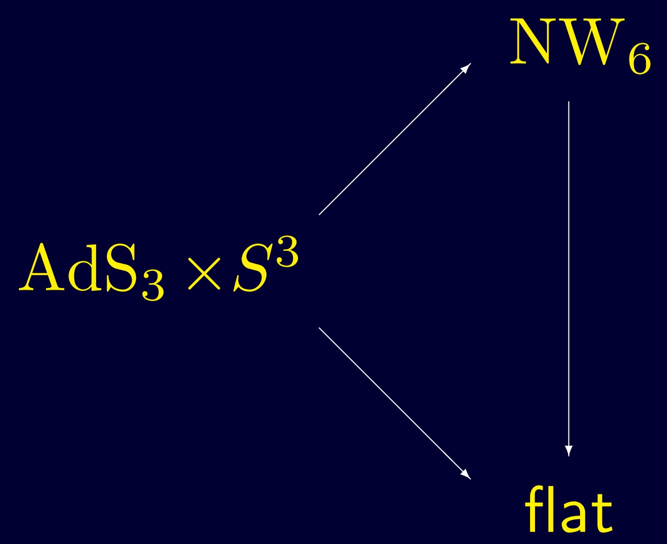

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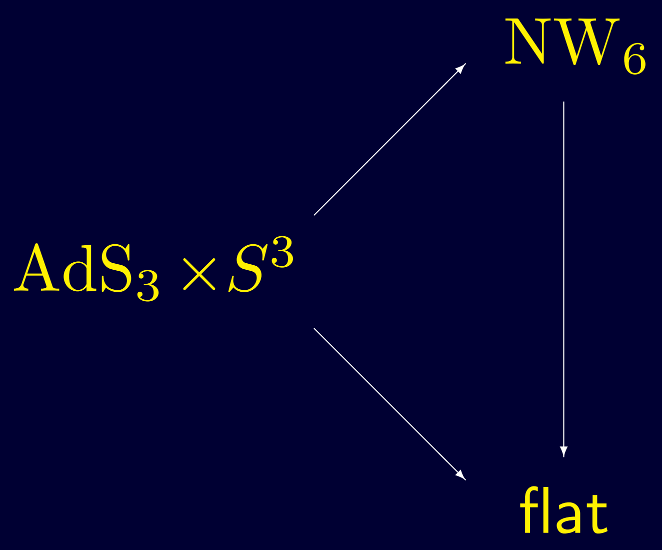

which in this case are *group contractions* à la Inönü–Wigner.

[Stanciu–FO `hep-th/0303212`]

[Back]

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(Unfortunate notation: a 2-Lie algebra is a Lie algebra.)

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(n-Lie algebras also appear naturally in the context of Nambu dynamics. [Nambu (1973)])

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[FO–Papadopoulos `math.AG/0211170`]

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[Blau–FO–Hull–Papadopoulos `hep-th/0110242`]

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Notice that g is a bi-invariant metric on a Lie group: a ten-dimensional version of the Nappi–Witten spacetime.

[Stanciu–FO `hep-th/0303212`]

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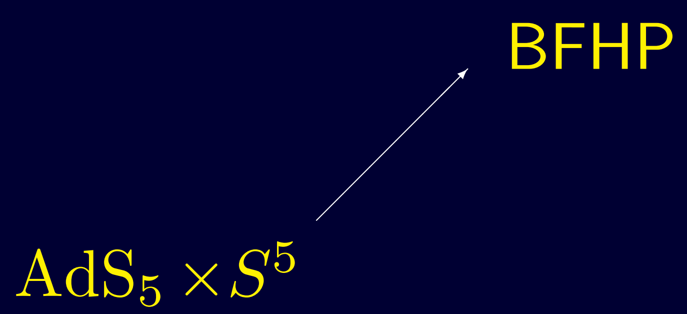

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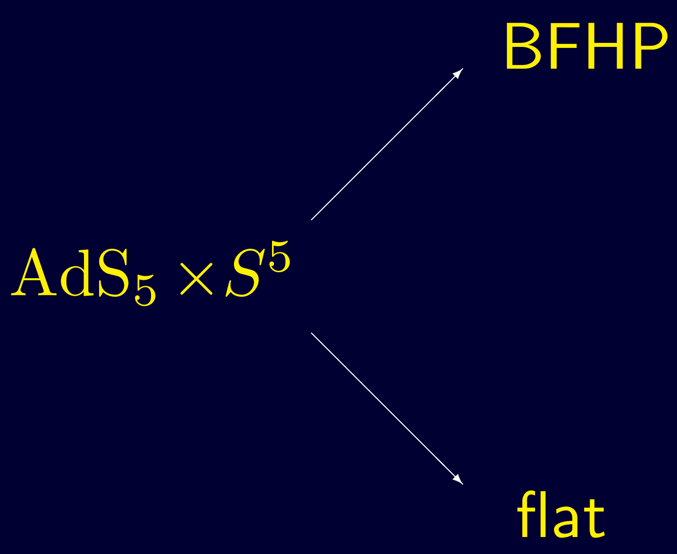

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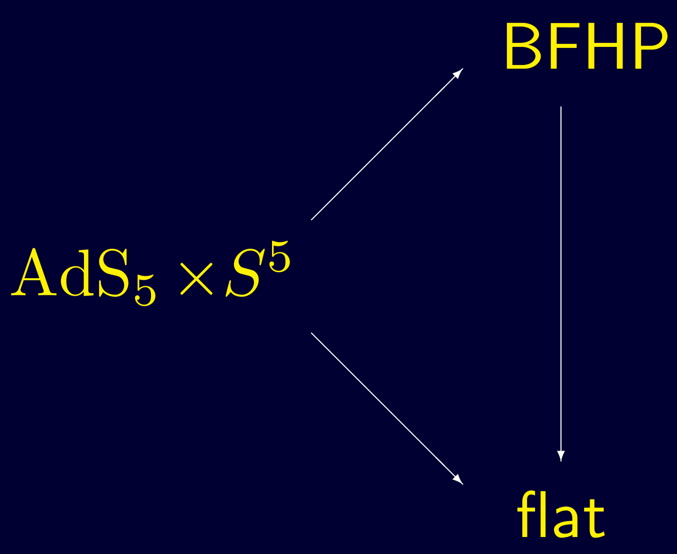

[Back]

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[Gauntlett–Gutowsky–Hull–Pakis–Reall `hep-th/0209114`]

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This now has a natural explanation.

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Thank you.