Killing superalgebras in supergravity

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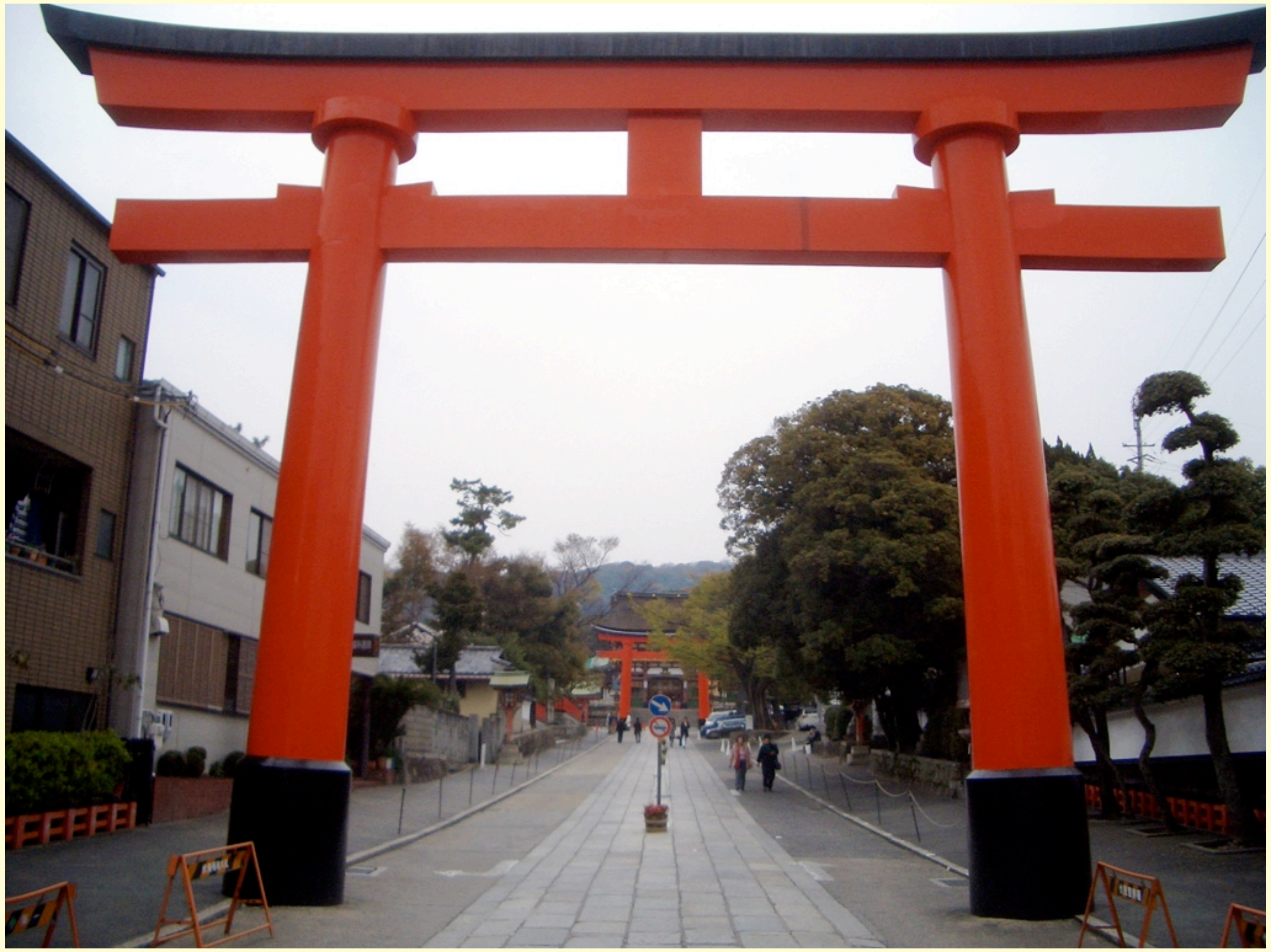
Introduction
(and Outline)



Basic physical question:

How does **supersymmetry** shape the geometry of a supergravity background?

Equivalent mathematical problem:

Classification of supersymmetric supergravity backgrounds.

But what do we mean by **classification**?

- ★ listing **all** possible backgrounds?
  - ✳ or only those with **sufficient** supersymmetry?
- ★ listing the possible **holonomy groups** of the superconnection?
- ★ determining special properties implied by supersymmetry? e.g., **homogeneity**

Some **"experimental"** evidence:

- ★ maximally supersymmetric backgrounds are **symmetric spaces**
- ★ all known backgrounds preserving > ½ supersymmetry are **homogeneous**
- ★ there are non homogeneous ½-BPS backgrounds; e.g., elementary branes

⇨ Natural conjecture:

All > ½ -BPS backgrounds are homogeneous!

What has been **proven** thus far?

- ✓ $d=10$ type I/heterotic supergravity
- ❖ $d=11$ and $d=10$ type IIA/B supergravities:
  - ➡ $> \frac{3}{4}$ implies homogeneity

These results use a geometric construction called the **Killing superalgebra** of a supergravity background.

This talk is based on the following work:

- ★ **hep-th/9808014**, w/ Bobby Acharya, Chris Hull & Bill Spence
  - ➡ applications to AdS/CFT
- ★ **hep-th/9902066**
  - ➡ Killing superalgebra for AdS x Y backgrounds
- ★ **hep-th/0409170**, w/ Patrick Meessen & Simon Philip
  - ➡ KSA for M-theory backgrounds and homogeneity
- ★ **hep-th/0703192**, w/ Emily Hackett-Jones & George Moutsopoulos
  - ➡ KSA and homogeneity of ten-dimensional backgrounds
- ★ **work in progress**, w/ Patricia Ritter
  - ➡ deformation theory of KSAs

Supergravity backgrounds

Generic supergravity fields:

Bosons	Fermions
lorentzian metric	gravitinos
gauge fields	gauginos
p-forms, scalars	dilatinos

Geometrically, a supergravity background consists of the following data:

- a ***d***-dimensional lorentzian spin manifold (M, g)
- a real spinor bundle S
- other bosonic fields F, ϕ,...

subject to field equations which generalise the Einstein and Maxwell equations.

The fermionic fields are set to zero, but their supersymmetry variations define the **Killing spinors**.

A background is said to be **supersymmetric** if it admits nonzero Killing spinors.

The gravitino variation gives rise to a differential equation:

$$\delta_\varepsilon \psi = \nabla \varepsilon + \cdots = D\varepsilon = 0$$

Connection on S ⇒ **Natural** question: which **holonomy** groups appear?

The other fermions give rise to algebraic equations:

$$\delta_\varepsilon \lambda = P\varepsilon = 0$$

A spinor ε is **Killing** if $\delta_\varepsilon(\text{fermions}) = 0$ ⇒

$$\begin{aligned} D\varepsilon &= 0 \\ P\varepsilon &= 0 \\ &\vdots \end{aligned}$$

$d=11$ supergravity

Fields $\quad g \qquad F \in \Omega^4 \qquad dF = 0$

Spinors are real and have 32 components

Field equations

$$\mathrm{Ricci}(g) = T(g, F) \qquad\qquad d \star F = -\tfrac{1}{2} F \wedge F$$

Killing spinors

$$D_X \varepsilon = \nabla_X \varepsilon + \tfrac{1}{6} \imath_X F \cdot \varepsilon + \tfrac{1}{12} X \wedge F \cdot \varepsilon = 0$$

Clifford product

$$\dim\{\varepsilon \mid D\varepsilon = 0\} = 32\nu$$

$\downarrow$

supersymmetry fraction

$\nu = 1$ $\Longleftrightarrow$ Maximally supersymmetric vacua

Figueroa-O'Farrill+Papadopoulos (2002)

$\nu = \frac{1}{2}$ $\Longleftrightarrow$ ½-BPS backgrounds
e.g., M2, M5, MKK, MW

Which fractions can appear? So far,

$$\frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \frac{1}{8}, \frac{5}{32}, \frac{3}{16}, \dots, \frac{1}{4}, \dots, \frac{3}{8}, \dots, \frac{1}{2}, \dots, \frac{9}{16}, \dots, \frac{5}{8}, \dots, \frac{11}{16}, \dots, \frac{3}{4}, \dots, 1$$

and only $\frac{31}{32}$ (supergravity preons) has been ruled out.

Gran+Gutowski+Papadopoulos+Roest (2006)
Figueroa-O'Farrill+Gadhia (2007)

d=10 heterotic supergravity

Fields $\quad g \quad \phi \quad H \in \Omega^3 \quad dH = 0 \quad F \in \Omega^2(\mathfrak{g})$

Spinors are real, chiral and have 16 components

Field equations follow from (string frame) lagrangian

$$e^{-2\phi}\left(R + 4|d\phi|^2 - \tfrac{1}{2}|H|^2 - \tfrac{1}{2}|F|^2\right)$$

Killing spinors

$$D\varepsilon = 0 \qquad d\phi \cdot \varepsilon + \tfrac{1}{2} H \cdot \varepsilon = 0 \qquad F \cdot \varepsilon = 0$$

spin connection with torsion H

Killing superalgebras

Lie superalgebras

(real) vector superspace

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

(super) antisymmetric even bracket

$$[\quad , \quad] : \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$$

$$[X, Y] = -(-1)^{XY} [Y, X]$$

obeying the Jacobi identity

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{XY} [Y, [X, Z]]$$

Let us unpack this data:

$[\quad , \quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0$ is an honest Lie bracket

$\therefore$ $\mathfrak{g}_0$ is a Lie algebra

$[\quad , \quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$ makes $\mathfrak{g}_1$ into a rep of $\mathfrak{g}_0$

$[\quad , \quad] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ is $\mathfrak{g}_0$-equivariant

and the only remaining identity is the odd-odd-odd Jacobi identity:

$[[X, X], X] = 0$ for all $X \in \mathfrak{g}_1$

The construction of the Killing superalgebra of a supergravity background requires:

1) identifying $\mathfrak{g}_0$ and $\mathfrak{g}_1$

2) identifying the brackets:

$$[\quad , \quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0$$

$$[\quad , \quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$$

$$[\quad , \quad] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$$

and

3) proving the Jacobi identities.

The last two also require showing that the brackets close on the correct space!

Details vary depending on the supergravity theory,
but the basic construction is universal.

$\mathfrak{g}_0$ is the Lie algebra of **infinitesimal symmetries**

i.e., Killing vector fields preserving all bosonic fields

e.g., in a ***d=11*** supergravity background (M, g, F)
vector fields X obeying $\mathcal{L}_X g = 0 = \mathcal{L}_X F$

$[\quad,\quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0$ is the Lie bracket of vector fields
(automatically obeys Jacobi)

$\mathfrak{g}_1$ is the space of **Killing spinors**

$[\quad , \quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$ is the **spinorial Lie derivative**

Lichnerowicz+Kosmann (1972)

X Killing $\Longleftrightarrow$ $\nabla X : Y \mapsto \nabla_Y X$ is skewsymmetric

$\rho : \mathfrak{so}(TM) \to \mathrm{End}(S)$ the spinor representation

$\mathcal{L}_X := \nabla_X + \rho(\nabla X)$ obeys the following properties:

1) $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$

Lie bracket of vector fields

commutator of endomorphisms

2) $[\mathcal{L}_X, \nabla_Y] = \nabla_{[X,Y]}$ any vector field

3) $\mathcal{L}_X(Y \cdot \varepsilon) = [X, Y] \cdot \varepsilon + Y \cdot \mathcal{L}_X \varepsilon$

$$X \in \mathfrak{g}_0 \Rightarrow \begin{array}{l} [\mathcal{L}_X, D_Y] = D_{[X,Y]} \\ \mathcal{L}_X(P\varepsilon) = P\mathcal{L}_X \varepsilon \end{array}$$

$$\therefore \quad \varepsilon \in \mathfrak{g}_1 \implies \mathcal{L}_X \varepsilon \in \mathfrak{g}_1$$

$$\therefore \quad [\quad , \quad] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$$

defined by $[X, \varepsilon] := \mathcal{L}_X \varepsilon$

is well-defined and, using ***1)***, obeys the even-even-odd Jacobi identity.

$[\quad , \quad] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ is defined by **squaring** spinors

$$[\varepsilon_1, \varepsilon_2] := X$$

where X has components $X^a = \bar{\varepsilon}_1 \gamma^a \varepsilon_2$

This is symmetric in lorentzian signature.

This is **not** completely trivial and requires a calculation!

X is always **causal**.

The Killing spinor equations imply that

$$\varepsilon_1, \varepsilon_2 \in \mathfrak{g}_1 \implies X \in \mathfrak{g}_0$$

(It suffices to show this for $\varepsilon_1 = \varepsilon_2$.)

The even-odd-odd Jacobi identity follows from properties of the spinorial Lie derivative.

It remains to prove the odd-odd-odd Jacobi identity:

$$\mathcal{L}_X \varepsilon = 0$$

where

$$X^a = \bar{\varepsilon}\gamma^a\varepsilon$$

This is an algebraic equation, which ought to follow from representation theory alone, but in practice requires a calculation.

The Killing superalgebra has been constructed (and shown to be a Lie superalgebra) for the following supergravity theories:

- ★ ***d=11*** supergravity
- ★ ***d=10*** type I/heterotic supergravities
- ★ ***d=10*** type IIA/IIB supergravities

Figueroa-O'Farrill+Meessen+Philip (2004)
Figueroa-O'Farrill+Hackett-Jones+Moutsopoulos (2007)

The explicit form of the Killing superalgebra is known for a number of supergravity backgrounds:

Minkowski vacuum $\Rightarrow$ Poincaré superalgebra

Freund-Rubin vacua:

$$\mathrm{AdS}_4 \times S^7 \Rightarrow \mathfrak{osp}(8|2)$$

$$\mathrm{AdS}_7 \times S^4 \Rightarrow \mathfrak{osp}(6,2|2)$$

$$\mathrm{AdS}_5 \times S^5 \Rightarrow \mathfrak{su}(2,2|4)$$

Plane wave vacua $\Rightarrow$ **contractions** of the above

Figueroa-O'Farrill+Papadopoulos (2001)
Blau+Figueroa-O'Farrill+Hull+Papadopoulos (2001,2002)
Hatsuda+Kamimura+Sakaguchi (2002)
Blau+Figueroa-O'Farrill+Papadopoulos (2002)

For purely gravitational backgrounds, the KSA is a Lie subsuperalgebra of the Poincaré superalgebra:

Background	canonical ideal
elementary branes (also multicentred)	translations along brane worldvolume
branes at conical singularities	translations along brane worldvolume
half-BPS plane wave and its generalisations	parallel null vector
Kaluza-Klein monopole and its generalisations	translations along Minkowski factor

$[\mathfrak{g}_1, \mathfrak{g}_1]$

For the near-horizon geometries of the branes at conical singularities, one obtains the **conformal superalgebras** predicted by AdS/CFT.

Acharya+Figueroa-O'Farrill+Hull+Spence (1998)

The Killing superalgebra can be calculated **without** explicit knowledge of the form of the Killing spinors, using the Bär **cone construction**.

Bär (1993)

This construction can also be used to classify supersymmetric Freund-Rubin backgrounds.

Figueroa-O'Farrill+Leitner+Simón (in preparation)

Homogeneity

A supergravity background is **homogeneous** if it admits a **transitive** action of a Lie group by **flux-preserving isometries**.

In (super)gravity we must work with **local** geometries — this requires a local version of homogeneity.

A supergravity background is **locally homogeneous** if, at every point, there exists a local frame consisting of **infinitesimal symmetries**.

$[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$ ⇒ The more supersymmetry a background preserves, the more infinitesimal isometries it admits.

Facts:

Maximally supersymmetric backgrounds are homogeneous — in fact, **symmetric**.

There are non-homogeneous half-BPS backgrounds.

⇒ There exists a **critical fraction** $\nu_c \geq \frac{1}{2}$ such that if $\nu > \nu_c$ the background is locally homogeneous.

All **known** supergravity backgrounds with $\nu > \frac{1}{2}$ are (locally) homogeneous — suggesting that $\nu_c = \frac{1}{2}$.

For d=11 and d=10 types IIA/B supergravities, one can **prove** that $\frac{1}{2} \leq \nu_c \leq \frac{3}{4}$, exploiting the representation theory of the Clifford algebra.

Killing spinors are determined by their value at any point, whence it is enough to show that

$\dim \mathfrak{g}_1$ large enough ⇒ $[\mathfrak{g}_1, \mathfrak{g}_1]^\perp = \mathbf{0}$

Linear algebra shows that $\dim \mathfrak{g}_1 > 24$ is large enough.

Figueroa-O'Farrill+Meessen+Philip (2004)
Figueroa-O'Farrill+Hackett-Jones+Moutsopoulos (2007)

Indeed,

$$V \in [\mathfrak{g}_1, \mathfrak{g}_1]^\perp \iff \bar{\varepsilon}_1 V^a \gamma_a \varepsilon_2 = 0 \qquad \forall \varepsilon_{1,2} \in \mathfrak{g}_1$$

whence Clifford multiplication by V defines a map

$$\widehat{V} : \mathfrak{g}_1 \to \mathfrak{g}_1^\perp$$

Since we know that $\nu_c \geq \frac{1}{2}$, we may assume that

$$\dim \mathfrak{g}_1 > \dim \mathfrak{g}_1^\perp$$

whence $\widehat{V}$ has nontrivial kernel $\Rightarrow$ $\widehat{V}$ is **null**

This means that the bilinear form on spinors

$$(\varepsilon_1, \varepsilon_2) \mapsto \overline{\varepsilon}_1 V^a \gamma_a \varepsilon_2$$

has rank **16**.

On the other hand, one can easily estimate the **maximum** rank of such a bilinear to be

$$2 \operatorname{codim} \mathfrak{g}_1$$

which shows that for a nonzero V to exist, the codimension of $\mathfrak{g}_1$ must be at least **8**.

Therefore, if $\dim \mathfrak{g}_1 > 24$, no such vector can exist and

$$[\mathfrak{g}_1, \mathfrak{g}_1]^\perp = \mathbf{0}$$

In the 2004 paper, we conjectured that the critical fraction was indeed **¾**.

However, this proof ignores the fact that the Killing spinors satisfy a differential equation — that is, ignores information about the **holonomy** of the superconnection.

A similar argument for ***d=10*** heterotic/type I supergravity would also suggest that $\nu_c \stackrel{?}{=} \frac{3}{4}$, but using the classification of parallelisable backgrounds one shows that $\nu_c = \frac{1}{2}$.

Figueroa-O'Farrill (2003)
川野+山口 (2003)
Figueroa-O'Farrill+川野+山口 (2003)
Figueroa-O'Farrill+Hackett-Jones+Moutsopoulos (2007)

Deformations

It is a generalised **belief** in string theory that supergravity backgrounds — that is, solutions to the supergravity field equations — can be deformed continuously to solutions of the field equations with quantum or α' corrections.

This belief justifies, from a string theory point of view, much of the research into supergravity.

For the purposes of this talk, we shall not question this belief.

Natural question:

What happens to the Killing superalgebra under quantum or α' corrections?

Possible answers:

- ★ The notion of KSA does **not** persist
- ✓ The notion persists:
  - ★ **not** as a Lie superalgebra
  - ✓ as a Lie superalgebra:
    - ★ of **different** dimension, or
    - ✓ of the same dimension

Let us make the **assumption** that it deforms as a Lie superalgebra of the same dimension.

Then, relative to some basis, the structure constants now depend on α' (or $\hbar$).

This is a well-known mathematical problem, which can be analysed using techniques of Lie (super)algebra **cohomology**, as developed by Chevalley+Eilenberg and, in the super case, by Leites.

A one-parameter family of Lie brackets is given by:

$$[X,Y]_t = \sum_{n\geq 0} t^n \Phi_n(X,Y)$$

where superantisymmetric!

$$\Phi_n : \Lambda^2\mathfrak{g} \to \mathfrak{g}$$

The Jacobi identity gives rise to an infinite number of quadratic equations involving the Φ_n, one for each power of the parameter.

The first equation is the Jacobi identity for Φ_0, corresponding to the undeformed Lie superalgebra.

The second equation says that Φ_1 is a 2-cocycle for the undeformed Lie superalgebra. If a coboundary, it is not a genuine deformation, but simply a t-dependent change of basis.

$\therefore \quad H^2(\mathfrak{g};\mathfrak{g})$ = space of **infinitesimal deformations**.

The remaining equations give an infinite number of obstructions to integrating an infinitesimal deformation, which can interpreted as cohomology classes one dimension higher.

$\therefore \quad H^3(\mathfrak{g};\mathfrak{g})$ = space of **obstructions**.

The calculations of these cohomology groups is a problem in linear algebra, but it can quickly grow **out of control** due to the size of the vector spaces involved.

By the **rigidity** of semisimple Lie algebras and of their representations, one can exploit the existence of semisimple factors of the undeformed Lie superalgebra to cut the computation **considerably** down in size.

Technically, one computes the cohomology groups using a method of successive approximations known as the **Hochschild-Serre spectral sequence**.

The end result of this method is the **factorisation theorem** of Hochschild–Serre–Binegar.

Let $I < \mathfrak{g}$ be an ideal such that $\mathfrak{s} := \mathfrak{g}/I$ is semisimple.

Then

$$H^n(\mathfrak{g};\mathfrak{g}) \cong \bigoplus_{i=0}^{n} \left(H^{n-i}(\mathfrak{s}) \otimes H^i(I;\mathfrak{g})^{\mathfrak{s}} \right)$$

We need only work with $\mathfrak{s}$-invariants.

In particular,

$$H^2(\mathfrak{g};\mathfrak{g}) \cong H^2(I;\mathfrak{g})^{\mathfrak{s}}$$

Center

$$H^3(\mathfrak{g};\mathfrak{g}) \cong H^3(I;\mathfrak{g})^{\mathfrak{s}} \oplus \left(H^3(\mathfrak{s}) \otimes \mathfrak{z} \right)$$

The resulting complexes are now much smaller and the calculations tractable by hand.

We have analysed the deformations of the Killing superalgebras associated to the simplest M-theory backgrounds.

The eleven-dimensional Poincaré superalgebra admits no deformations: it is **rigid**. (This contrasts sharply with four dimensions, where the Poincaré superalgebra deforms to the **de Sitter** superalgebras.)

This **suggests** that the Minkowski vacuum admits no quantum corrections.

Similarly the KSA of the M5-brane and of the Freund-Rubin vacua are also rigid, suggesting that these backgrounds receive no quantum corrections either.

On the other hand, the M2-brane KSA admits a deformation, whose bosonic subalgebra contains the isometry algebra of **anti-de Sitter space**. This suggests that under quantum corrections, the worldvolume of the M2-brane gets **curved**.

Similarly, the half-BPS M-wave and Kaluza-Klein monopole admit a unique deformation. The maximally supersymmetric plane wave admits at least one deformation, which corresponds to the inverse to the contraction induced by the plane-wave limit. (The analysis has still to be completed.)

One has to be careful, however, to extract predictions from this analysis, since it is based on the assumption that the KSA persists (and does not drop in dimension) under quantum corrections.

A better understanding of the structure of the quantum-corrected supergravities is necessary to make further progress.

どうもありがとうございました。