Homogeneous kinematical spacetimes

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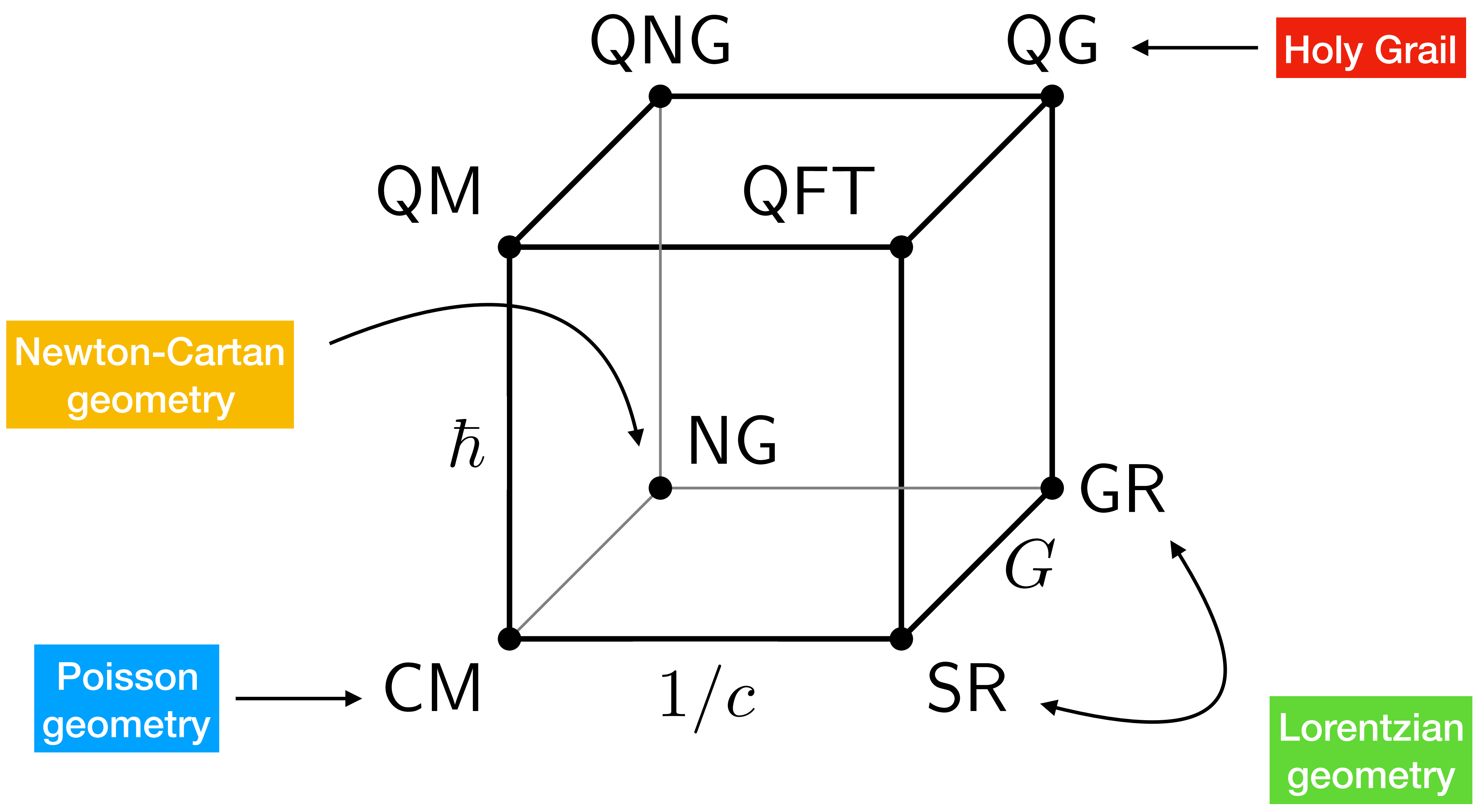
Based on…

- **1711.06111**
- **1711.07363**
- **1802.04048** with Tomasz Andrzejewski
- **1809.01224** with Stefan Prohazka
- **1905.00034** with Ross Grassie and Stefan Prohazka
- **1908.11278** with Ross Grassie

Part 1
Main results

Motivation


QNG
QG
Holy Grail
QM
QFT
Newton-Cartan geometry
$\hbar$
NG
GR
G
Poisson geometry
CM
$1/c$
SR
Lorentzian geometry

Motivation

Maximally symmetric lorentzian manifolds — (anti) de Sitter and Minkowski spacetimes — play an important rôle in contemporary theoretical physics: GR, QFT, AdS/CFT,…

There is a desire to explore “non-relativistic” limits of these theories, in view of its applications to flat space holography, condensed matter,…

Natural question: What are the “non-relativistic” analogues of these spacetimes?

Dramatis personae

Minkowski spacetime (D+1 dimensional)

$\mathbb{A}^{D+1}$ **with metric** $dx_1^2 + dx_2^2 + \cdots + dx_D^2 - c^2 dx_{D+1}^2$

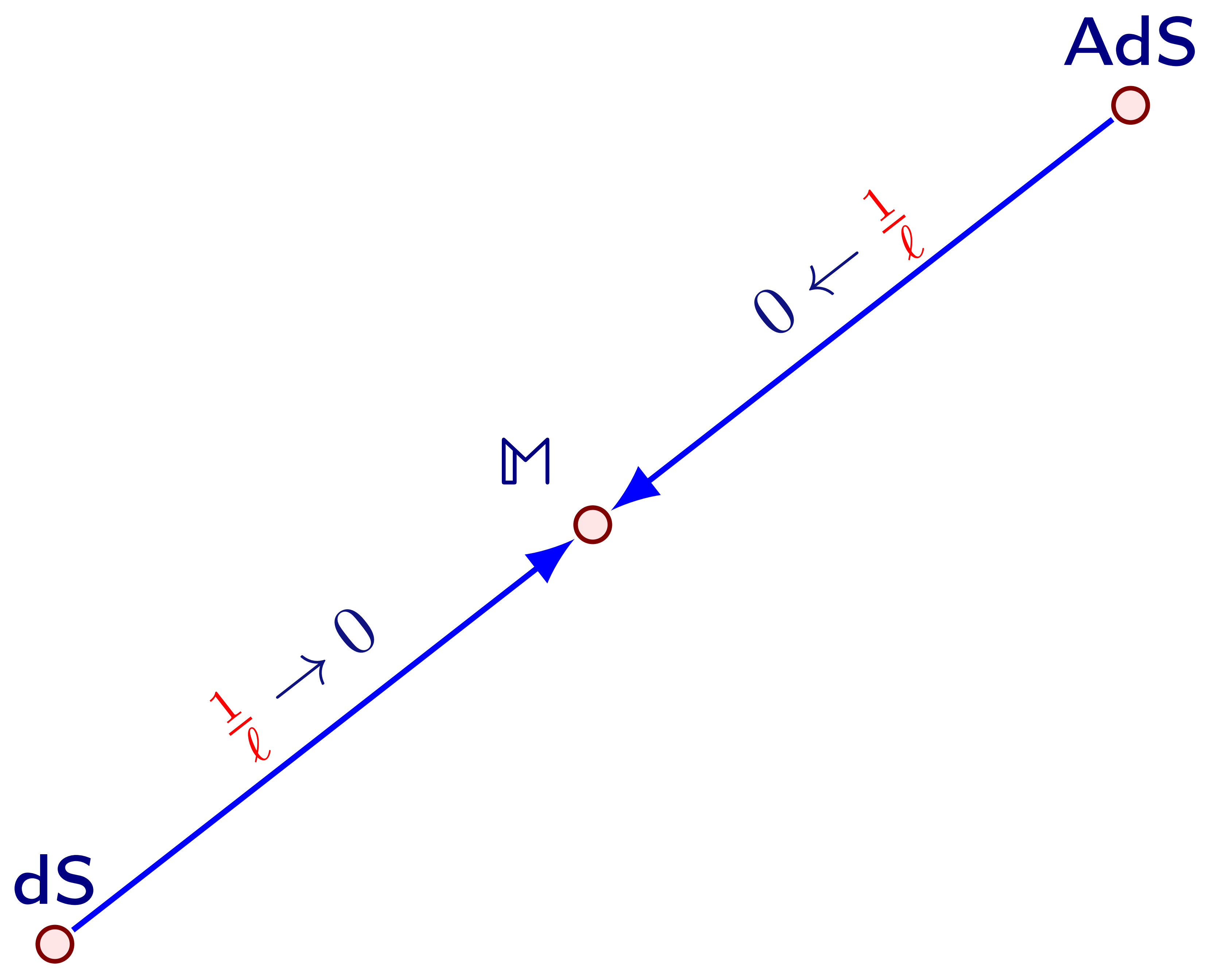
de Sitter spacetime (D+1 dimensional)

$x_1^2 + x_2^2 + \cdots + x_D^2 + x_{D+1}^2 - x_{D+2}^2 = \ell^2$ **in** $\mathbb{R}^{D+1,1}$

Anti de Sitter spacetime (D+1 dimensional)

$x_1^2 + x_2^2 + \cdots + x_D^2 - x_{D+1}^2 - x_{D+2}^2 = -\ell^2$ **in** $\mathbb{R}^{D,2}$

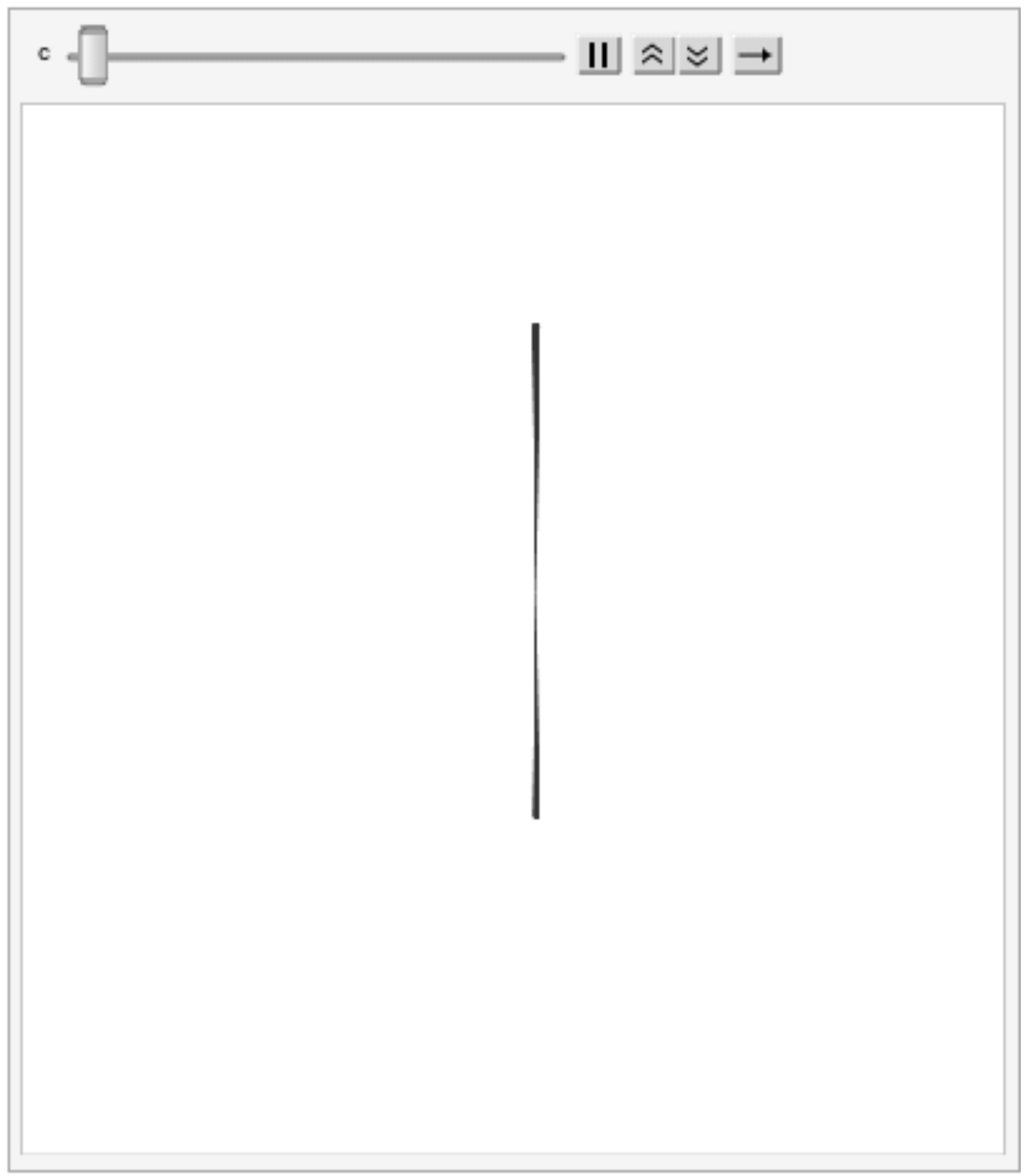
Zero-curvature limits



Non- and ultra-relativistic limits

$$\lim_{c \to 0}$$

Ultra-relativistic

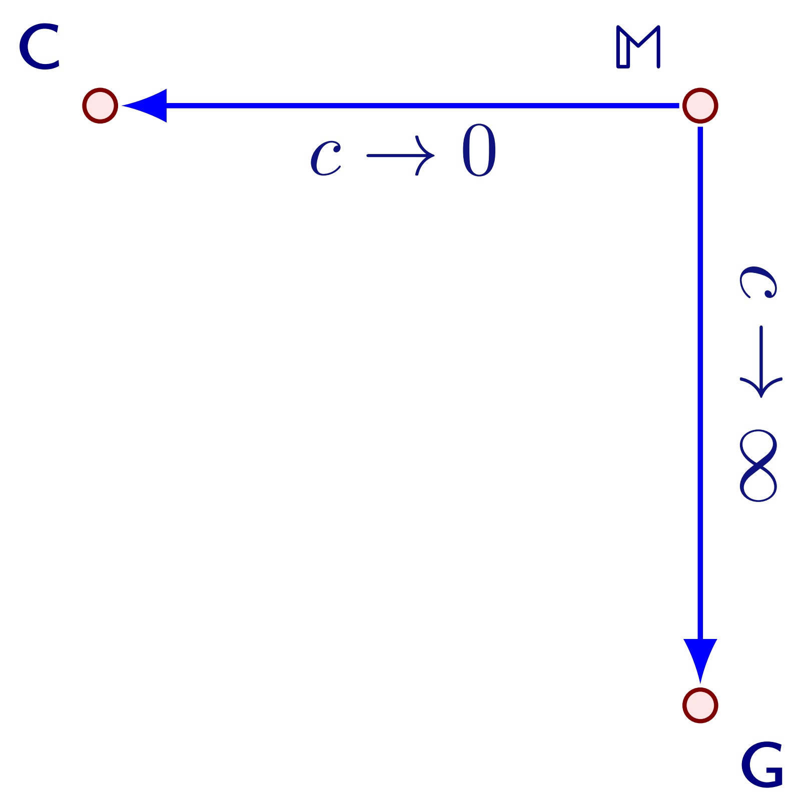


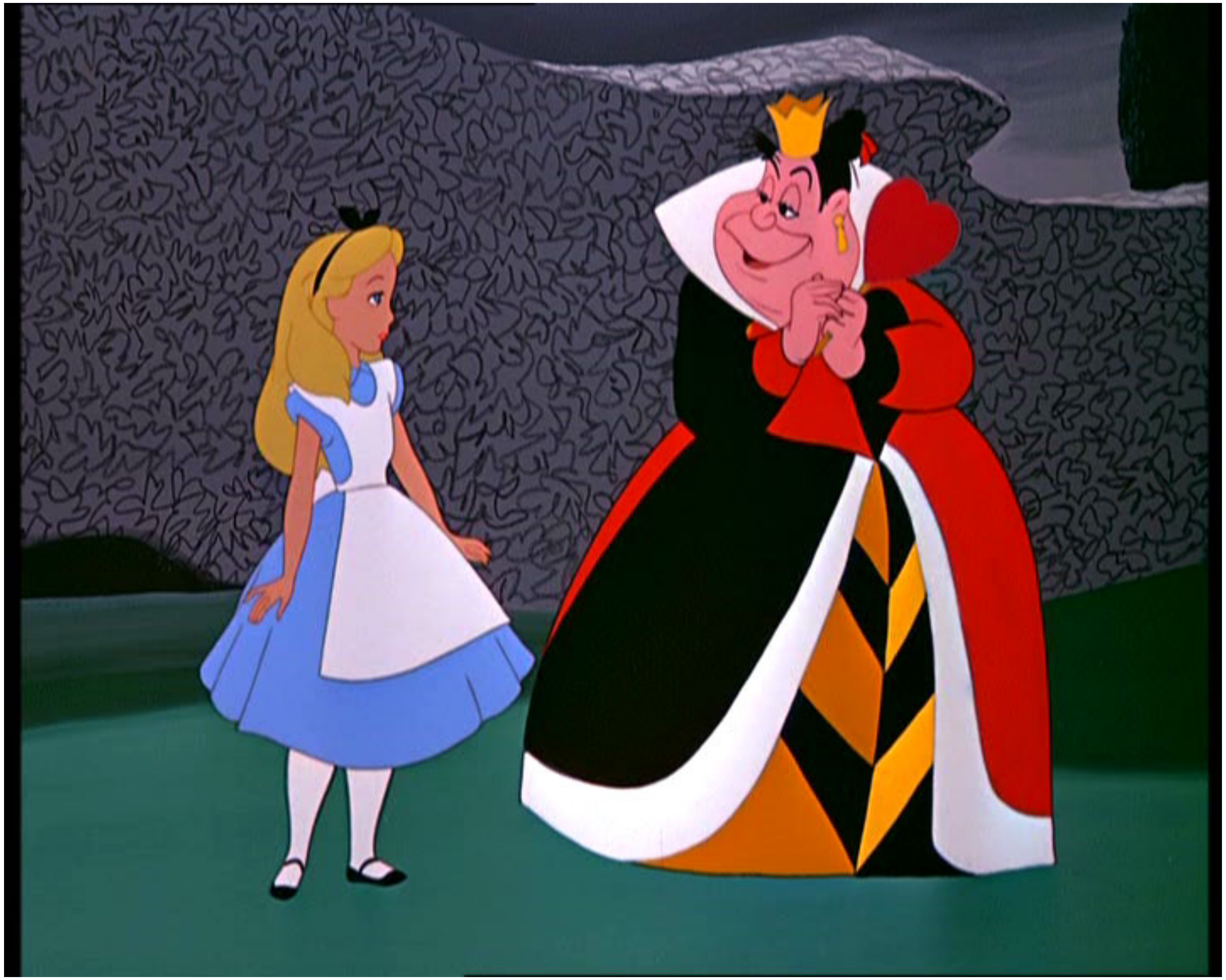
$$\lim_{c \to \infty}$$

Non-relativistic

Beyond lorentzian geometry

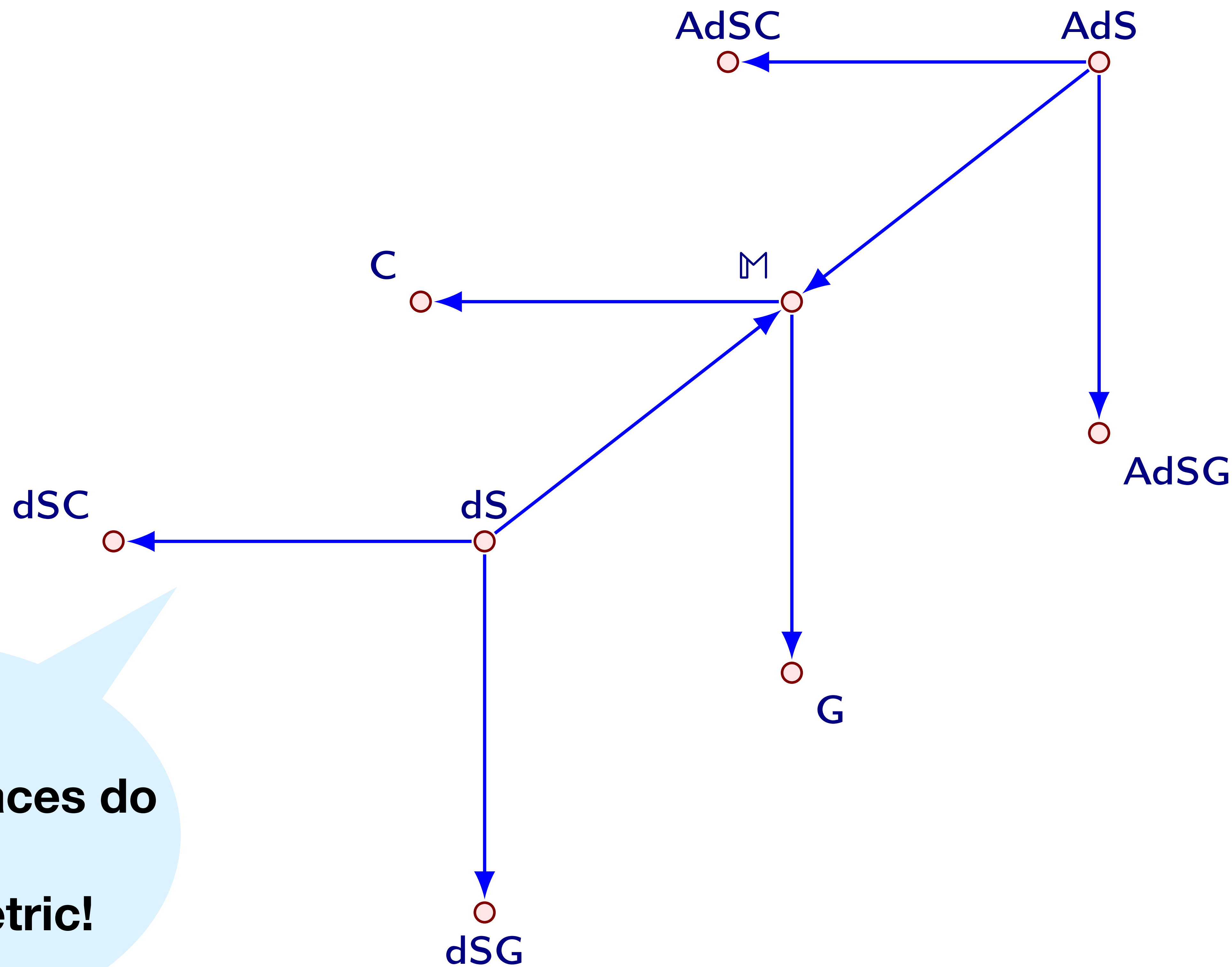
- The **non-relativistic limit of Minkowski spacetime is Galilean spacetime**
- **The ultra-relativistic limit of Minkowski spacetime is Carrollian spacetime** [Lévy-Leblond 1965]



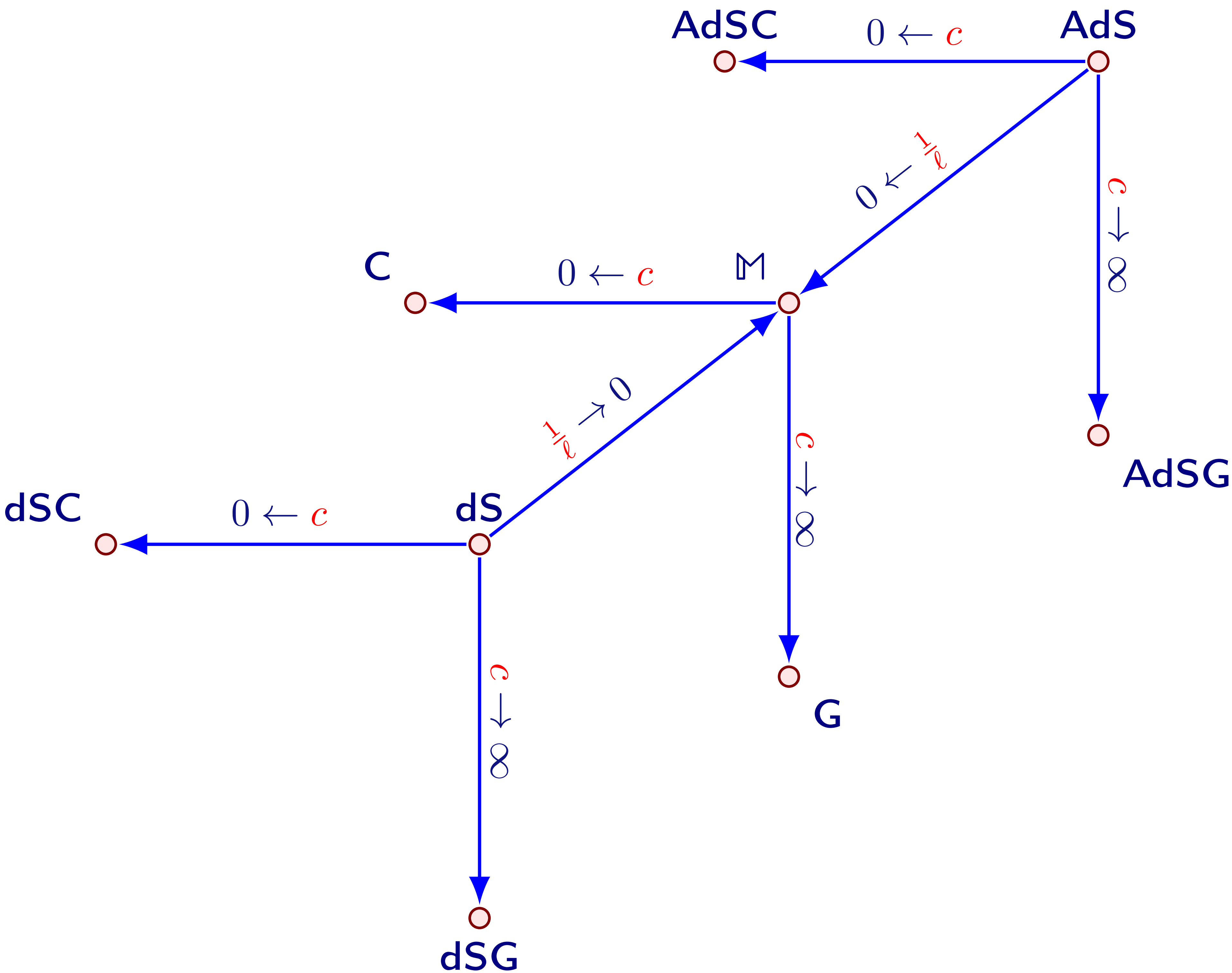


“My dear, here we must run as fast as we can, just to stay in place.”

- **(Anti) de Sitter spacetimes also have such limits: galilean (A)dS and carrollian (A)dS**



galilean and carrollian spaces do not inherit a lorentzian metric!

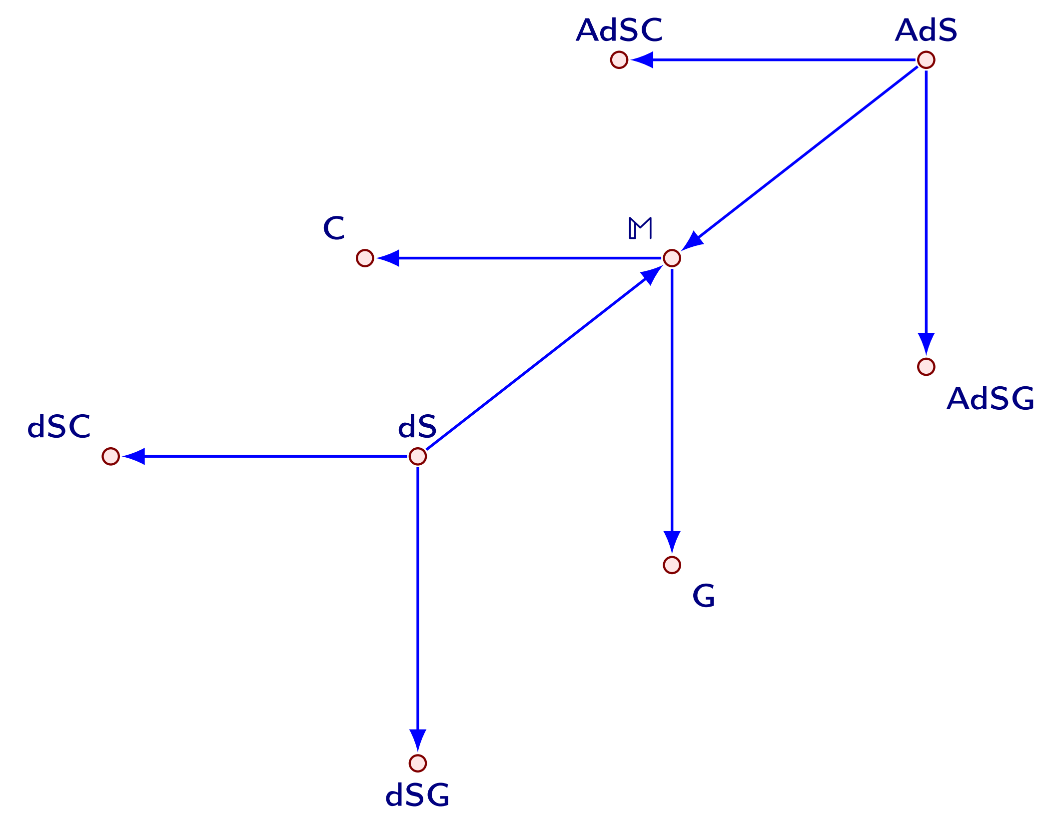


AdSC
AdS
$0 \leftarrow c$
$0 \leftarrow \frac{1}{\ell}$
$c \to \infty$
AdSG
C
$\mathbb{M}$
$0 \leftarrow c$
$\frac{1}{\ell} \to 0$
$c \to \infty$
G
dSC
dS
$0 \leftarrow c$
$c \to \infty$
dSG

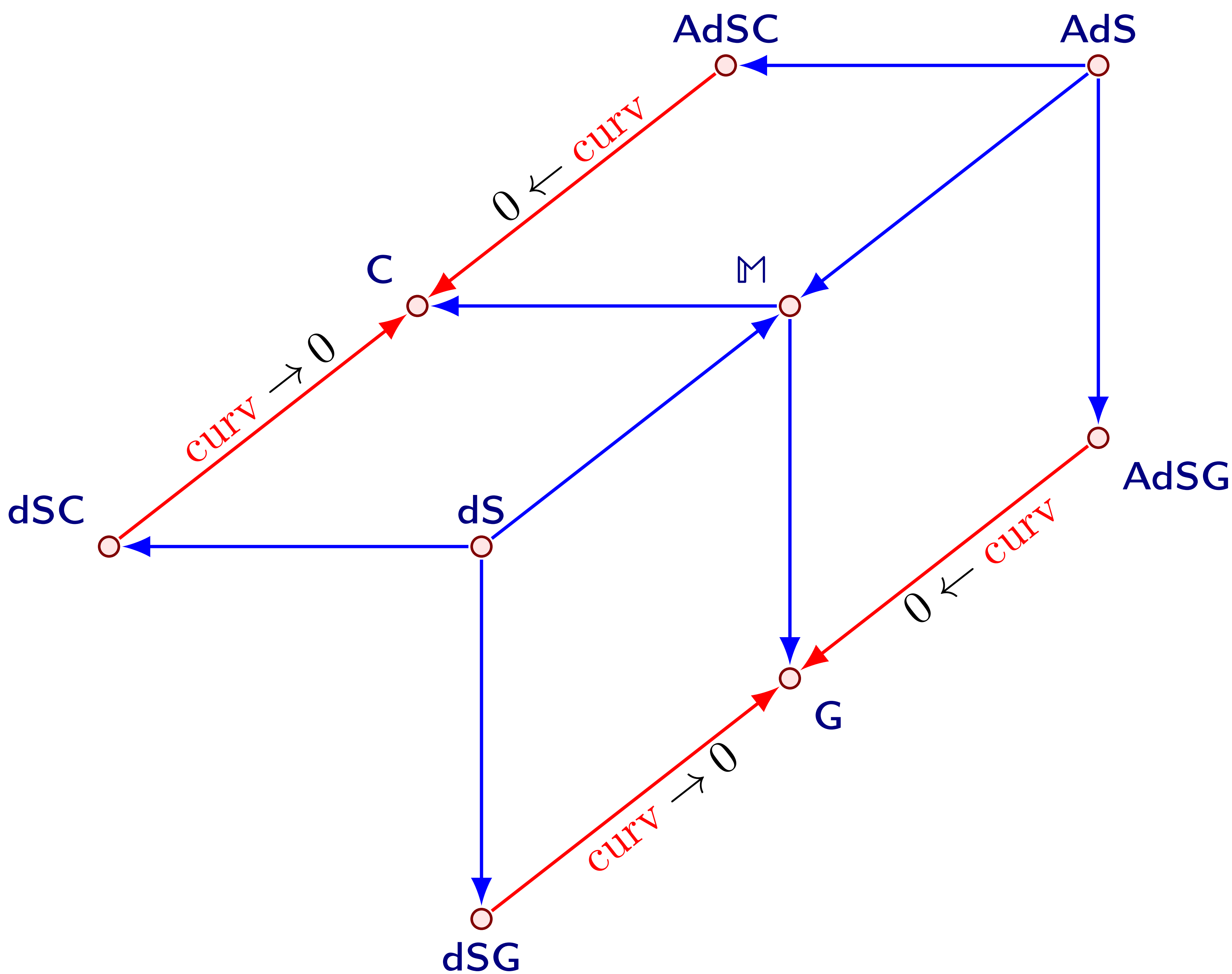
Symmetric spaces of kinematical Lie groups

- These 9 spacetimes: **M, AdS, dS, C, G, dSC, AdSC, dSG, AdSG** (together with the riemannian symmetric spaces **E, S, H**) are symmetric homogeneous spaces of **kinematical Lie groups** (with **D**-dimensional space isotropy)

- Symmetric homogeneous spaces admit canonical torsion-free invariant connections. For **C, G, dSC, AdSC, dSG, AdSG** these are **not** metric connections. We can nevertheless still take the zero curvature limit of **dSC, AdSC, dSG, AdSG**. (**C** and **G** are already flat.)



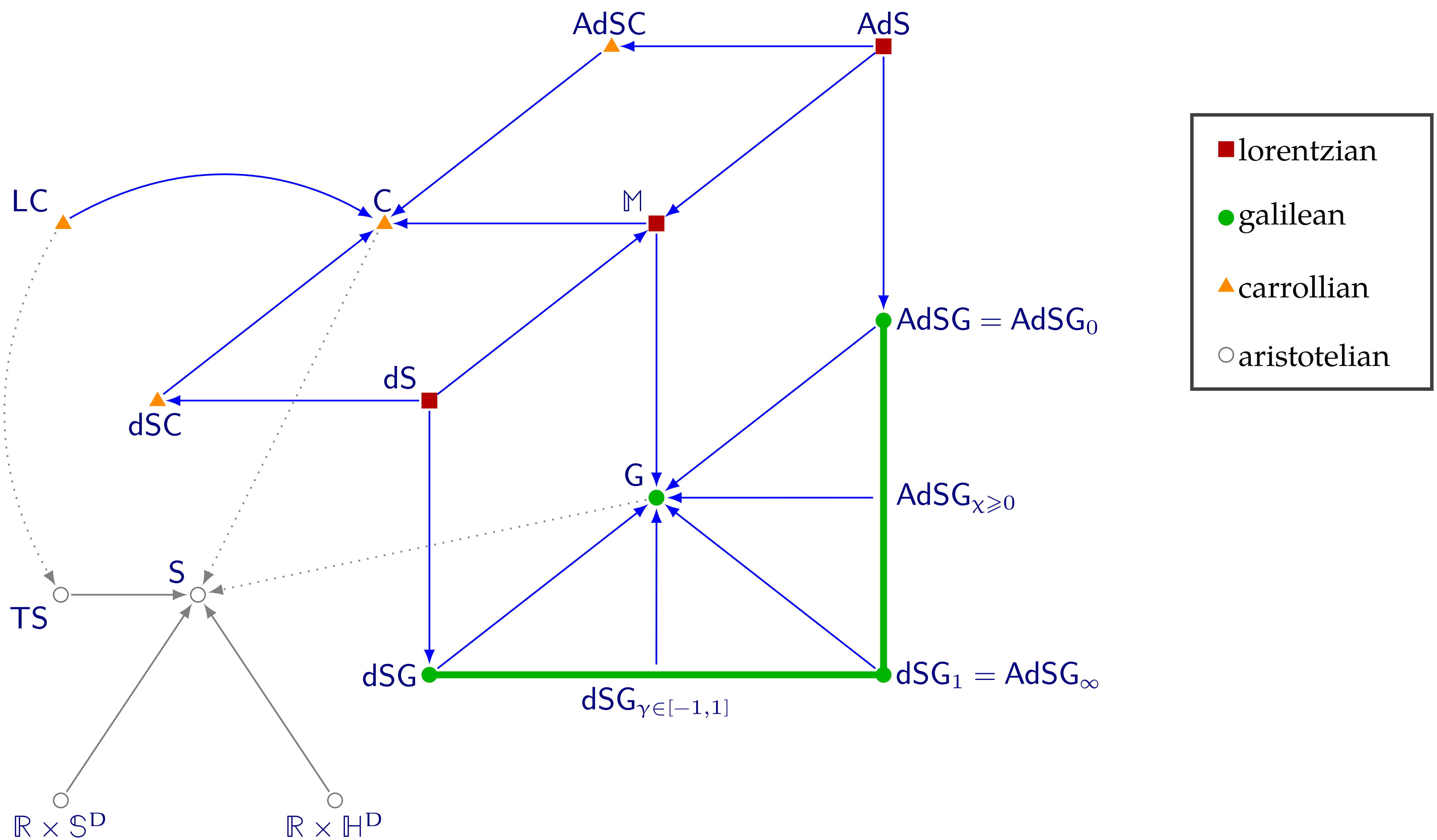
State of (prior) art



Highlights of new results

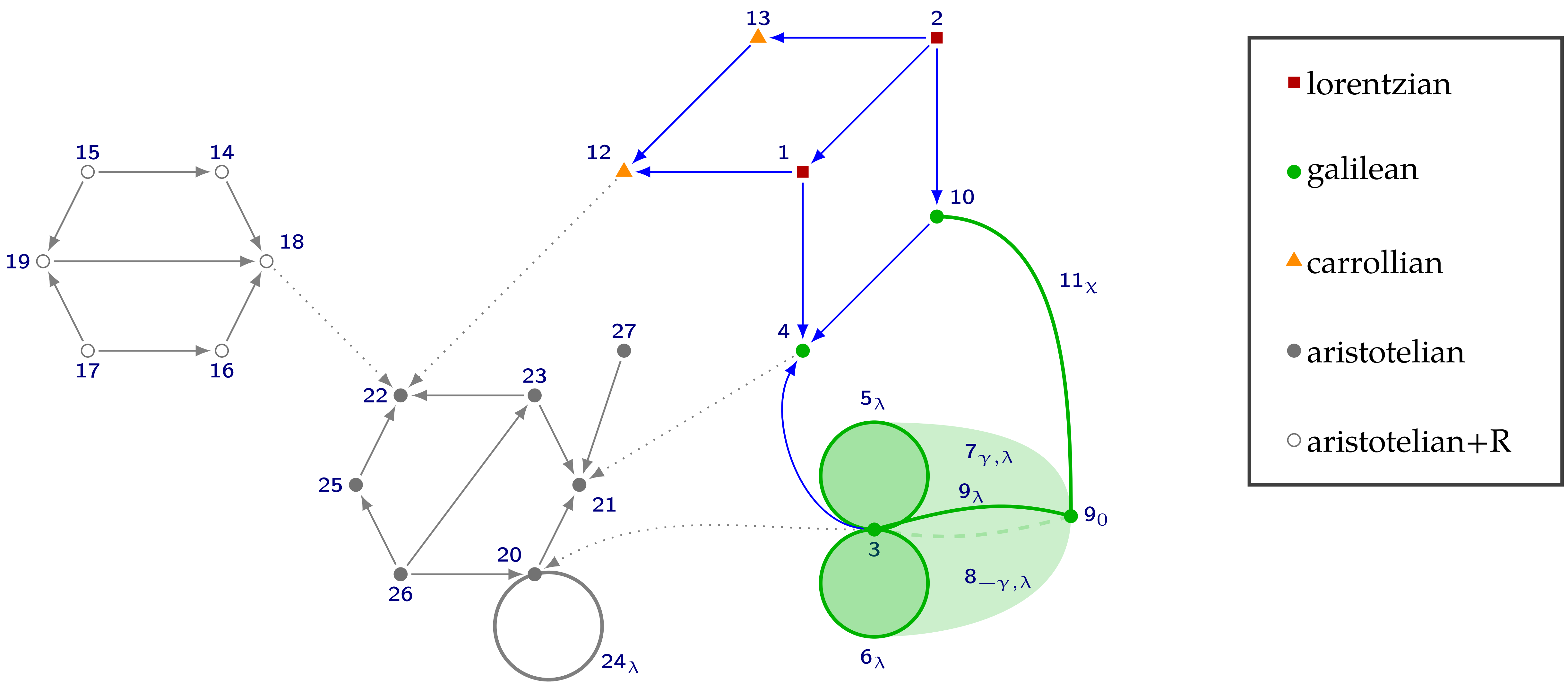
- Classification of (**D+1**)-dimensional, **simply-connected, spatially isotropic, homogeneous,** kinematical spacetimes
- Classification of (**D+1**)-dimensional **aristotelian** spacetimes
- **Limits** between the spacetimes
- Proof that the orbits of **boosts** are generically **non-compact** (except in riemannian and aristotelian "spacetimes", of course)
- Determination of (**infinite-dimensional**) Lie algebras of (conformal) **symmetries**
- Classification of **(3+1)**-dimensional **homogeneous** kinematical **super**spaces and their **limits**

D≥3

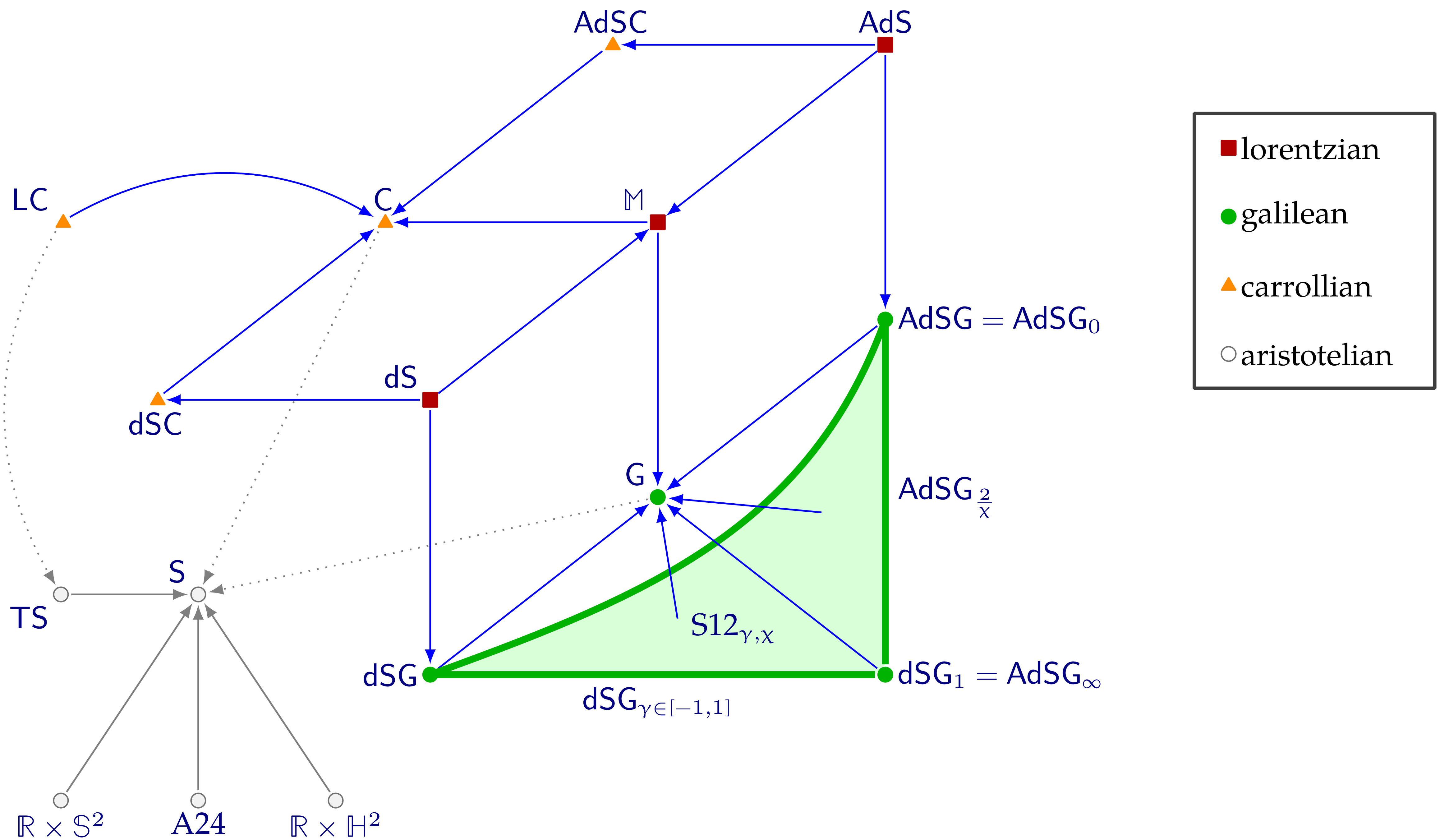


AdSC
AdS
LC
C
𝕄
dS
dSC
G
AdSG = AdSG$_0$
AdSG$_{\chi\geqslant 0}$
S
TS
dSG
dSG$_1$ = AdSG$_\infty$
dSG$_{\gamma\in[-1,1]}$
$\mathbb{R}\times\mathbb{S}^D$
$\mathbb{R}\times\mathbb{H}^D$
lorentzian
galilean
carrollian
aristotelian

D=3 N=1 superspaces



D=2



AdSC
AdS
LC
C
$\mathbb{M}$
AdSG = AdSG_0
dS
dSC
G
$\mathsf{AdSG}_{\frac{2}{\chi}}$
S
TS
$\mathsf{S12}_{\gamma,\chi}$
dSG
$\mathsf{dSG}_1 = \mathsf{AdSG}_\infty$
$\mathsf{dSG}_{\gamma\in[-1,1]}$
$\mathbb{R}\times\mathbb{S}^2$
A24
$\mathbb{R}\times\mathbb{H}^2$
lorentzian
galilean
carrollian
aristotelian

Part 2
(Some) technical details

Kinematical Lie algebras

- The Lie algebra of a kinematical Lie group (with **D**-dimensional space isotropy) is a **kinematical Lie algebra**
- A **kinematical Lie algebra** (with **D**-dimensional space isotropy) is a real Lie algebra $\mathfrak{k}$ of dimension **(D+1)(D+2)/2** such that
  - $\mathfrak{so}(D) \subset \mathfrak{k}$
  - under $\mathfrak{so}(D)$, $\mathfrak{k} = \mathfrak{so}(D) \oplus 2\,V \oplus S$
    - **V** = **D**-dimensional **vector** rep of $\mathfrak{so}(D)$
    - **S** = 1-dimensional **scalar** rep of $\mathfrak{so}(D)$

Typically, we write the generators as

$$J_{ab} \qquad B_a \qquad P_a \qquad H$$

$\mathfrak{so}(D)$ → J_{ab}

"boosts" → B_a

"spatial translations" → P_a

"time translation" → H

The reason for the " " is that the geometrical/physical interpretation can only be given when the Lie algebra acts on a spacetime.

$$[\boldsymbol{J}, \boldsymbol{J}] = \boldsymbol{J} \qquad [\boldsymbol{J}, \boldsymbol{B}] = \boldsymbol{B} \qquad [\boldsymbol{J}, \boldsymbol{P}] = \boldsymbol{P} \qquad [\boldsymbol{J}, H] = 0$$

Examples

Simple Lie algebras $\mathfrak{so}(D+1,1)$ $\mathfrak{so}(D,2)$ $\mathfrak{so}(D+2)$

$$\mathfrak{p} = \mathfrak{so}(D,1) \ltimes \mathbb{R}^{D,1}$$

Poincaré

$$\mathfrak{e} = \mathfrak{so}(D+1) \ltimes \mathbb{R}^{D+1}$$

Euclidean

$$[H, \boldsymbol{B}] = \boldsymbol{P}$$

Galilean

$$[\boldsymbol{B}, \boldsymbol{P}] = H$$

Carroll

More examples

$$[H, \boldsymbol{P}] = \boldsymbol{P} \qquad [H, \boldsymbol{B}] = \gamma \boldsymbol{B} \qquad \gamma \in [-1, 1]$$

$\gamma = -1$ **Newton-Hooke**

$$[H, \boldsymbol{B}] = \chi \boldsymbol{B} + \boldsymbol{P} \qquad [H, \boldsymbol{P}] = \chi \boldsymbol{P} - \boldsymbol{B} \qquad \chi \geq 0$$

$\chi = 0$ **Newton-Hooke**

These are all in **D>3.** For **D=3** and **D=2** there are others due to the existence of

ϵ_{abc} and ϵ_{ab}

Classifications

- **D=0** There is only one 1-dimensional Lie algebra
- **D=1** There are no rotations, so any 3-dimensional Lie algebra is kinematical **[Bianchi 1898]**
- **D=3** **[Bacry+Lévy-Leblond 1968] [Bacry+Nuyts 1986]**
- **D≥3** **[JMF 2017]**
- **D=2** **[Andrzejewski+JMF 2018]**

} **using deformation theory**

Homogeneous spacetimes

$\mathcal{K}/\mathcal{H}$ **Klein pair** $(\mathfrak{k}, \mathfrak{h})$

$\mathcal{K}$ Lie group with kinematical Lie algebra $\mathfrak{k}$

$\mathcal{H}$ closed Lie subgroup with Lie algebra $\mathfrak{h}$

$\mathrm{Span}\{\boldsymbol{J}, \boldsymbol{B}\}$

∃ choice of basis

Theorem There is a one-to-one correspondence between (isomorphism classes of) simply-connected homogeneous spacetimes and (isomorphism classes of) *geometrically realisable*, *effective* Klein pairs.

Caveat

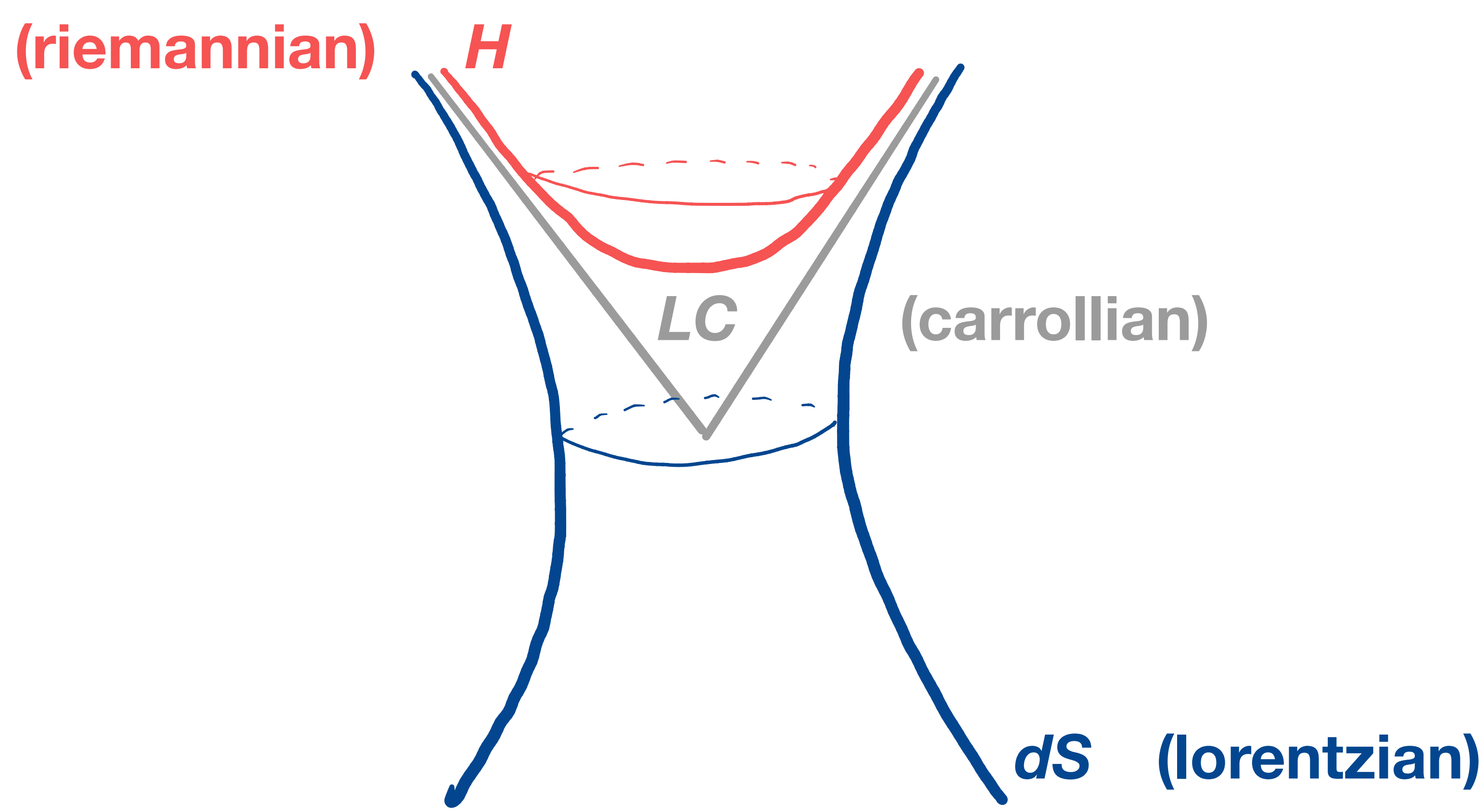
- A kinematical Lie algebra $\mathfrak{k}$ may possess
  - **no** homogeneous spacetimes,
  - a **unique** homogeneous spacetime, or
  - **more than one** homogeneous spacetimes:

$\mathfrak{so}(D+1,1)$ $\left\{\right.$ **de Sitter**, **hyperbolic space**, **carrollian light cone**

$\mathfrak{p}$ $\left\{\right.$ **Minkowski**, **carrollian AdS**

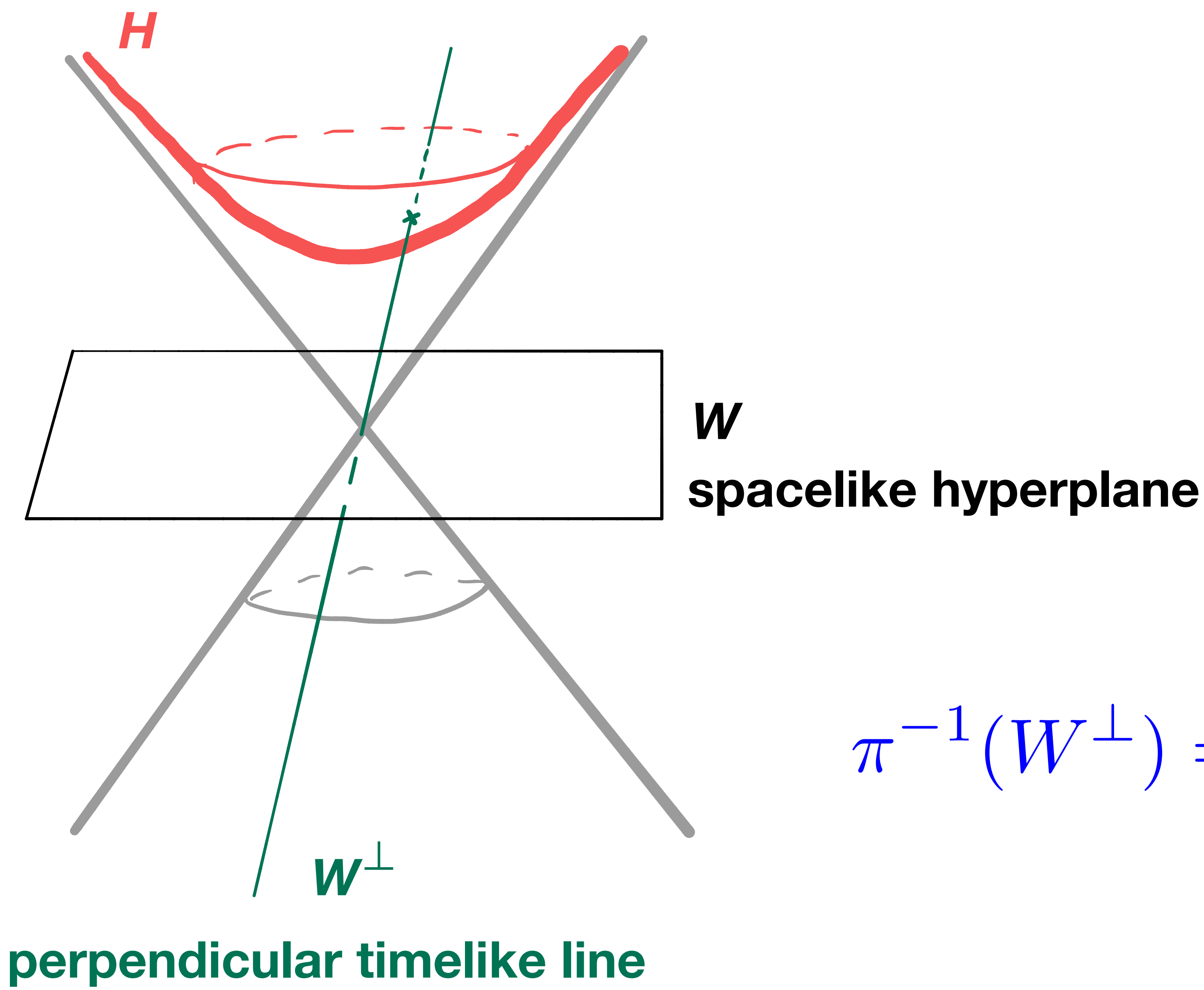
$\mathfrak{e}$ $\left\{\right.$ **Euclidean**, **carrollian dS**

$\mathfrak{so}(1,D+1)$



𝔭

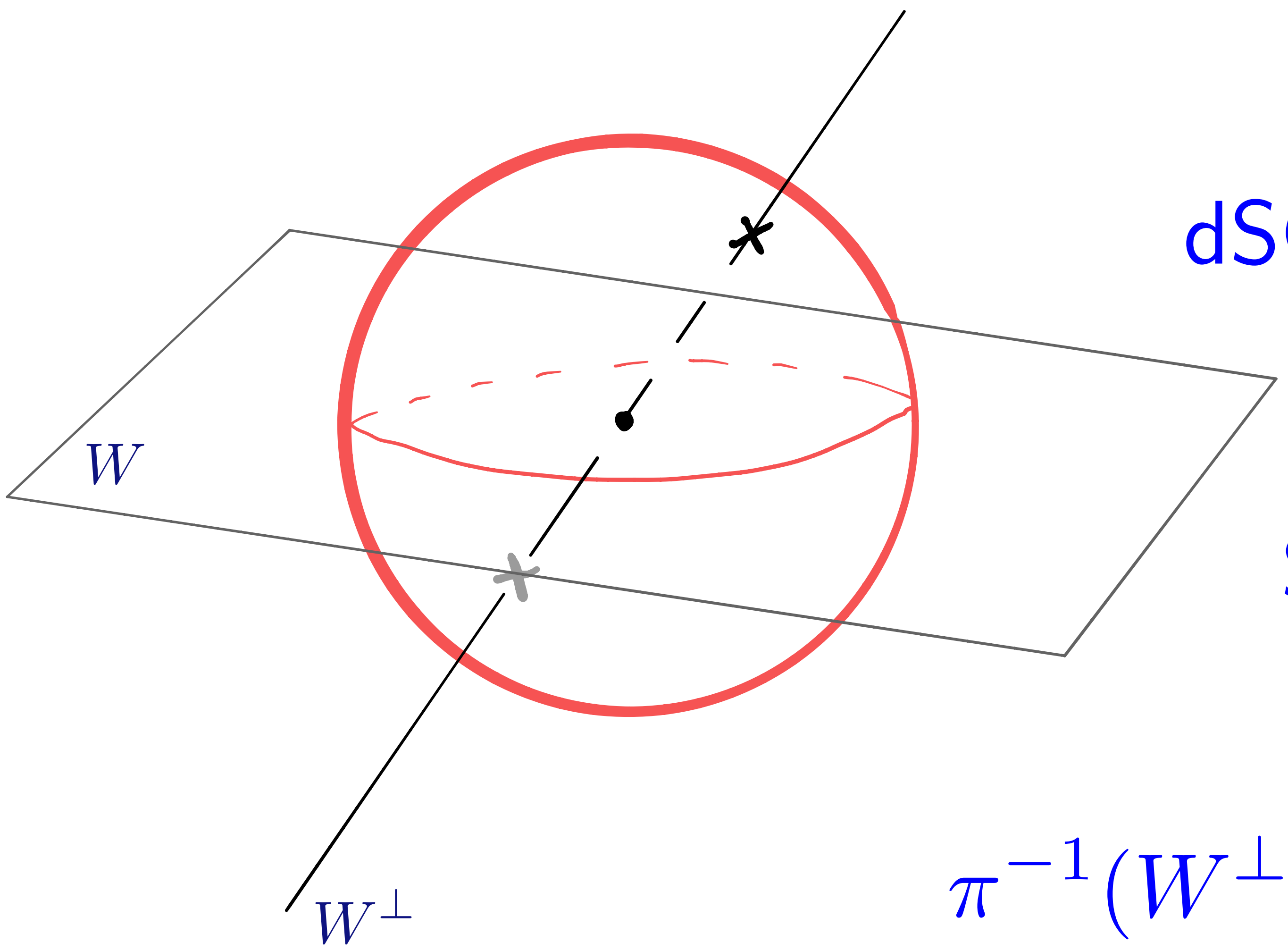
AdSC is the space of affine spacelike hyperplanes in Minkowski spacetime



$$\mathrm{AdSC}^{D+1} \xrightarrow{\pi} \mathrm{H}^D$$

$$\pi^{-1}(W^\perp) = \{v + W \mid v \in W^\perp\}$$

dSC is (the double cover of) the affine grassmannian of hyperplanes in euclidean space



$$\begin{array}{ccc} \mathsf{dSC}^{D+1} & \longrightarrow & \mathrm{Graff}(D, D+1) \\ \downarrow \bar{\pi} & & \downarrow \pi \\ \mathsf{S}^D & \longrightarrow & \mathbb{RP}^D \end{array}$$

$$\pi^{-1}(W^\perp) = \{ v + W \mid v \in W^\perp \}$$

Classifications

- Geometrically realisable, effective Klein pairs for kinematical Lie algebras **[JMF+Prohazka 2018]**

- Aristotelian ("no boosts") Lie algebras and their spacetimes **[JMF+Prohazka 2018]**

- Boosts act with generically non-compact orbits in all spacetimes except the aristotelian ($\nexists$ boosts) and the riemannian symmetric spaces ("boosts" = rotations) **[JMF+Grassie+Prohazka 2019]**

Invariant structures

$\mathcal{M} = \mathcal{K}/\mathcal{H}$ simply-connected homogeneous kinematical spacetime

$\mathcal{K}$ simply-connected

$\mathcal{H}$ closed, connected

Theorem There is a one-to-one correspondence between $\mathcal{K}$-invariant tensor fields on $\mathcal{M}$ and $\mathfrak{h}$-invariant tensors on $\mathfrak{k}/\mathfrak{h}$.

invariant *metric* $\left(S^2(\mathfrak{k}/\mathfrak{h})^*\right)^{\mathfrak{h}}$

invariant *cometric* $\left(S^2(\mathfrak{k}/\mathfrak{h})\right)^{\mathfrak{h}}$

invariant one-form $\left((\mathfrak{k}/\mathfrak{h})^*\right)^{\mathfrak{h}}$

invariant vector field $(\mathfrak{k}/\mathfrak{h})^{\mathfrak{h}}$

- With the exception of some “exotic” 2-dimensional spacetimes, the others fall into one of several classes determined by their invariant structure:
  - **riemannian:** invariant positive-definite metric
  - **lorentzian:** invariant lorentzian metric
  - **galilean:** invariant “clock” one-form $\boldsymbol{\tau}$ and spatial cometric $\boldsymbol{h}$, with $\boldsymbol{h}(\boldsymbol{\tau},-)=0$
  - **carrollian:** invariant vector field $\boldsymbol{\kappa}$ and spatial metric $\boldsymbol{b}$, with $\boldsymbol{b}(\boldsymbol{\kappa},-)=0$
  - **aristotelian:** simultaneously invariant galilean and carrollian structures: $\boldsymbol{\tau}$, $\boldsymbol{\kappa}$, $\boldsymbol{h}$, $\boldsymbol{b}$

Carrollian = null hypersurfaces

- Carrollian manifolds may be embedded as null hypersurfaces in a lorentzian manifold: e.g., $\mathscr{I}^{\pm}$
  - **C** embeds in Minkowski spacetime as $x^{+} = 0$

    [Duval+Gibbons+Horvathy+Zhang 2014]
  - **LC** embeds in Minkowski spacetime as the future lightcone*

    * except in **D=1** since the lightcone is not simply-connected
  - **dSC** embeds in de Sitter spacetime
  - **AdSC** embeds in anti de Sitter spacetime

Galilean = null reduction

$(\widetilde{M}, g)$ **lorentzian manifold**

$\xi \in \mathscr{X}(\widetilde{M})$ $\quad g(\xi, \xi) = 0$ $\quad \mathscr{L}_\xi g = 0$ **nowhere-vanishing**

$\pi : \widetilde{M} \to M$ **null reduction (assumed smooth)**

$\xi^\flat = \pi^* \tau$ $\quad \tau \in \Omega^1(M)$ **clock one-form**

$h \in \Gamma(S^2 TM)$ **spatial cometric**

$$h(\alpha, \beta) = g^{-1}(\pi^* \alpha, \pi^* \beta) \qquad \forall \alpha, \beta \in \Omega^1(M)$$

$$\pi_* \xi = 0 \implies h(\tau, -) = 0$$

Symmetries

- Symmetries of riemannian and lorentzian manifolds are always finite-dimensional and the same is true for aristotelian manifolds

- Symmetries of galilean and carrollian manifolds need not be finite-dimensional

- Same holds for conformal symmetries

Carrollian symmetries

$(\mathcal{M}, \kappa, b)$ homogeneous carrollian spacetime

A vector field ξ is a **carrollian Killing vector** if

$$[\xi, \kappa] = 0 \qquad \text{and} \qquad \mathcal{L}_\xi b = 0$$

A vector field ξ is a **carrollian conformal Killing vector** if

$$[\xi, \kappa] = -\lambda\kappa \qquad \text{and} \qquad \mathcal{L}_\xi b = 2\lambda b$$

$$\exists\, \lambda \in C^\infty(\mathcal{M})$$

Symmetries of C

The Lie algebra $\mathfrak{ckv}$ of carrollian Killing vectors of **C** is the semidirect product:

$$0 \longrightarrow C^\infty(\mathbb{E}^D) \longrightarrow \mathfrak{ckv} \longrightarrow \mathfrak{e} \longrightarrow 0$$

The Lie algebra $\mathfrak{cckv}$ of carrollian conformal Killing vectors of **C** depends on the dimension **D.**

density line bundle

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}_{D\geq 3} \longrightarrow \mathfrak{so}(D+1,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}_{D=2} \longrightarrow \mathscr{O}(\mathbb{C}) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}_{D=1} \longrightarrow C^\infty(\mathbb{R}) \longrightarrow 0$$

For **D=3** it is isomorphic to the BMS Lie algebra. **[Duval+Gibbons+Horvathy 2014]**

Carrollian symmetries of the light-cone

For **D≥2** the Lie algebra of carrollian symmetries of the light-cone is just the finite-dimensional kinematical Lie algebra **$\mathfrak{so}$(D+1,1)**, but for **D=1** it is the "wronskian" Lie algebra

$$C^\infty(\mathbb{R}) \qquad [f,g] = fg' - f'g$$

The Lie algebra of conformal carrollian symmetries of the light-cone is a semidirect product of Lie algebra of carrollian symmetries by the abelian ideal of sections of the density line bundle:

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\mathsf{LC}}_{D\geq 2} \longrightarrow \mathfrak{so}(D+1,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\mathsf{LC}}_{D=1} \longrightarrow C^\infty(\mathbb{R}) \longrightarrow 0$$

[Duval+Gibbons+Horvathy 2014]

Symmetries of (A)dSC

The Lie algebras of carrollian Killing vectors of **(A)dSC** are semidirect products:

$$0 \longrightarrow C^\infty(\mathbb{S}^D) \longrightarrow \mathfrak{ckv}_{\mathsf{dSC}} \longrightarrow \mathfrak{so}(D+1) \longrightarrow 0$$

$$0 \longrightarrow C^\infty(\mathbb{H}^D) \longrightarrow \mathfrak{ckv}_{\mathsf{AdSC}} \longrightarrow \mathfrak{so}(D,1) \longrightarrow 0$$

The Lie algebras of carrollian conformal Killing vectors of **(A)dSC** are semidirect products, which depend on **D**.

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{(A)dSC}}_{D\geq 3} \longrightarrow \mathfrak{so}(D+1,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{dSC}}_{D=2} \longrightarrow \mathfrak{so}(3,1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{AdSC}}_{D=2} \longrightarrow \mathscr{O}(\mathbb{H}) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{dSC}}_{D=1} \longrightarrow C^{\infty}(S^1) \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\mathscr{L}) \longrightarrow \mathfrak{cckv}^{\text{AdSC}}_{D=1} \longrightarrow C^{\infty}(\mathbb{R}) \longrightarrow 0$$

Future directions

- Exhibit the torsional galilean spacetimes as null reductions.
- Explore geodesics of invariant connections on these spacetimes.
- The BMS-like Lie algebras associated to **AdSC** extend the Poincaré algebra. Do they play a rôle in flat space holography?
- BMS-like **super**algebras should arise as (conformal) supersymmetries in our homogeneous superspaces.
- There are limits from these spacetimes to spacetimes without spatial isotropy (but with "Lorentz" isotropy), resulting in "pseudo-carrollian" spacetimes such as Ashtekar—Hansen's **Spi**. This landscape is largely unexplored. **[Gibbons 2019]**