Lie superalgebra deformations & supersymmetry

(joint work with Paul de Medeiros, Andrea Santi & Andrew Beckett)

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& work in progress

In Japanese grammar there are two related notions: subject & topic.
The subject of this talk is as in the title, but the topic is the geometric construction of Lie (super)algebras. In a nutshell, the class of constructions I will discuss in this talk are of the following kind. Let (M,g) be a spin manifold (typically lorentzian) and $\$ \to M$ a spinor bundle. On $\$$ we have a Spin-invariant inner product (eg: symplectic) $(\,,\,)$ and a Koszul connection $\mathcal{D}$ on $\$$. When the construction works, one gets a Lie (super)algebra structure on a $\mathbb{Z}_2$-graded vector space $\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$, where the Killing spinors $\mathfrak{k}_{\bar 1} := \{ \varepsilon \in \Gamma(\$) \mid \mathcal{D}\varepsilon = 0\}$ and $\mathfrak{k}_{\bar 0} = \{ X \in \mathfrak{X}(M) \mid \mathcal{L}_X \mathcal{D} = 0 \}^*$, which extends the Lie bracket of vector fields on $\mathfrak{k}_{\bar 0}$. The bracket $[X,\varepsilon] = \mathcal{L}_X \varepsilon$, the spinorial Lie derivative in the PhD thesis of Yvette Kossmann-Schwarzbach, and $[\varepsilon_1, \varepsilon_2]$ is induced from the Clifford product $TM \times \$ \to \$$ by duality:
$\$ \times \$ \to TM \quad : \quad g([\varepsilon_1,\varepsilon_2], X) = (\varepsilon_1, X\cdot\varepsilon_2) \quad \forall X \in \mathfrak{X}(M)$.

I will give you two examples of this construction.

① $(M,g) = S^{15}$ with round metric $\quad \$ = \mathrm{Spin}(16) \times_{\mathrm{Spin}(15)} \Delta$ ← irreducible spinor rep. of Spin(15) with invariant symmetric inner product and $\mathcal{D}_X\varepsilon := \nabla_X \varepsilon - \frac{1}{2} X\cdot\varepsilon$.
Then $\mathfrak{k}_{\bar 0} \cong \underline{\mathfrak{spin}}(16)$ and $\mathfrak{k}_{\bar 1} \cong \Delta_+$ ← pos-chirality half-spinor irrep of Spin(16) and hence $\mathfrak{k} \cong \mathfrak{e}_8$, as $\mathbb{Z}_2$-graded Lie algebra (not superalg.)
A similar construction with S^8 leads to $\mathfrak{k} \cong \mathfrak{f}_4$ and with S^7 one gets $\mathfrak{k} \cong \mathfrak{b}_4 \cong \underline{\mathfrak{spin}}(9)$. Curiously, $S^7 \to S^{15} \to S^8$ is the octonionic Hopf fibration (itself explained via spinors in 9+1 dimensions)

② (M,g) 4d Minkowski spacetime, $\$ = M \times \Delta$ ← real spinor rep of $\underline{\mathfrak{spin}}(3,1) \cong \underline{\mathfrak{sl}}(2,\mathbb{C})$, $\Delta \cong \mathbb{R}^4$, symplectic. Take $\mathcal{D} = \nabla$.
Then $\mathfrak{k} \cong (N=1, D=4)$ Poincaré superalgebra.

* after the talk, Boris Kruglikov pointed out that for $\mathcal{D} = \nabla$, $\mathfrak{k}_{\bar 0}$ would be the affine group and this doesn't act on spinors. So my "clever" idea of defining $\mathfrak{k}_{\bar 0}$ in this way is not quite correct. I suppose $X \in \mathfrak{X}(M)$ must be a conformal Killing vector field so that $\mathcal{L}_X$ on spinors is defined <u>and</u> in addition $\mathcal{L}_X \mathcal{D} = 0$.

Hopefully these two examples show that the construction is not without interest. The main question, though, is where do we get $\mathscr{D}$ from? A natural source of $\mathscr{D}$'s are supergravity theories

eg: $D=11$ SUGRA (M,g) lorentzian spin 11-dim'l manifold, $F \in \Omega^4(M)$, $\$ \to M$ rank-32 $\mathbb{R}$ symplectic.

$$\mathscr{D}_X = \nabla_X + \tfrac{1}{6}\iota_X F + \tfrac{1}{12} X^\flat \wedge F$$

If $dF=0$, $\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$, with $\mathfrak{k}_{\bar 0} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = \mathcal{L}_X F = 0\}$, is a LSA called the Killing superalgebra of (M,g,F)

<u>Theorem</u> (JMF + Hustler '12) (Conjectured by Meessen '04)

dim $\mathfrak{k}_{\bar 1} > \frac{1}{2}$ rank $\$ \implies \mathfrak{k}_{\bar 0} \xrightarrow{ev_p} T_pM$ is surjective $\forall p \in M$ ie: (M,g,F) is locally homogeneous

(JMF + Santi, '16) $\implies (M,g,F)$ obey the field equations $E=0$ where $E \in \Omega^1(M;\operatorname{End}\$)$ is $E(X) = \sum_i e^i \cdot R^{\mathscr{D}}_{e_i,X}$

clifford

curvature of $\mathscr{D}$

dual frames

Example: (M,g) $D=11$ Minkowski, $F=0$ $\implies \mathfrak{k} \cong (D=11)$ Poincaré superalgebra.

My motivation is to extend supersymmetry beyond Minkowski spacetime. The LSAs are then interpreted as supersymmetry algebras. Spacetime supersymmetry appeared in 1971 Gol'fand & Likhtman, where $N=1$ $D=4$ Poincaré superalgebra appeared for the first time. In 1977, Zumino constructed supersymmetric theories in AdS_4 with $\mathfrak{k} \cong \mathfrak{osp}(1,4)$.

There are at least two roads we can take (and this explains the delay in deciding the subject of my talk):

1. sacrifice lorentzian metric; e.g., kinematical supersymmetries. With Ross Grassie, we recently classified homogeneous kinematical superspacetimes [1908.11278]
2. (Today!) keep lorentzian metric and determine suitable $\mathscr{D}$s.

Finally, we reach the subject of the talk.

Fact: the Killing superalgebra of a supergravity background is filtered and its associated graded Lie superalgebra is a graded subalgebra of the Poincaré superalgebra. Let (V,η) be a Lorentzian vector space and let $\mathfrak{so}(V)$ and $C\ell(V)$ denote the Lie algebra of skew-symmetric endomorphisms and the Clifford algebra ($v^2 = -\eta(v,v)\mathbb{1}$), respectively.

The Poincaré superalgebra

spinor rep of $\mathfrak{so}(V)$ (or $C\ell(V)$)

$$\mathfrak{p}(V) := \underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V}$$

← degrees making $\mathfrak{p}(V)$ into a $\mathbb{Z}$-graded Lie superalgebra

with $[A,B] = A\circ B - B\circ A$, $[A,s] = As$, $[A,v] = Av$, $[v,w] = 0$, $[v,s] = 0$ and $[s,s] \in V$ is defined by $\eta([s,s],v) = \langle s, v\cdot s\rangle$, for $\langle\,,\rangle$ a $\mathrm{Spin}(V)$-invariant inner product, for all $A,B \in \mathfrak{so}(V)$, $s \in S$ and $v,w \in V$.

KSAs are filtered deformations of graded subalgebras $\mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} < \mathfrak{p}(V)$.

As in pretty much every deformation theory, the infinitesimal deformations (and obstructions to integrating an infinitesimal deformation) are governed by a cohomology theory. In this case, it's generalised Spencer cohomology.

[Cheng-Kac, '98] Let $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ be a graded LSA, and let $\mathfrak{g}_- = \bigoplus_{k<0} \mathfrak{g}_k$. Then $\mathfrak{g}$ is a $\mathfrak{g}_-$-module under the restriction to $\mathfrak{g}_-$ of ad. The Chevalley-Eilenberg differential ∂ in $C^\bullet(\mathfrak{g}_-;\mathfrak{g})$ has degree 0, because $\mathfrak{g}$ is graded (so $[\,,]$ has degree 0). Also, ∂ is $\mathfrak{g}_0$-equivariant. Therefore we can refine $C^\bullet(\mathfrak{g}_-;\mathfrak{g})$ by degree: $C^{d,n}(\mathfrak{g}_-;\mathfrak{g})$ and Cheng-Kac prove that infinitesimal filtered deformations of $\mathfrak{g}$ are controlled by $\mathfrak{g}_0$-invariant classes in $H^{d,2}(\mathfrak{g}_-;\mathfrak{g})$ for the minimal even $d>0$. Typically, this is $d=2$.

since we want to preserve the $\mathbb{Z}_2$-grading.

Let $\mathfrak{g} = \mathfrak{p} = \mathfrak{so}(V)\oplus S \oplus V$, where $\mathfrak{g}_- = S\oplus V$. The cochains $C^{2,2}(\mathfrak{p}_-;\mathfrak{p})$ decompose into three components:

$$\Lambda^2 V^* \otimes V \quad \oplus \quad V^*\otimes S^* \otimes S \quad \oplus \quad \odot^2 S^* \otimes \mathfrak{so}(V)$$

$\mathscr{D} - \nabla \in \Gamma\left(\mathrm{Spin}(M)\times_{\mathrm{Spin}(V)} (V^*\otimes S^*\otimes S)\right) \Rightarrow$ the $V^*\otimes S^*\otimes S$-component gives a candidate $\mathscr{D}$.

In collaboration with various subsets of Andrea Santi, Paul de Medeiros and Andrew Beckett, we have calculated $H^{2,2}(\mathfrak{p}_-;\mathfrak{p})$ for a variety of Poincaré superalgebras in different dimensions.

d=11 [with AS]

$H^{2,2} \cong \Lambda^4 V$ and $\mathcal{D}$ is the connection in d=11 supergravity

$\uparrow$ $\mathfrak{so}(V)$-mod

$\mathfrak{h}$ is a LSA if $dF=0$ $\quad (\mathfrak{h}_{\bar 0} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = \mathcal{L}_X F = 0\})$

$\mathcal{D}$ flat $\Rightarrow$ Minkowski spacetime, $F=0$

[JMF + Papadopoulos, '01]

$AdS_4 \times S^7$, $F = d\mathrm{vol}_{AdS_4}$

$AdS_7 \times S^4$, $F = d\mathrm{vol}_{S^4}$

Cahen-Wallach symmetric space, F = null

d=4 [with AS + PdM]

$H^{2,2} \cong V \oplus 2\mathbb{R}$ (φ, a, b), $\mathcal{D}_X = \nabla_X + (\varphi \wedge X)\cdot \mathrm{vol} - 2g(\varphi, X)\mathrm{vol} - X\cdot(a + b\,\mathrm{vol})$ $\Rightarrow$ $\mathfrak{h}$ is always a LSA $(\mathfrak{h}_{\bar 0} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = \mathcal{L}_X \varphi = \mathcal{L}_X a = \mathcal{L}_X b = 0\})$

↳ "old minimal off-shell N=1 D=4 SUGRA"

[cf. Festuccia + Seiberg '11]

$\mathcal{D}$ flat $\Rightarrow$ Minkowski spacetime $(\varphi = a = b = 0)$

AdS_4 $(\varphi = 0,\ a, b \in \mathbb{R},\ a^2 + b^2 > 0)$

$AdS_3 \times \mathbb{R}$ — $\nabla\varphi = 0$, φ spacelike

$\mathbb{R} \times S^3$ — $\nabla\varphi = 0$, φ timelike

Nappi-Witten pp-wave — $\nabla\varphi = 0$, φ null

d=6 [with AS + PdM]

d=5 [with AB]

Beyond supergravity!