Geometric M-theory backgrounds

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Aim

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Aim: To classify/characterise supergravity backgrounds with classical geometric interpretation.

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- parallelisable backgrounds in ten-dimensional string theory [hep-th/0305079, FO-Kawano-Yamaguchi hep-th/0308141]

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3

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∃ parallel spinors does <u>not</u> imply Ricci-flatness

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Possible *G* follow from orbit decomposition of the spinor representation under Spin(1, n)

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[Bryant math.DG/0004073]

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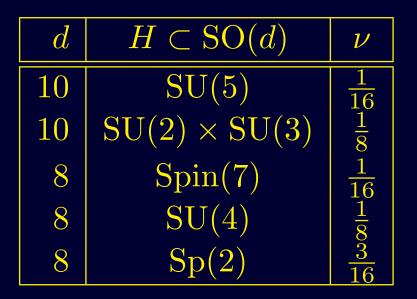
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[Sorkin, Gross–Perry (1983); Han–Koh (1985)]

 $\overline{M} = \mathbb{R}^{1,10-d} \times K^d$

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d	$H \subset \mathrm{SO}(d)$	ν
8	$\operatorname{Sp}(1) \times \operatorname{Sp}(1)$	$\frac{1}{4}$
7	G_2	$\frac{1}{8}$
6	${ m SU}(3)$	$\frac{1}{4}$
4	$\mathrm{SU}(2) \cong \mathrm{Sp}(1)$	$\frac{1}{2}$
0	{1}	1

[Wang (1989)]

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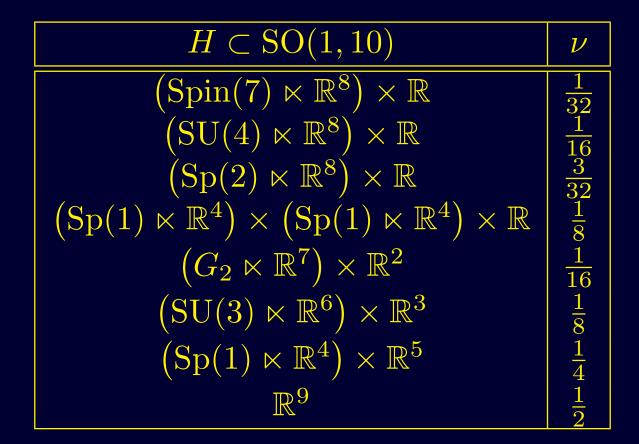
[Hull (1984)]

Indecomposable non-static backgrounds

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$H \subset \mathrm{SO}(1, 10)$	ν
$(\operatorname{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$\frac{1}{32}$
$(\mathrm{SU}(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$\frac{1}{16}$
$(\operatorname{Sp}(2)\ltimes\mathbb{R}^8)\times\mathbb{R}$	$\frac{\overline{3}}{32}$
$(\operatorname{Sp}(1) \ltimes \mathbb{R}^4) \times (\operatorname{Sp}(1) \ltimes \mathbb{R}^4) \times \mathbb{R}^4$	$\frac{1}{8}$
$(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$	$\frac{1}{16}$
$(\mathrm{SU}(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$	$\frac{1}{8}$
$(\operatorname{Sp}(1)\ltimes\mathbb{R}^4) imes\mathbb{R}^5$	$\frac{1}{4}$
\mathbb{R}^9	$\frac{1}{32} \frac{1}{16} \frac{32}{32} \frac{1}{8} \frac{1}{16} \frac{1}{18} \frac{1}{14} \frac{1}{2}$

Indecomposable non-static backgrounds



[Leistner math.DG/0309274]

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• $AdS_5 \times S^5$ in IIB

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• (M, g) and (N, h) Einstein with scalar curvatures $\pm \frac{4}{3}f^2$ and $\mp \frac{7}{6}f^2$, respectively.

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with $\lambda = \pm \frac{f}{6}$ and $\pm \frac{f}{12}$, respectively.

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[Bär (1993)]

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[Baum (2000), Baum-Leitner math.DG/0305063]

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For d = 4, 5 the most general such metric is known. [hep-th/9904124]

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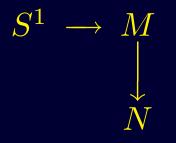
[Leitner math.DG/0302024]

Lorentzian Einstein Sasaki geometry

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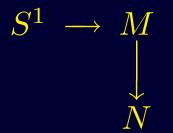
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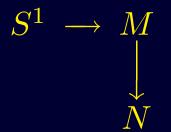
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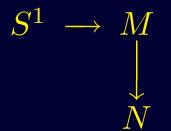


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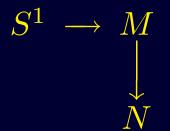


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- ullet circle bundle is associated to spin bundle on N
- circles are timelike, hence "Gödel-like"

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Others thus far lack a clear physical interpretation.

Common sector of type II string theory:

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$$\int_{M} e^{-2\phi} \left(R + 4 |d\phi|^2 - \frac{1}{2} |H|^2 \right) d\text{vol}_g$$

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This turns M into a Lie group with bi-invariant metric.

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[Cartan–Schouten (1926), Wolf (1970)]

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Only the first two have dH = 0.

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[Cahen–Parker (1977)]

They are all Lie groups with bi-invariant metrics, whence dH = 0.

Parallelisable building blocks

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Space	Torsion	
AdS_3	dH = 0	$ H ^2 < 0$
$\mathbb{R}^{1,n}, n \geq 0$	H = 0	
$\mathbb{R}^n, \ n \geq 1$	H = 0	
S^3	dH = 0	$ H ^2 > 0$
S^7	dH eq 0	$ H ^2 > 0$
SU(3)	dH = 0	$ H ^2 > 0$
$CW_{2n+2}(J)$	dH = 0	$ H ^2 = 0$

Ten-dimensional parallelisable geometries

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 $\begin{array}{l} \operatorname{AdS}_{3} \times S^{7} \\ \operatorname{AdS}_{3} \times S^{3} \times \mathbb{R}^{4} \\ \mathbb{R}^{1,0} \times S^{3} \times S^{3} \times S^{3} \\ \mathbb{R}^{1,2} \times S^{7} \\ \mathbb{R}^{1,6} \times S^{3} \\ \operatorname{CW}_{10}(J) \\ \operatorname{CW}_{6}(J) \times S^{3} \times \mathbb{R} \\ \operatorname{CW}_{4}(J) \times S^{3} \times S^{3} \\ \operatorname{CW}_{4}(J) \times \mathbb{R}^{6} \end{array}$

 $\begin{aligned} &\operatorname{AdS}_{3} \times S^{3} \times S^{3} \times \mathbb{R} \\ &\operatorname{AdS}_{3} \times \mathbb{R}^{7} \\ &\mathbb{R}^{1,1} \times \operatorname{SU}(3) \\ &\mathbb{R}^{1,3} \times S^{3} \times S^{3} \\ &\mathbb{R}^{1,9} \\ &\operatorname{CW}_{8}(J) \times \mathbb{R}^{2} \\ &\operatorname{CW}_{6}(J) \times \mathbb{R}^{4} \\ &\operatorname{CW}_{4}(J) \times S^{3} \times \mathbb{R}^{3} \end{aligned}$

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The case of linear dilaton was analysed by Kawano and Yamaguchi.

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For non-dilatonic backgrounds, this equation has solutions if and only if $|H|^2 = 0$; which restricts the possible geometries.

Spacetime	Supersymmetry
$\operatorname{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$	16
$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	16
$CW_{10}(J)$	16,18(A),20,22(A),24(B),28(B)
$\mathrm{CW}_8(J) \times \mathbb{R}^2$	16,20
$\mathrm{CW}_6(J) \times \mathbb{R}^4$	16,24
$\mathrm{CW}_4(J) imes \mathbb{R}^6$	16
$\mathbb{R}^{1,9}$	32

Turning on the linear dilaton all backgrounds are now half-BPS

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 $\overline{\mathrm{AdS}_{3} \times S^{3} \times S^{3} \times \mathbb{R}}$ $\overline{\mathrm{AdS}_{3} \times S^{3} \times \mathbb{R}^{4}}$ $\mathbb{R}^{1,1} \times \mathrm{SU}(3)$ $\mathbb{R}^{1,3} \times S^{3} \times S^{3}$ $\mathbb{R}^{1,6} \times S^{3}$ $\mathbb{R}^{1,9}$

 $\begin{array}{l} \mathrm{CW}_{10}(J)\\ \mathrm{CW}_8(J)\times\mathbb{R}^2\\ \mathrm{CW}_6(J)\times\mathbb{R}^4\\ \mathrm{CW}_6(J)\times S^3\times\mathbb{R}\\ \mathrm{CW}_4(J)\times S^3\times\mathbb{R}^3\\ \mathrm{CW}_4(J)\times\mathbb{R}^6\end{array}$

[Kawano-Yamaguchi hep-th/0306038]

All these backgrounds are exact string backgrounds

Turning on the linear dilaton all backgrounds are now half-BPS:

 $AdS_3 \times S^3 \times S^3 \times \mathbb{R}$ $AdS_3 \times S^3 \times \mathbb{R}^4$ $\mathbb{R}^{1,1} \times SU(3)$ $\mathbb{R}^{1,3} \times S^3 \times S^3$ $\mathbb{R}^{1,6} \times S^3$ $\mathbb{R}^{1,9}$

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All these backgrounds are exact string backgrounds: coupling a WZW model for (M, g, H) to a Liouville field theory for ϕ .

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action

$$\int_{M} e^{-2\phi} \left(R + 4|d\phi|^2 - \frac{1}{2}|H|^2 - \frac{N}{2}|F|^2 \right) d\text{vol}_g$$

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$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	8
$\mathrm{CW}_{10}(J)$	8,10,12,14
$\mathrm{CW}_8(J) \times \mathbb{R}^2$	8,10
$\mathrm{CW}_6(J) \times \mathbb{R}^4$	8,12
$\mathrm{CW}_4(J) imes \mathbb{R}^6$	8
$\mathbb{R}^{1,9}$	16

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Thank you.