Symmetric M-theory backgrounds

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Eleven-dimensional supergravity

(M, g, F)

(M^{11}, g) lorentzian spin manifold

$F \in \Omega^4(M) \qquad dF = 0$

$d \star F = \frac{1}{2} F \wedge F$

$\mathrm{Ric}(X, Y) = \frac{1}{2} \langle \iota_X F, \iota_Y F \rangle - \frac{1}{6} |F|^2 g(X, Y)$

Not unlike Einstein-Maxwell theory

One line of research: finding solutions!

Particularly interesting are the **supersymmetric** solutions; i.e., those admitting **Killing spinors**:

$$\nabla_X \varepsilon + \tfrac{1}{6}\imath_X F \cdot \varepsilon + \tfrac{1}{12} X \wedge F \cdot \varepsilon = 0$$

$$\dim\{\text{Killing spinors}\} = 32\nu$$

known $\nu \in \left\{0, \frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \frac{1}{8}, \frac{5}{32}, \frac{3}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{9}{16}, \frac{5}{8}, \frac{11}{16}, \frac{3}{4}, 1\right\}$

ruled out $\nu \in \left\{\frac{30}{32}, \frac{31}{32}\right\}$

Fact: all known solutions with $\nu > \frac{1}{2}$

are (locally) ***homogeneous!***

(A Lie group acts locally transitively by isometries and preserving ***F***.)

Theorem (JMF + Meessen + Philip, 2004)

$$\nu > \tfrac{3}{4} \implies \text{locally homogeneous}$$

Homogeneity Conjecture

$$\nu > \tfrac{1}{2} \implies \text{locally homogeneous}$$

Classification of homogeneous backgrounds?

Presumably involves classifying 11-dimensional homogeneous lorentzian manifolds

Probably **hard**, except for special cases: semisimple isometry group,...

First step: classify the symmetric backgrounds.

Lorentzian symmetric spaces

A lorentzian (locally) symmetric space is locally isometric to

$$M_0 \times M_1 \times \cdots \times M_k$$

M_0 lorentzian indecomposable

$M_{i>0}$ riemannian irreducible

Question: How many 11-dimensional LSS?

Indecomposable LSS

Theorem (Cahen + Wallach, 1970)
Proof (Cahen + Parker, 1977)

A simply-connected, indecomposable lorentzian symmetric space is isometric to one of:

$\mathbb{R}$ with metric $-dt^2$

the universal cover of ***de Sitter*** or ***anti de Sitter*** space

a ***Cahen-Wallach*** pp-wave

Local models

n-dimensional de Sitter space

$$-x_0^2 + x_1^2 + \cdots x_n^2 = 1/\kappa^2 \qquad \subset \mathbb{R}^{n,1}$$

n-dimensional anti de Sitter space

$$-x_0^2 + x_1^2 + \cdots x_{n-1}^2 - x_n^2 = -1/\kappa^2 \qquad \subset \mathbb{R}^{n-1,2}$$

n-dimensional Cahen-Wallach space

$$u_1^2 + u_2^2 = 1 \qquad 2u_1 v_1 + 2u_2 v_2 = \sum_{i=1}^{n-2} A_{ij} x_i x_j \qquad \subset \mathbb{R}^{n,2}$$

Irreducible RSS

Well-known list due to Élie Cartan.

They come in pairs: one compact and one not.

In dimension ≤ 10, we have the usual suspects: spheres, projective spaces, grassmannians,...

n	1	2	3	4	5	6	7	8	9	10
#	1	2	2	4	4	6	2	12	4	8

A gas of RSS

$i_n =$ number of irreducible n-dimensional RSS

$r_n =$ number of n-dimensional RSS

$$\prod_{n=1}^{\infty} \frac{1}{1 - i_n t^n} = \sum_{n=0}^{\infty} r_n t^n$$

n	1	2	3	4	5	6	7	8	9	10
r_n	1	3	5	13	21	47	73	161	253	497

ℓ_n = number of indecomposable n-dimensional LSS

$$\ell_n = \begin{cases} 1 & n = 1 \\ 2 & n = 2 \\ 3 & n > 2 \end{cases}$$

N = number of 11-dimensional LSS

$$N = \sum_{n=1}^{11} \ell_n r_{11-n}$$

And the answer is...

ℓ_n = number of indecomposable n-dimensional LSS

$$\ell_n = \begin{cases} 1 & n = 1 \\ 2 & n = 2 \\ 3 & n > 2 \end{cases}$$

N = number of 11-dimensional LSS

$$N = \sum_{n=1}^{11} \ell_n r_{11-n}$$

And the answer is...

N = 1978

Field equations simplify:

$$\nabla F = 0 \implies d \star F = 0$$

We are left with:

$$F \wedge F = 0$$

$$\mathrm{Ric}(X,Y) = \tfrac{1}{2} \langle \iota_X F, \iota_Y F \rangle - \tfrac{1}{6} |F|^2 g(X,Y)$$

Some special cases:

$$F = 0 \implies \mathrm{Ric} = 0 \implies \mathrm{CW}_d(A) \times \mathbb{R}^{11-d}$$

$$\mathrm{tr}(A) = 0$$

$$-dt^2 + \mathrm{RSS}_{10} \implies \mathbb{R}^{10,1} \quad \& \quad F = 0$$

$$\mathrm{CW}_d(A) \times \mathrm{RSS}_{11-d} \implies \mathrm{CW}_d(A) \times \mathbb{R}^{11-d}$$

$$F = dx^- \wedge \varphi \quad \exists \varphi \in \Lambda^3 \mathbb{R}^{d-2}$$

$$\mathrm{tr}(A) = \tfrac{1}{2}|\varphi|^2$$

$$\nexists \quad \mathrm{dS}_d \times \mathrm{RSS}_{11-d}$$

We are left with ***568*** cases of the form

$$\mathrm{AdS}_{2\leq d\leq 7} \times \mathrm{RSS}_{11-d}$$

d=7 conforms to the ***Freund-Rubin*** ansatz:

$$\mathrm{AdS}_7 \times \begin{cases} S^4 \\ \mathbb{CP}^2 \\ S^2 \times S^2 \end{cases}$$

$$F = f\nu_4 \qquad \mathrm{Ric}_7 = -\tfrac{1}{6} f^2 g_7 \qquad \mathrm{Ric}_4 = \tfrac{1}{3} f^2 g_4$$

There are **no** ***d=6*** backgrounds.

d=5 starts to get interesting:

$$\mathrm{AdS}_5 \times \begin{cases} \mathbb{CP}^3 \\ \mathrm{Gr}^+_{\mathbb{R}}(2,5) \end{cases} \qquad F = \tfrac{1}{2} f \omega^2$$

$$\mathrm{Ric}_5 = -\tfrac{1}{2} f^2 g_5 \qquad \mathrm{Ric}_6 = \tfrac{1}{2} f^2 g_6$$

$$\mathrm{AdS}_5 \times H^2 \times \begin{cases} \mathbb{CP}^2 \\ S^4 \end{cases} \qquad F = f \nu_4$$

$$\mathrm{Ric}_5 = -\tfrac{1}{6} f^2 g_5 \qquad \mathrm{Ric}_2 = -\tfrac{1}{6} f^2 g_2 \qquad \mathrm{Ric}_4 = \tfrac{1}{3} f^2 g_4$$

$\mathrm{AdS}_5 \times H^2 \times S^2 \times S^2$ also exists,
but it belongs to a more interesting family.

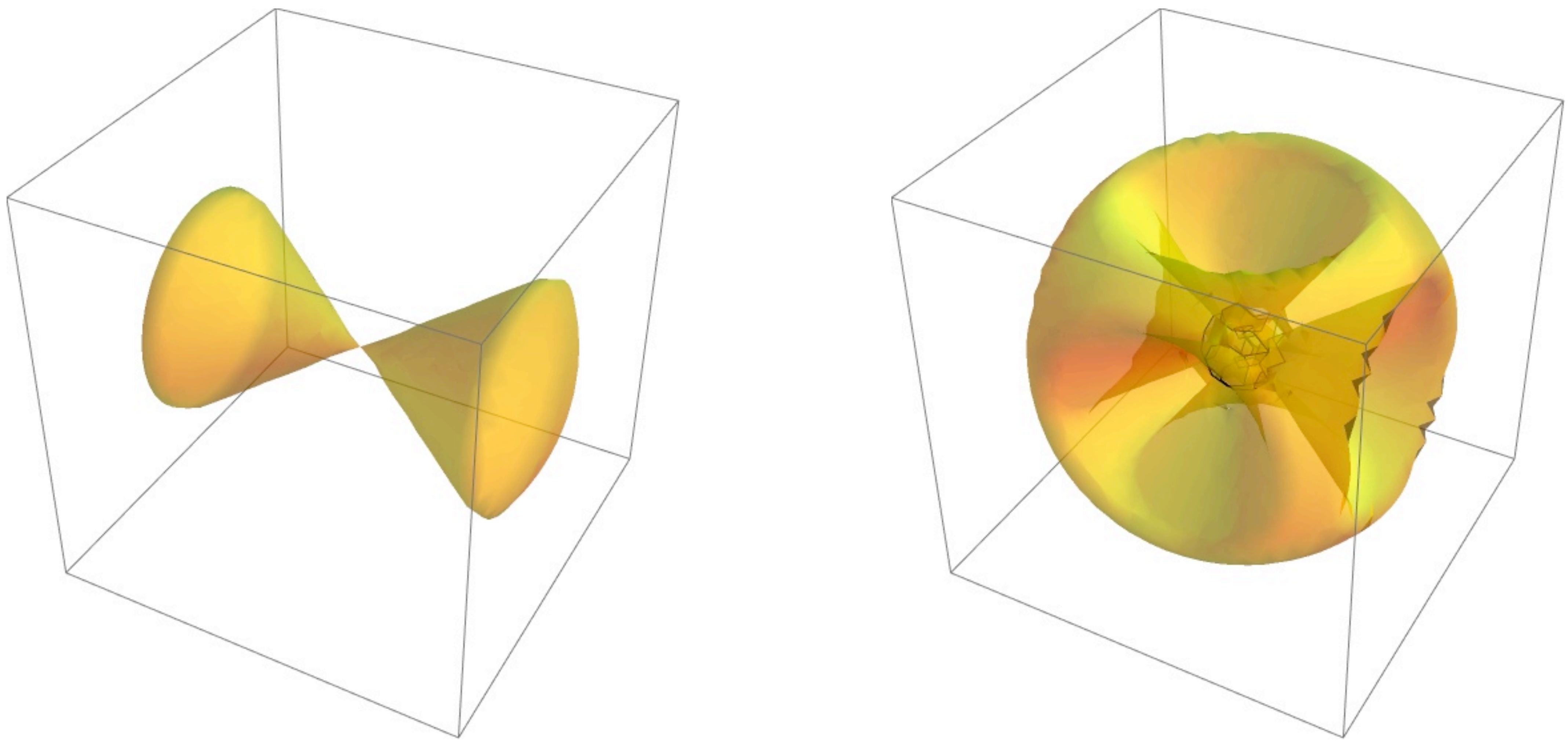
To explain the picture, one should notice that the field equations are invariant under a ***homothetic*** transformation:

$$g \mapsto t^2 g \qquad F \mapsto t^3 F$$

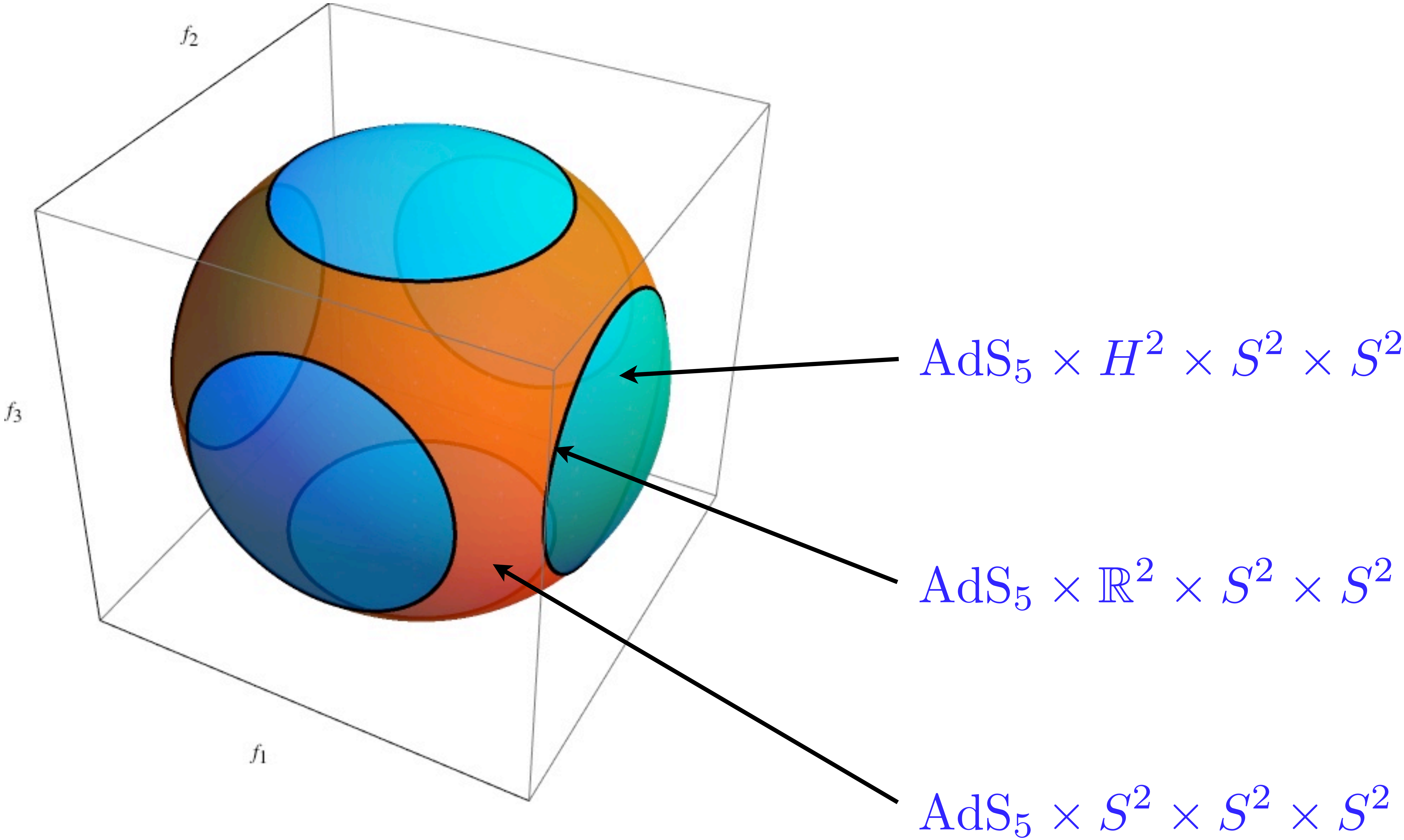
(This is true in general, not just for symmetric backgrounds.)

Therefore moduli spaces are naturally cones.

e.g.,



Conical regions are uniquely determined by their intersection with the unit sphere.



$$F = f_1 \nu_1 \wedge \nu_2 + f_2 \nu_2 \wedge \nu_3 + f_3 \nu_1 \wedge \nu_3$$

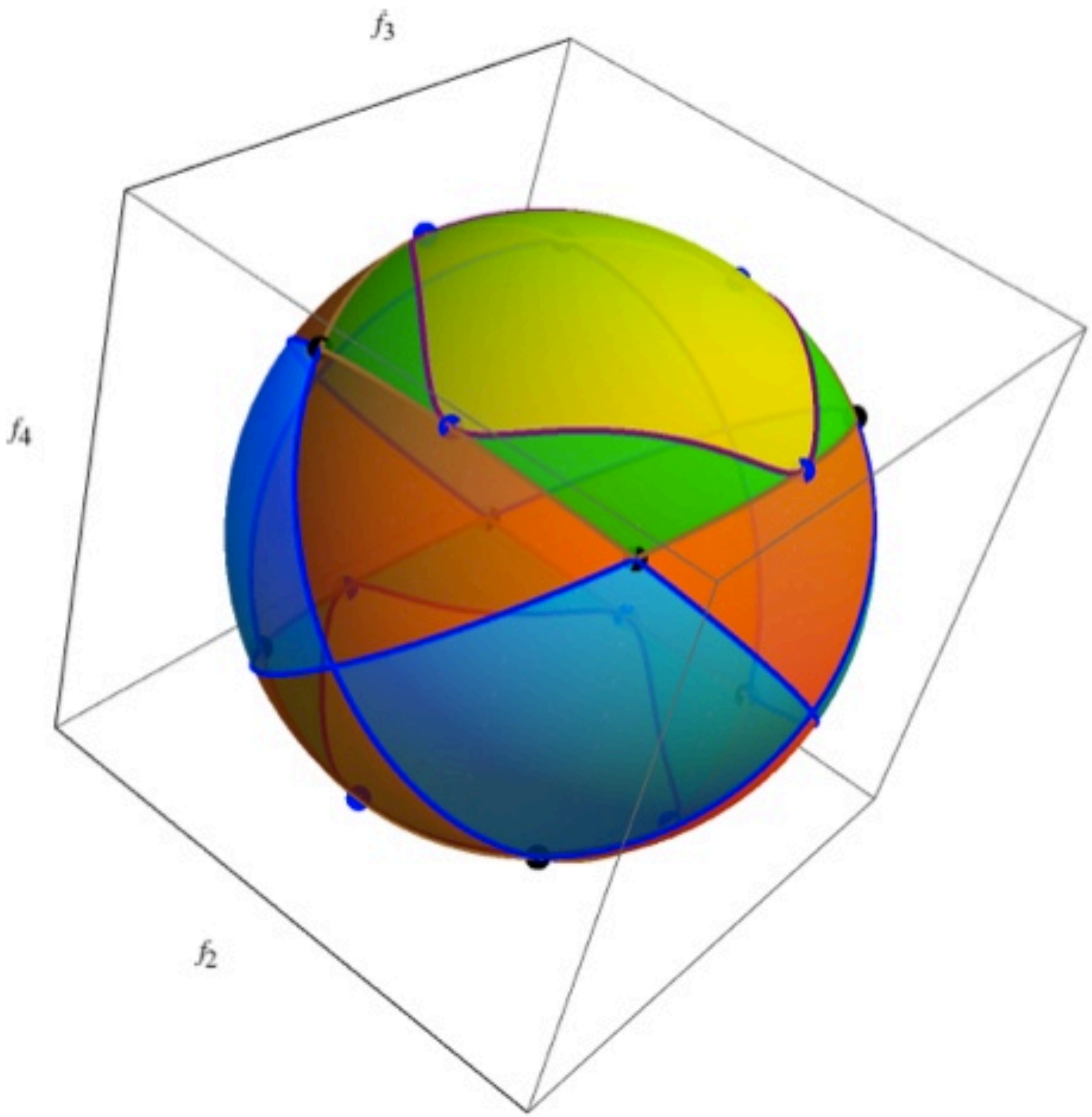
d=4 was studied exhaustively in the early 1980s, in the context of ***Kaluza-Klein supergravity***.

$$\mathrm{AdS}_4 \times \begin{cases} S^7 \\ S^5 \times S^2 \\ \mathrm{SLAG}_3 \times S^2 \\ S^4 \times S^3 \\ \mathbb{CP}^2 \times S^3 \\ S^2 \times S^2 \times S^3 \end{cases} \qquad F = f\nu_4$$

$$\mathrm{AdS}_4 \times H^3 \times \begin{cases} S^4 \\ \mathbb{CP}^2 \\ S^2 \times S^2 \end{cases} \qquad F = f\omega_4$$

The real "fun" starts with ***d=3***.

There are 24 geometries, some with ***F***-moduli.



- $AdS_3 \times S^2 \times T^6$
- $AdS_3 \times \mathbb{C}P^2 \times T^4$
- $AdS_3 \times T^4 \times S^2 \times S^2$
- $AdS_3 \times \mathbb{C}P^2 \times S^2 \times T^2$
- $AdS_3 \times \mathbb{C}P^2 \times H^2 \times T^2$
- $AdS_3 \times \mathbb{C}P^2 \times H^2 \times H^2$
- $AdS_3 \times \mathbb{C}H^2 \times S^2 \times S^2$
- $AdS_3 \times \mathbb{C}P^2 \times S^2 \times H^2$
- $AdS_3 \times \mathbb{C}P^2 \times S^2 \times S^2$

***d*=2** is total madness.

There are $\geq$60 geometries, some with high dimensional ***F*-**moduli.

(Some of them give rise to pretty pictures, though.)

Among the most interesting examples is

$$\mathrm{AdS}_2 \times \mathrm{SLAG}_4$$

$\mathrm{SLAG}_4 \cong \mathrm{SU}(4)/\mathrm{SO}(4)$

$\Omega =$ parallel 4-form

$\Omega \wedge \Omega = 0$

(I wish I understood this 4-form more conceptually.)

$$F = f\Omega \qquad \mathrm{Ric}_2 = -3f^2 g_2 \qquad \mathrm{Ric}_9 = f^2 g_9$$

Another interesting example is

$$\mathrm{AdS}_2 \times S^1 \times \mathrm{G}_{\mathbb{C}}(2,4)$$

$\mathrm{G}_{\mathbb{C}}(2,4) \cong \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$ compact HSS

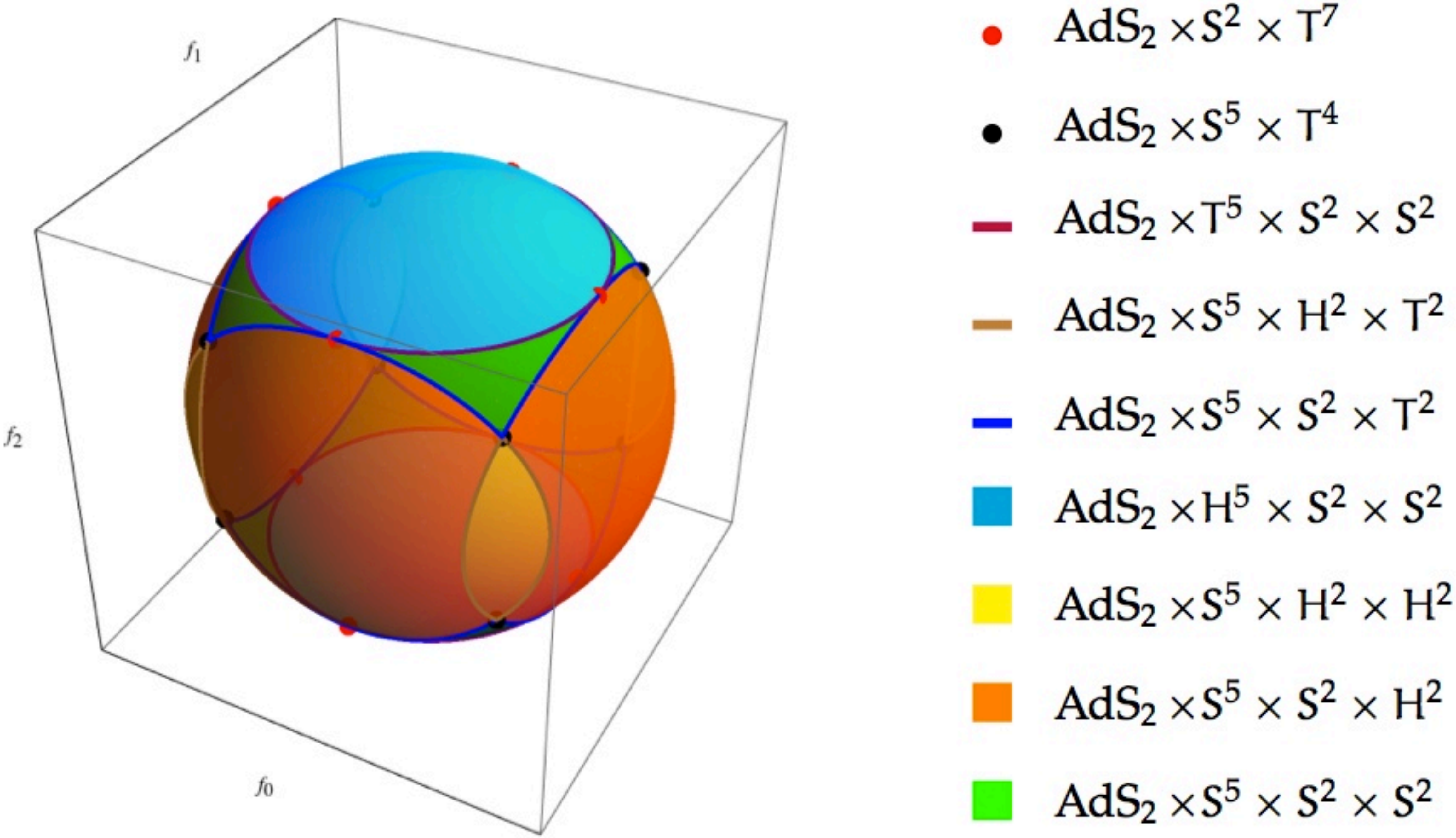
$\omega = \text{Kähler form}$

$\Omega^{(1)}, \Omega^{(2)} = \text{self-dual parallel 4-forms} \quad \Omega^{(1)} \wedge \Omega^{(2)} = 0$

$\Omega^{(1)} + \Omega^{(2)} = \frac{1}{2}\omega^2 \qquad \omega \wedge \Omega^{(1)} = \omega \wedge \Omega^{(2)} = \frac{1}{4}\omega^3$

$$F = f\left(\sqrt{\tfrac{3}{2}}\nu \wedge \omega \pm (\Omega^{(1)} - \Omega^{(2)})\right)$$

$\mathrm{Ric}_2 = -3f^2 g_2 \qquad \mathrm{Ric}_8 = \frac{3}{2}f^2 g_8$



f_1
f_2
f_0
$AdS_2 \times S^2 \times T^7$
$AdS_2 \times S^5 \times T^4$
$AdS_2 \times T^5 \times S^2 \times S^2$
$AdS_2 \times S^5 \times H^2 \times T^2$
$AdS_2 \times S^5 \times S^2 \times T^2$
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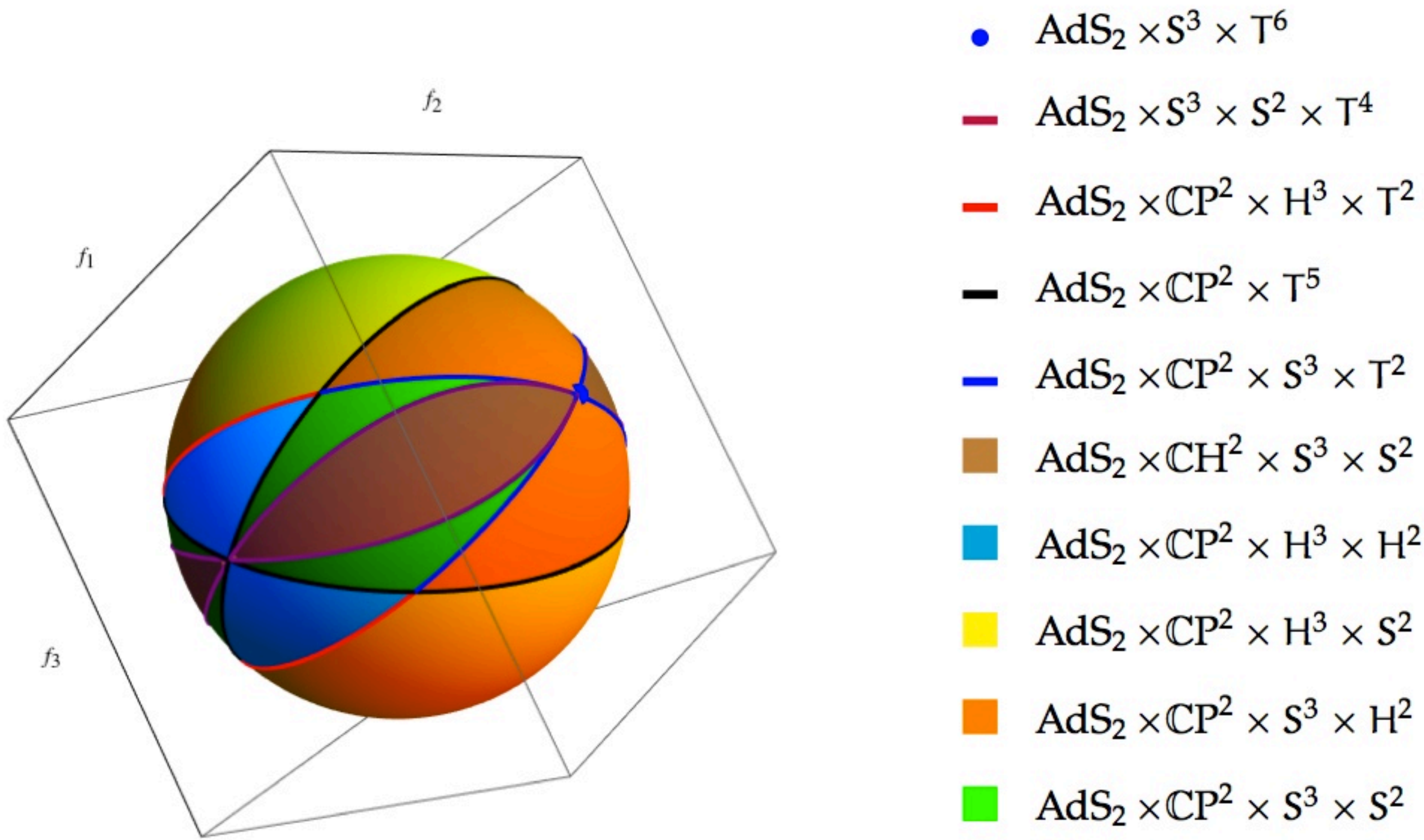

f_2
f_1
f_3
$AdS_2 \times S^3 \times T^6$
$AdS_2 \times S^3 \times S^2 \times T^4$
$AdS_2 \times \mathbb{CP}^2 \times H^3 \times T^2$
$AdS_2 \times \mathbb{CP}^2 \times T^5$
$AdS_2 \times \mathbb{CP}^2 \times S^3 \times T^2$
$AdS_2 \times \mathbb{CH}^2 \times S^3 \times S^2$
$AdS_2 \times \mathbb{CP}^2 \times H^3 \times H^2$
$AdS_2 \times \mathbb{CP}^2 \times H^3 \times S^2$
$AdS_2 \times \mathbb{CP}^2 \times S^3 \times H^2$
$AdS_2 \times \mathbb{CP}^2 \times S^3 \times S^2$

Next step: **supersymmetry** of the symmetric backgrounds.

All maximally supersymmetric backgrounds are symmetric:

$$\mathrm{AdS}_4 \times S^7$$

$$\mathrm{AdS}_7 \times S^4$$

$$\mathrm{CW}_{11}(A) \quad \exists A$$

$$\mathbb{R}^{10,1}$$

Supersymmetry of the other symmetric backgrounds is still ***in progress***.

Thank you!