Kinematical (super)spacetimes

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1711.06111
1711.07363
1802.04048

} JMP

1809.01224
1905.00034
1908.11278

} JHEP

} >300 journal pages

"What are the possible kinematics?"

Bacry & Lévy-Leblond (1968)
Bacry & Nuyts (1986)

Their answer:

- definition and classification of kinematical Lie groups
- observation that every such G acts transitively on some 4-dimensional kinematical spacetime
- these include the maximally symmetric lorentzian spacetimes:
  - Minkowski $\mathbb{M}$ Poincaré
  - de Sitter dS $SO(d+1,1)$
  - anti de Sitter AdS $SO(d,2)$

but much more in addition

Kinematical Lie groups

Kinematical Lie algebra (with D-dim'l space isotropy) $\mathfrak{g}$ real LA $\dim = \frac{(D+2)(D+1)}{2}$

$$\mathfrak{g} \supset \mathfrak{so}(D) \qquad \mathfrak{g} = \mathfrak{so}(D) \oplus 2V \oplus S$$

D-dim'l vector rep. (V)

1-dim'l scalar rep. (S)

Basis:

J_{ab} — $\mathfrak{so}(D)$ "rotations"

B_a — "boosts"

P_a — "spatial translations"

H — time translation

Why the " " ? The geometrical/physical interpretation depends on the geometric realisation of $\mathfrak{g}$ as vector fields on some spacetime. A given KLG may act on more than one spacetime.

The brackets $[J_{ab}, -]$ are fixed, but the other brackets are only constrained by Jacobi $\Rightarrow$ $\mathfrak{so}(D)$-equivariance

Classifications

1898 Bianchi — Every 3-dim'l LA is kinematical ($D=1$)

1968 Bacry + Lévy-Leblond — $D=3$ KLAs with parity & time-reversal

1986 Bacry + Nuyts — $D=3$ KLAs

2017 JMF — $D\geqslant 3$ KLAs

2018 Andrzejewski + JMF — $D=2$ KLAs

2019 JMF + Grassie — $D=3$ $N=1$ KLSAs

$$\mathfrak{s} = \underbrace{\mathfrak{s}_{\bar 0}}_{D=3\ \text{KLA}} \oplus \underbrace{\mathfrak{s}_{\bar 1}}_{\mathfrak{so}(3)\ \text{spinor rep.}}$$

Remark

$D=2,3$ richer due to $\mathfrak{so}(D)$-equivariant

$\Lambda^2 V \to S$ $\quad \epsilon_{ab}$

$\Lambda^2 V \to V$ $\quad \epsilon_{abc}$

Some well-known KLAs (generic $D \geq 2$)

$$
\begin{tikzcd}
\mathfrak{so}(D,2) \arrow[r] & \text{Poincaré} \arrow[r] \arrow[ddr] & \text{Carroll} \arrow[dr] & \\
\mathfrak{so}(D+1,1) \arrow[ur] \arrow[dr] & & & \text{Static} \\
\mathfrak{so}(D+2) \arrow[r] & \text{Euclidean} \arrow[uur] \arrow[r] & \text{Galilean} \arrow[ur] &
\end{tikzcd}
$$

+ Newton-Hooke, ... $\longrightarrow$ contractions

Galilean $\quad [H, B_a] = P_a$

Carroll $\quad [B_a, P_b] = \delta_{ab} H$

D=3 N=1 Kinematical Lie superalgebras

$$
\begin{tikzcd}
\mathfrak{osp}(1,4) \arrow[r] & \text{Poincaré} \arrow[r] \arrow[dr] & \text{Carroll} \arrow[r] & \text{Static} \\
 & & \text{Galilean} \arrow[ur] &
\end{tikzcd}
$$

+ many more

Kinematical spacetimes

G a kinematical Lie group (w/ D-dim'l space isotropy)

$H \subset G$ a closed subgroup with Lie algebra

$$\underset{\substack{\parallel \\ \mathfrak{so}(D) \oplus V}}{\mathfrak{h}} \subset \mathfrak{g}$$

$M = G/H$ $(D+1)$-dim'l spatially-isotropic kinematical spacetime

Theorem There is a 1-1 correspondence

$$\{\text{simply-connected } G/H\}/\cong \quad \longleftrightarrow \quad \{(\mathfrak{g},\mathfrak{h}) \mid \text{effective \& realisable}\}/\cong$$

no ideals of $\mathfrak{g}$ in $\mathfrak{h}$

easy to check

$\exists\, G$ with LA $\mathfrak{g}$ s.t. connected H gen'd by $\mathfrak{h}$ is closed

not so easy to check

Classification of kinematical spacetimes

JMF + Prohazka '18

Several classes:

Riemannian $\mathbb{S}^{D+1}$, $\mathbb{E}^{D+1}$, $\mathbb{H}^{D+1}$

Lorentzian $\mathbb{M}^{D+1}$, dS_{D+1}, AdS_{D+1}

Carrollian C, dSC, $AdSC$ + LC

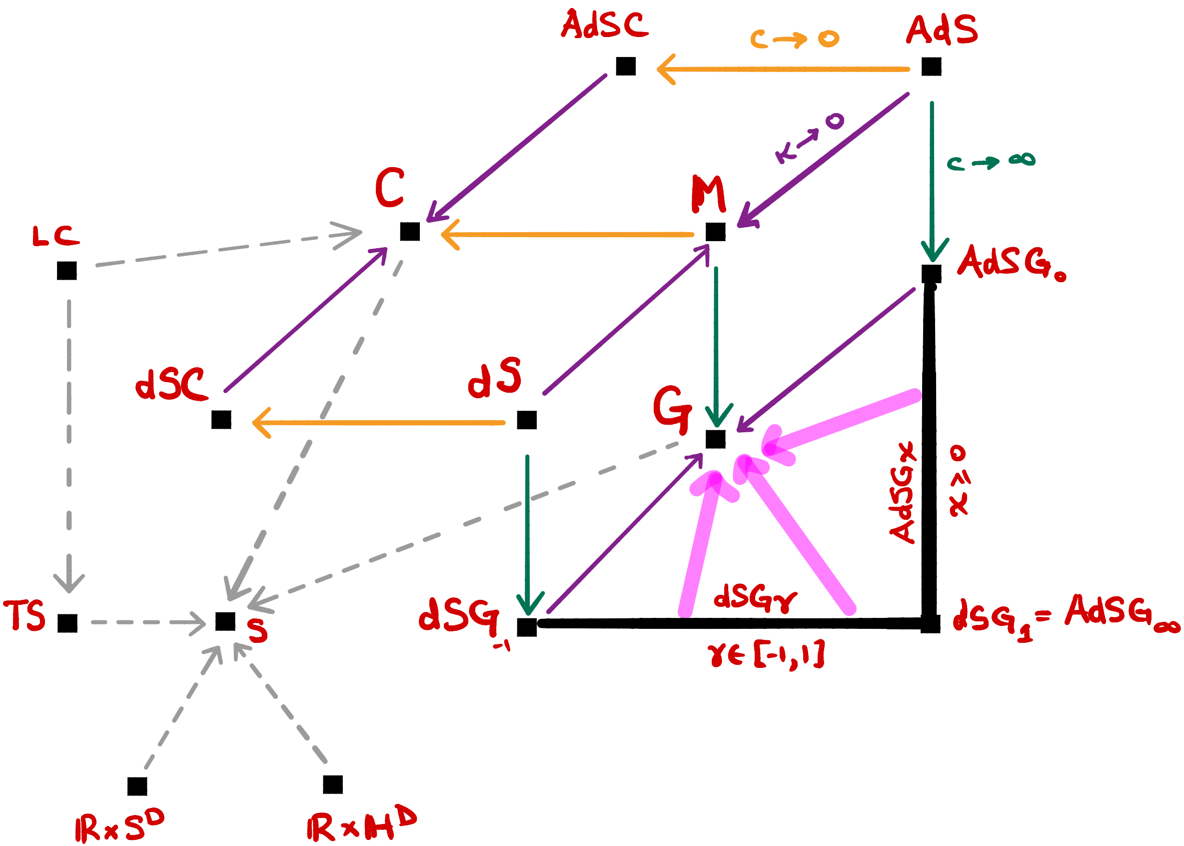
Galilean G, dSG_γ, $AdSG_\chi$

$\gamma \in [-1,1]$ $\chi \in [0,\infty)$

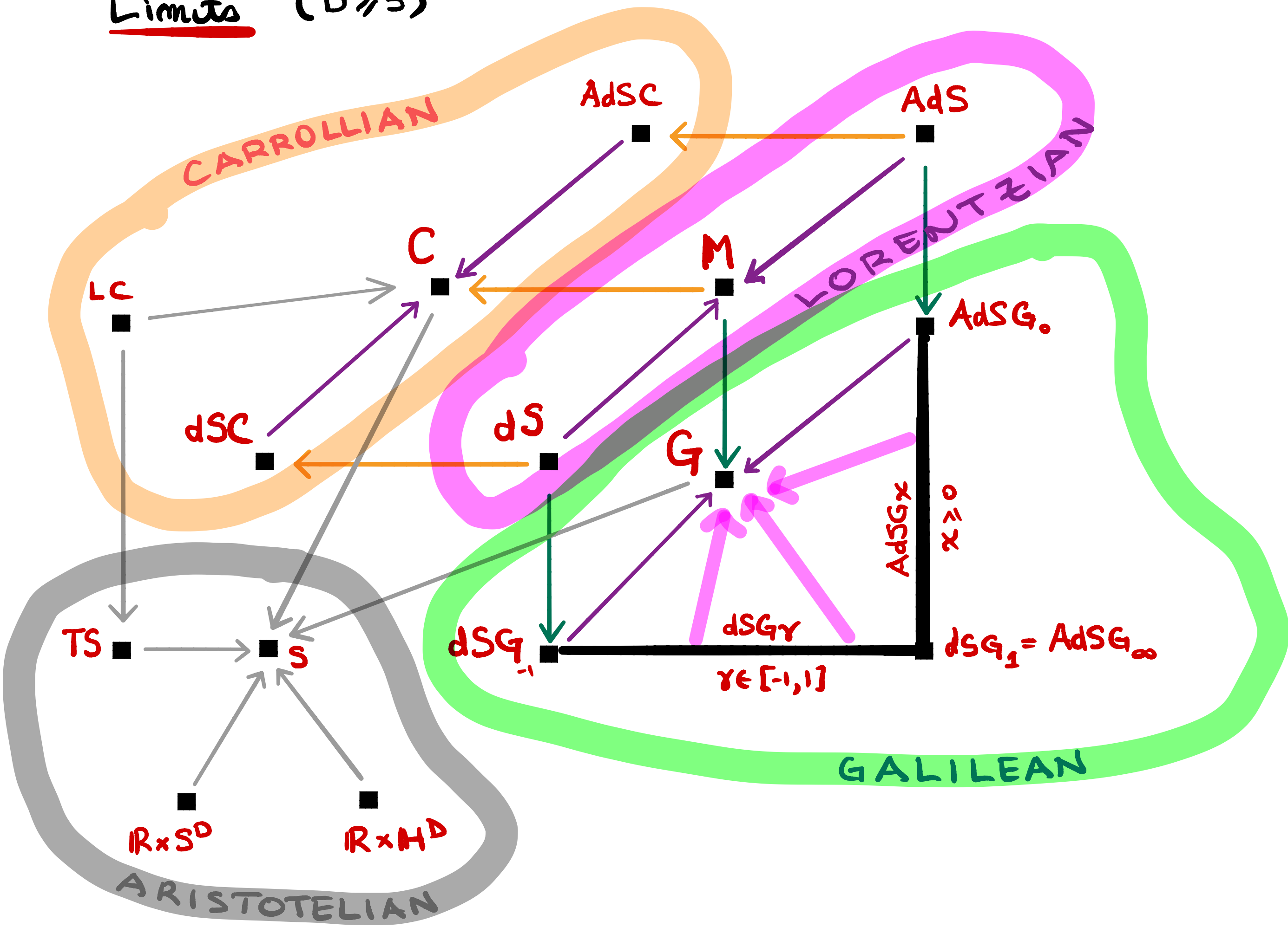
Aristotelian S, $\mathbb{R}\times S^D$, $\mathbb{R}\times \mathbb{H}^D$ + TS

no boosts

Limits (D≥3)

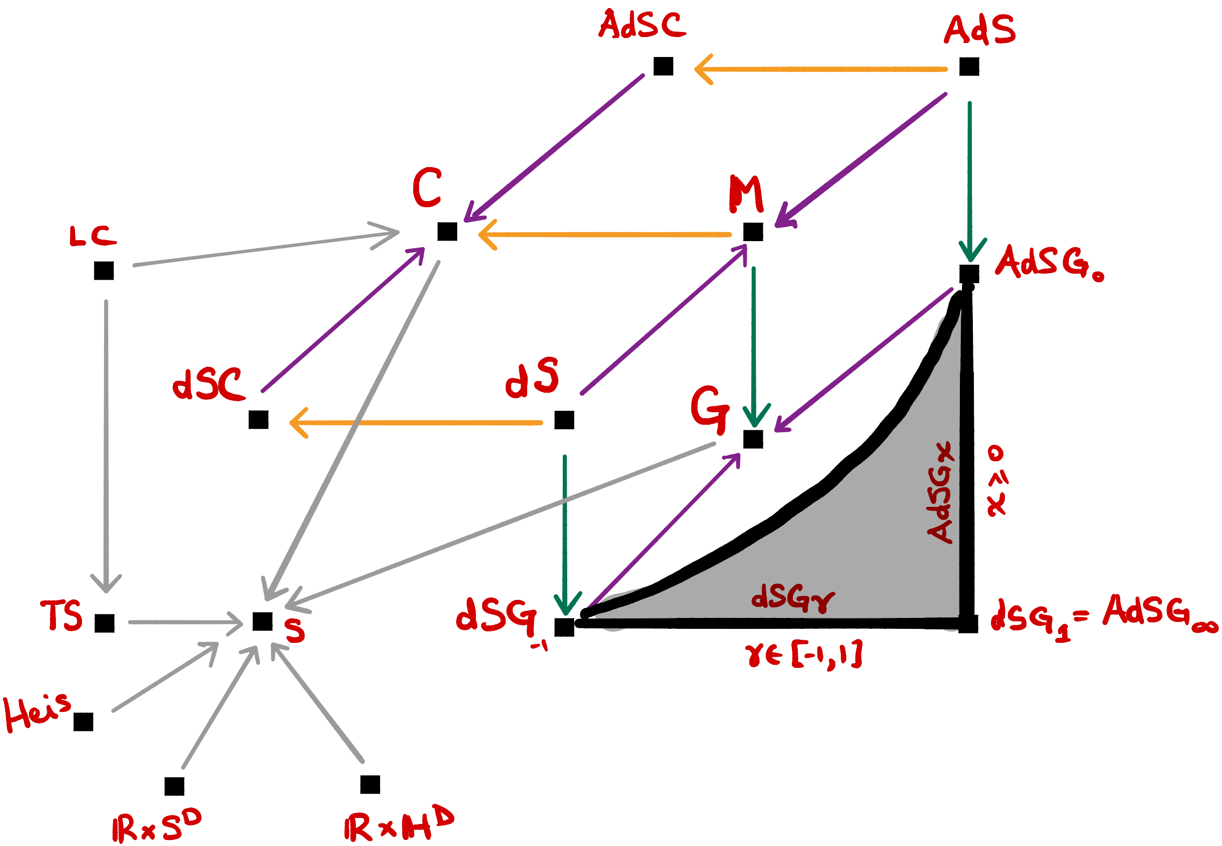

AdSC
AdS
c→0
κ→0
c→∞
C
M
LC
AdSG₀
dSC
dS
G
AdSGχ
χ≥0
TS
S
dSG₋₁
dSGγ
dSG₁= AdSG∞
γ∈[-1,1]
ℝ×S^D
ℝ×H^D

Limits (D⩾3)



CARROLLIAN
LORENTZIAN
GALILEAN
ARISTOTELIAN
AdSC
AdS
C
M
LC
dSC
dS
G
AdSG₀
AdSGχ
χ ⩾ 0
dSGγ
dSG₋₁
dSG₁ = AdSG∞
γ ∈ [-1,1]
TS
S
ℝ×S^D
ℝ×H^D

Limits (D=2)


AdSC
AdS
C
M
LC
AdSG$_0$
dSC
dS
G
AdSG$_\chi$
$\chi \geq 0$
TS
S
dSG$_{-1}$
dSG$_\gamma$
dSG$_1$ = AdSG$_\infty$
$\gamma \in [-1,1]$
Heis
$\mathbb{R}\times S^D$
$\mathbb{R}\times H^D$

Geometrical properties (Generic D)

- All but LC are reductive
- Flat symmetric : $\mathbb{M}$, $\mathbb{E}$, G, C, Static
- Non-flat symmetric : dS, AdS, $\mathbb{S}$, $\mathbb{H}$, AdSG, dSG, AdSC, dSC, $\mathbb{R}\times\mathbb{S}^D$, $\mathbb{R}\times\mathbb{H}^D$
- Reductive torsional : $dSG_{\gamma\in[-1,1]}$, $AdSG_{\chi>0}$, TS
- Except for riemannian ($\mathbb{E}$, $\mathbb{S}$, $\mathbb{H}$) and aristotelian (Static, TS, $\mathbb{R}\times\mathbb{S}^D$, $\mathbb{R}\times\mathbb{H}^D$) boosts are non-compact (generically)

Invariants

Riemannian / Lorentzian : metric g

Galilean : clock one-form τ spatial cometric γ $\gamma(\tau,-)=0$

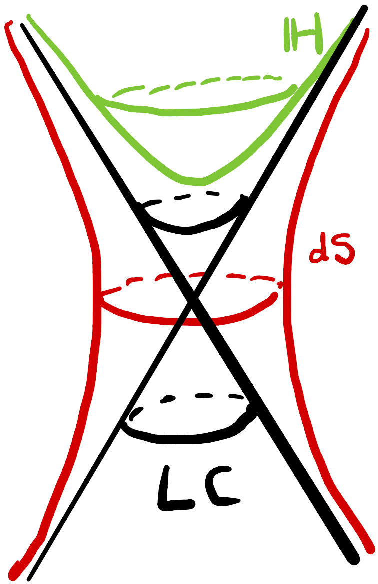
Carrollian : carrollian vector field κ spatial metric h $h(\kappa,-)=0$

Aristotelian : τ, κ, γ, h

All but LC have invariant connections

Examples

$\mathbb{H}^{D+1}$, dS_{D+1} , LC_{D+1} all share $\mathfrak{g} = \underline{\mathfrak{so}}(D+1,1)$

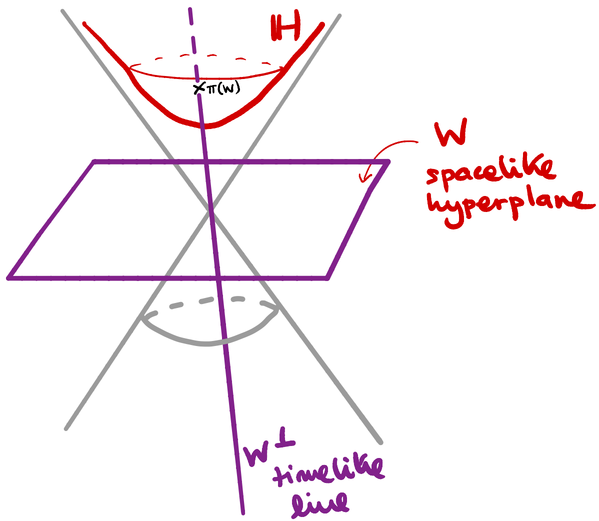


Examples

$\mathbb{M}$ & AdS C share g = Poincaré

$AdSC^{D+1} \xleftarrow{}$ affine spacelike hyperplanes in $\mathbb{M}^{D+1}$

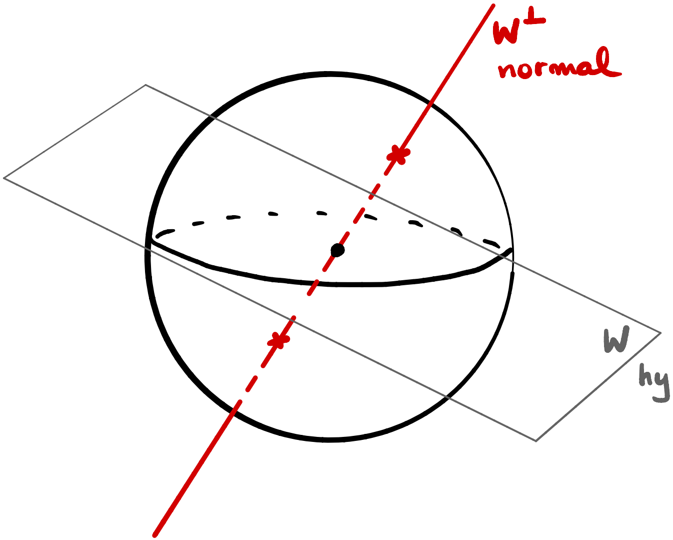
$$\begin{array}{c} AdSC^{D+1} \\ \downarrow \pi \\ \mathbb{H}^D \end{array}$$



$$\pi^{-1}(W^\perp) = \left\{ \begin{array}{l} \text{affine hyperplanes} \\ \text{parallel to } W \end{array} \right\}$$

Examples

$\mathbb{E}$ & dSC share g = Euclidean



Grassmannian of affine hyperplanes

$$\begin{tikzcd} dSC^{D+1} \arrow[r] \arrow[d, "\tilde{\pi}"] & \mathrm{GrAff} \arrow[d, "\pi"] \\ S^D \arrow[r] & \mathbb{RP}^D \end{tikzcd}$$

$\pi^{-1}(W^\perp) = \left\{ \begin{array}{l} \text{affine hyperplanes} \\ \text{parallel to } W \end{array} \right\}$

Symmetries

G/H has symmetry G but the G-invariant structure might have additional symmetries: often ∞-dimensional.

e.g., conformal symmetries of the carrollian structure on LC define an ∞-dim'l LA isomorphic for $D=2$ with the BMS algebra.

Duval + Gibbons + Horvathy '14

This persists in $D \geqslant 3$ giving a possible definition of higher-dimensional BMS algebras; although link with asymptotic symmetries of AF spacetimes is unclear.

JMF + Grassie + Prohazka '19

$D = 3(+1) \quad N = 1$

Kinematical superspacetimes

JMF + Grassie '19

Kinematical Lie supergroups $\longleftrightarrow$ Kinematical super Harish-Chandra pairs $(G, \mathfrak{s})$

Kinematical Lie group

Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\bar 0} \oplus \mathfrak{s}_{\bar 1}$, $\mathfrak{s}_{\bar 0} = \mathfrak{g}$

$\mathfrak{g}$-rep $\int$ ing to G-rep

$M = G/H$ is "superised" to a split supermanifold.

M simply-connected : take G simply-connected and H connected.

The $\mathfrak{g}$-rep $\mathfrak{s}_{\bar 1}$ lifts to a G-rep which restricts to an H-rep.

Let $E = G \times_H \mathfrak{s}_{\bar 1}$. The sections of $\Lambda^\bullet E \to M$ are the "superfunctions" of a split supermanifold $\mathcal{M}$, superising M.

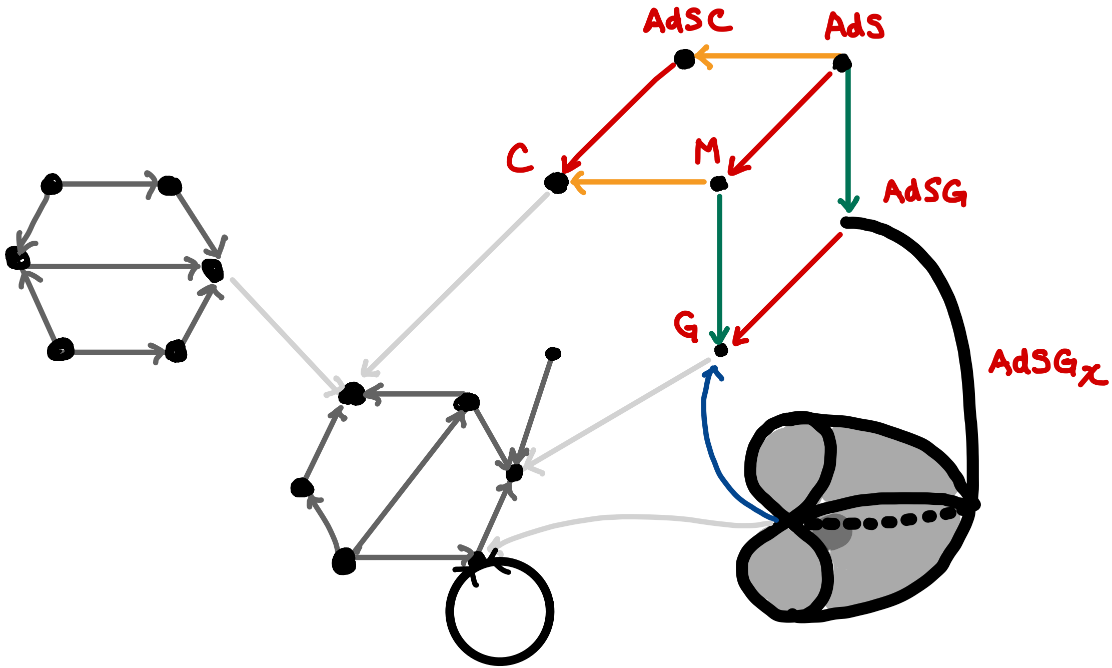
The infinitesimal description of $\mathcal{M}$ is a super pair $(\mathfrak{s}, \mathfrak{h})$.

Realisability of $(\mathfrak{s}, \mathfrak{h})$ is realisability of $(\mathfrak{g}, \mathfrak{h})$, but $(\mathfrak{s}, \mathfrak{h})$ can be effective, even if $(\mathfrak{g}, \mathfrak{h})$ is not $\Rightarrow$ R-symmetry

$D=3(+1) \quad N=1$

Kinematical superspacetimes

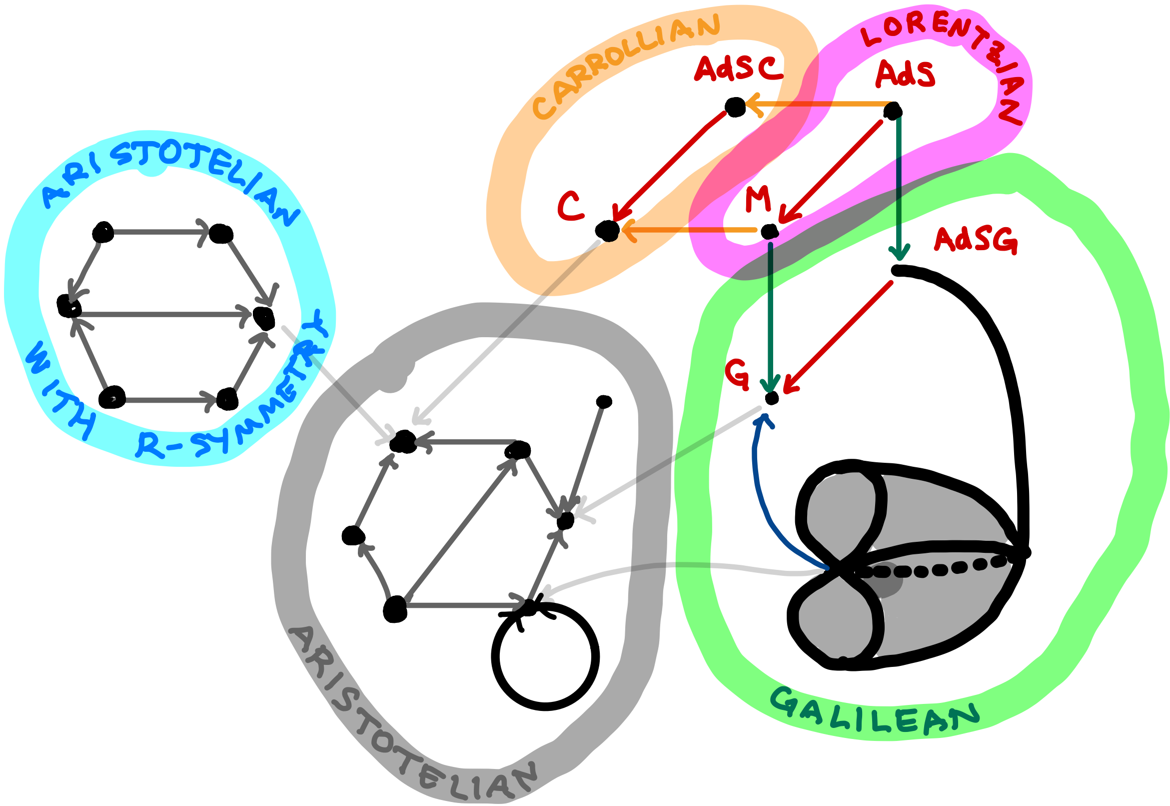
JMF + Grassie '19



$D=3(+1)$ $N=1$

Kinematical superspacetimes

JMF + Grassie '19



Future directions

- Are there BMS-superalgebras associated to the D=3 N=1 kinematical superspaces? We would hope to recover the super-BMS algebra in Awada + Gibbons + Shaw ('85) as Carrollian superconformal symmetries.

- The kinematical (super)spaces are homogeneous and play the rôle of Klein geometries in what could be called a "kinematical Erlangen programme".

  It would be interesting to explore the associated Cartan geometries.

- The representation theory of most kinematical Lie (super)groups is still largely unexplored. This would appear to be a pre-requisite to a full study of the natural field theories which can exist on a kinematical (super) spacetime.