*If triangles could speak, they would say that God is **eminently triangular.***

Spinoza

Killing superalgebras in supergravity

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Wilhelm Killing (1847-1923)

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work in progress

- Killing spinors in supergravity
- Superalgebras from Killing spinors
- Examples
- Geometric interlude
- Uses of Killing superalgebras
- Further directions

Killing spinors in supergravity

Supergravity backgrounds

- local lorentzian metric **g**
- possibly extra bosonic fields Φ
- real spinor bundle **S**
- a connection **D** on **S**, depending on **g** and Φ

Killing spinors

$$\delta_\varepsilon \psi = \nabla \varepsilon + \cdots = D\varepsilon = 0$$

$$\delta_\varepsilon \lambda = P\varepsilon = 0$$

Killing spinors are **D**-parallel sections of the subbundle of **S** defined by **P**.

K={Killing spinors}   dim **K** ≤ rank **S**

d=11 supergravity

Fields g $F \in \Omega^4$ $dF = 0$

Spinors are real and have 32 components

Field equations

$$\mathrm{Ricci}(g) = T(g, F) \qquad d \star F = -\tfrac{1}{2} F \wedge F$$

Killing spinors

$$D_X \varepsilon = \nabla_X \varepsilon - \tfrac{1}{6} \imath_X F \cdot \varepsilon + \tfrac{1}{12} X \wedge F \cdot \varepsilon = 0$$

Clifford product

d=10 heterotic supergravity

Fields $\quad g \quad \phi \quad H \in \Omega^3 \quad dH = 0 \quad F \in \Omega^2(\mathfrak{g})$

Spinors are real, chiral and have 16 components

Field equations follow from (string frame) lagrangian

$$e^{-2\phi}\left(R + 4|d\phi|^2 - \tfrac{1}{2}|H|^2 - \tfrac{1}{2}|F|^2\right)$$

Killing spinors

$$D\varepsilon = 0 \qquad d\phi \cdot \varepsilon + \tfrac{1}{2}H \cdot \varepsilon = 0 \qquad F \cdot \varepsilon = 0$$

↑ spin connection with torsion H

Superalgebras from Killing spinors

Killing superalgebra

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

$$\mathfrak{k}_1 = \left\{ \text{Killing spinors} \right\}$$

$$\mathfrak{k}_0 = [\mathfrak{k}_1, \mathfrak{k}_1]$$

$$\langle [\varepsilon_1, \varepsilon_2], V \rangle = (\varepsilon_1, V \cdot \varepsilon_2)$$

$\mathfrak{g}_0$ consists of **infinitesimal automorphisms** of the background: Killing vectors which preserve the bosonic fields, hence the connection **D**

For example, the Killing superalgebra of the flat vacuum is the translation ideal of the Poincaré superalgebra:

$$[Q, Q] = P$$

It has been shown to be a **Lie superalgebra** in all cases.

Symmetry superalgebra

A background may have more automorphisms than those generated by the Killing spinors.
e.g. Minkowski space is also invariant under Lorentz transformations.
Taking all infinitesimal automorphisms into account, one obtains the **symmetry superalgebra** of the background.

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$$

$\mathfrak{s}_1 =$ {Killing spinors}

$\mathfrak{s}_0 =$ {infinitesimal autos}

$$\mathfrak{s}_1 = \mathfrak{k}_1$$

$$\mathfrak{s}_0 > [\mathfrak{s}_1, \mathfrak{s}_1] = \mathfrak{k}_0$$

The Killing superalgebra is the canonical ideal of the symmetry superalgebra.

The symmetry superalgebra of the Minkowski vacuum is the Poincaré superalgebra.

Maximal superalgebra

The Killing superalgebra of the Minkowski vacuum admits a **maximal central extension**:

$$[Q, Q] = P + Z$$

where **Z** can be identified with **brane charges**.

Geometrically, **Z** are **all** the parallel forms constructed from the Killing spinors.

Townsend calls this the **M-algebra**.

Is **M** the M of M-theory? or the M of Minkowski?

Let’s explore the second possibility.

Is there a “maximal extension” of the Killing superalgebra of a background?

Let’s try to make this notion precise.

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a Lie superalgebra

$[-,-] : S^2\mathfrak{g}_1 \to \mathfrak{g}_0$

is in general neither injective or surjective.

If surjective we say that **g** is **odd-generated**.

e.g., the Killing superalgebra is odd-generated.

If injective, we say that **g** is **full**.

If both we say that it is **minimally full**.

The translation ideal of the Poincaré superalgebra is odd-generated.

The Poincaré superalgebra itself is not.

The M-algebra is both odd-generated and full, so it is minimally full.

Adding the Lorentz generators to the M-algebra yields a full algebra, but not minimally so.

We *tentatively* define the **maximal superalgebra** of a supergravity background to be a *minimally full* ***extension*** of its Killing superalgebra.

The M-algebra is the maximal superalgebra of the Minkowski background.

The extra bosonic generators in the maximal superalgebra need **not** be central.

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1$$

$\mathfrak{m}_1 =$ **{**Killing spinors**}**

$$\mathfrak{m}_0 = [\mathfrak{m}_1, \mathfrak{m}_1] \cong S^2\mathfrak{m}_1$$

} minimally full

But it is also an **extension**:

$\mathfrak{m}_1 = \mathfrak{k}_1 \qquad \mathfrak{k}_0 < \mathfrak{m}_0$

$[-,-]_{\mathfrak{m}} : \mathfrak{m}_0 \otimes \mathfrak{m}_1 \to \mathfrak{m}_1$ restricts to

$$[-,-]_{\mathfrak{k}} : \mathfrak{k}_0 \otimes \mathfrak{m}_1 \to \mathfrak{m}_1$$

Minimally full Lie superalgebras have a very simple structure:

$$[Q_a, Q_b] = Z_{ab}$$
$$[Z_{ab}, Q_c] = \omega_{bc} Q_a + \omega_{ac} Q_b$$
$$[Z_{ab}, Z_{cd}] = \omega_{bc} Z_{ad} + \omega_{bd} Z_{ac} + \omega_{ac} Z_{ac} + \omega_{ad} Z_{bc}$$

for some $\omega \in \left(\Lambda^2 \mathfrak{m}_1^*\right)^{\mathfrak{m}_0}$

This is the result of a calculation of Kamimura and Sakaguchi.

For the M-algebra, $\omega = 0$

If ω is nondegenerate, this superalgebra is **orthosymplectic**.

The only invariant of ω is its rank, which is even.

All cases can be obtained from the orthosymplectic case by contractions.

Imposing in addition that it be an extension of the Killing superalgebra, it is often possible to

- determine the algebra uniquely, assuming it exists;
- or in some cases to prove that it does not exist.

Examples

The maximal superalgebra of the maximally supersymmetric Freund-Rubin backgrounds of eleven-dimensional and IIB supergravities is **osp(1|32)**.

$$\begin{cases} \dim\left(\Lambda^2 \mathfrak{m}_1{}^*\right)^{\mathfrak{k}_0} = 1 & \text{and } \omega \text{ is symplectic} \\ \mathfrak{k}_0 & \text{is not abelian} \end{cases}$$

The extra bosonic generators **Z** are constructed geometrically, as are the brackets.

The maximally supersymmetric plane waves do **not** admit a maximal superalgebra.

$$\mathfrak{m}_1^\perp = \{Q \in \mathfrak{m}_1 | \omega(Q, -) = 0\}$$

$$\mathfrak{m}_1^\perp \subset (\mathfrak{m}_1)^{\mathfrak{k}_0}$$

but for the plane waves, $(\mathfrak{m}_1)^{\mathfrak{k}_0} = 0$ and

$$\mathfrak{z}(\mathfrak{k}_0) \neq 0 \implies \mathfrak{z}(\mathfrak{m}_0) \neq 0 \implies \mathfrak{m}_1^\perp \neq 0$$

But what about the Penrose limit of the orthosymplectic superalgebra?

It is **not** full!

$$S^2\mathfrak{m}_1 \ncong [\mathfrak{m}_1, \mathfrak{m}_1]$$

This answers (finally! but negatively) a question posed by Mohab Abou Zeid in 2002.

A geometrical interlude

At its most basic, a supergravity background is a geometry with a privileged notion of spinor.

Round spheres too are such geometries, possessing **geometric Killing spinors**:

$$\nabla_X \varepsilon = \lambda X \cdot \varepsilon$$

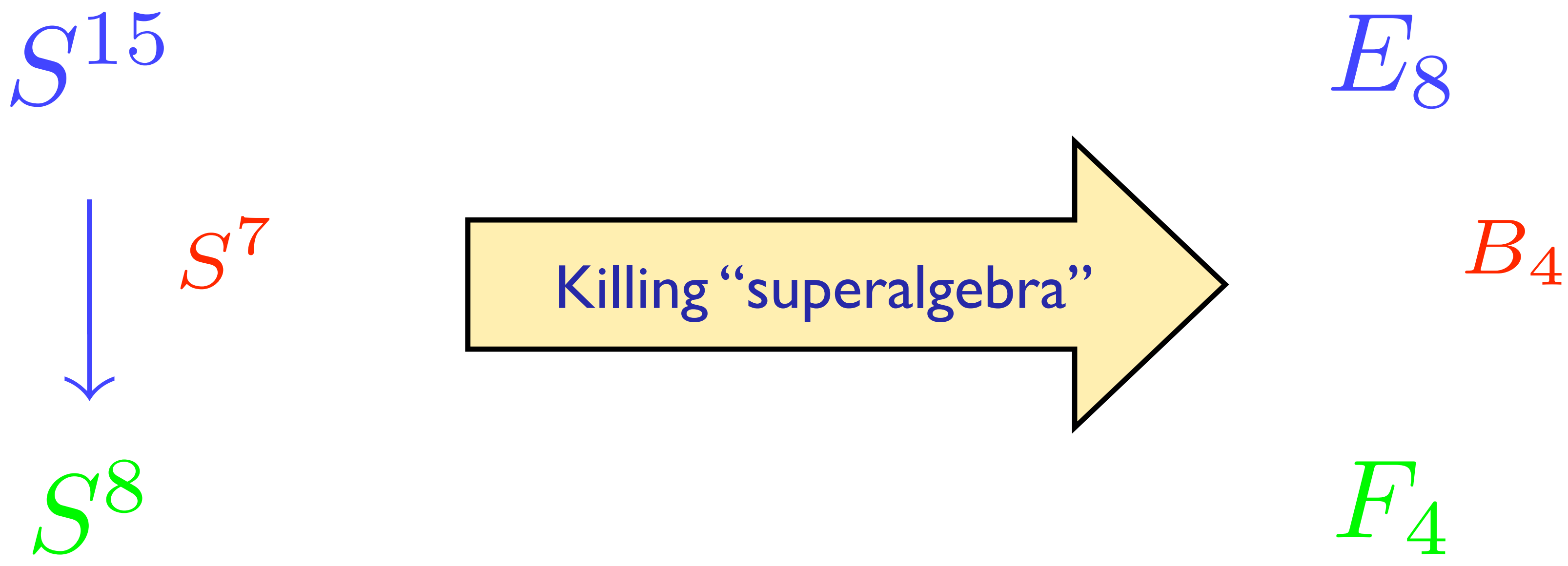
Is there a Killing superalgebra for the spheres?

In general we do not expect a Lie **super**algebra, just a $\mathbb{Z}/2$-graded Lie algebra:

$$[\varepsilon_1, \varepsilon_2] = -[\varepsilon_2, \varepsilon_1]$$

Also, in general, only 3/4 of the Jacobi identity is satisfied.

However...


S^{15}
S^{7}
S^{8}
Killing “superalgebra”
E_8
B_4
F_4

Last Updated: Monday, 19 March 2007, 12:28 GMT

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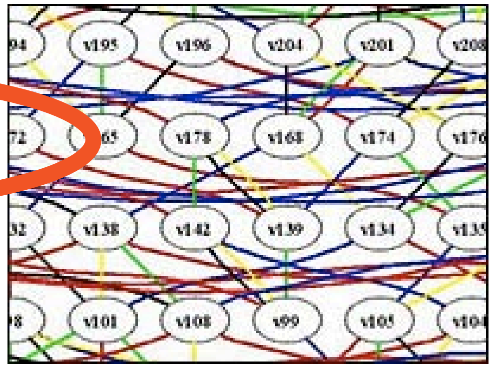
248-dimension maths puzzle solved

An international team of mathematicians has detailed a vast complex numerical "structure" which was described more than a century ago.



The structure is described in the form of a vast matrix

Mapping the 248-dimensional structure, called E8, took four years of work and produced more data than the Human Genome Project, researchers said.

E8 is a "Lie group", a means of describing symmetrical objects.

The team said their findings may assist fields of physics which use more than four dimensions, such as string theory.

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Surfer dude stuns physicists with theory of everything

By Roger Highfield, Science Editor
Last Updated: 6:01pm GMT 14/11/2007

Have your say   Read comments

An impoverished surfer has drawn up a new theory of the universe, seen by some as the Holy Grail of physics, which has received rave reviews from scientists.

The E8 pattern (left), Garrett Lisi surfing (middle) and out of the water (right)

Garrett Lisi, 39, has a doctorate but no university affiliation and spends most of the year surfing in Hawaii, where he has also been a hiking guide and bridge builder (when he slept in a jungle yurt).

In winter, he heads to the mountains near Lake Tahoe, Nevada, where he snowboards. "Being poor sucks," Lisi says. "It's hard to figure out the secrets of the universe when you're trying to figure out where you and your girlfriend are going to sleep next month."

Despite this unusual career path, his proposal is remarkable because, by the arcane standards of particle physics, it does not require highly complex mathematics.

Uses of Killing superalgebras

Early uses

- Test of AdS/CFT correspondence, by checking the symmetries across the correspondence for general AdS backgrounds.

- The Penrose limit interpretation of maximally supersymmetric waves arose as an attempt to understand their Killing superalgebras, which are contractions of the KSAs of AdSxS.

Homogeneity conjecture

All **known** supergravity backgrounds preserving more than half of the supersymmetry are (locally) **homogeneous**.

The homogeneity conjecture states that all such backgrounds are (locally) homogeneous.

This has been proved for type I/heterotic backgrounds.

Using the Killing superalgebra one can show that eleven-dimensional and ten-dimensional type II backgrounds preserving > 3/4 of the supersymmetry are (locally) homogeneous.

I believe the conjecture is true, but a proof probably requires a better understanding of the holonomy of **D** than the one we have at present.

Further directions

BPS objects in curved backgrounds

Using the maximal superalgebra, one can now begin to study the possible supersymmetric backgrounds asymptotic to one with such a maximal superalgebra.

The fact that the maximal superalgebra is not in general a central extension means the extra generators can no longer be interpreted simply as charges.

Quantum/stringy corrections

It is a generalised **belief** in string theory that supergravity backgrounds can be deformed continuously to solutions of the field equations with quantum or α' corrections.

This belief justifies, from a string theory point of view, much of the research into supergravity.

We shall not question it today.

Natural question:

What happens to Killing superalgebras under quantum/stringy or α' corrections?

Possible answers:

- ★ The notion does **not** persist
- ✓ The notion persists:
  - ★ **not** as a Lie superalgebra
  - ✓ as a Lie superalgebra:
    - ★ of **different** dimension, or
    - ✓ of the same dimension

Let us make the **assumption** that it deforms as a Lie superalgebra of the same dimension.

This is a well-known mathematical problem, which can be analysed using techniques of Lie (super)algebra **cohomology**.

We have analysed the deformations of the symmetry superalgebras associated to the simplest ten- and eleven-dimensional backgrounds.

The eleven-dimensional Poincaré superalgebra admits no deformations: it is **rigid**. (This contrasts sharply with four dimensions, where the Poincaré superalgebra deforms to the **de Sitter** superalgebras.)

This agrees with the fact that the Minkowski vacuum admits no quantum corrections.

Similarly the SSAs of the M5-brane and of the Freund-Rubin vacua are also rigid.

On the other hand, the M2-brane KSA admits a deformation, whose bosonic subalgebra contains the isometry algebra of **anti-de Sitter space**. This suggests that under quantum corrections, the worldvolume of the M2-brane gets **curved**.

Of course, the deformation could be due to a classical solution which has escaped notice. We have not ruled this out yet!

Similarly, the half-BPS M-wave and Kaluza-Klein monopole admit a unique deformation. This latter one does not seem to be realisable in supergravity.

The center of the U(2) isometry of Taub-NUT acts trivially on the Minkowski factor of the Kaluza-Klein monopole. Upon deformation, it acts by homotheties,reminiscent of a nongeometric background.

Moreover, the deformations seem to be consistent with Kaluza-Klein reduction, strongly suggesting a geometric origin. The question remains whether it is the geometry of supergravity that these deformations are probing. The MKK result would seem to indicate otherwise.

In summary, the Killing superalgebras seem to encode also information beyond the supergravity approximation.

Decoding this information is work in progress.

Is our algebra the measure
Of that unexhausted treasure
That affords the purest pleasure,
Ever found when it is sought?

James Clerk Maxwell