Homogeneous Geometry

(EMPG preseminar 23/1/2019)

The aim of this preseminar is to introduce some of the language of homogeneous geometry. All manifolds will be C^∞ and finite-dimensional.

A Lie group G acts on a manifold M if there exists a C^∞ map

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, m) &\longmapsto g \cdot m \end{aligned}$$

satisfying two conditions:
- if $e \in G$ is the identity, then $e \cdot m = m \quad \forall m \in M$.
- $\forall g_1, g_2 \in G, m \in M \qquad g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$

Let $m \in M$ and let $G_m = \{g \in G \mid g \cdot m = m\}$. This is a closed subgroup of G called the stabilizer of m. The intersection $N = \bigcap_{m \in M} G_m$ of all stabilizers is a normal subgroup of G. If $N = \{e\}$ the action is said to be effective and if N is discrete, the action is locally effective. If $m \in M$, we let $G \cdot m = \{g \cdot m \mid g \in G\}$ denote the orbit of m under G. The orbit $G \cdot m$ is diffeomorphic to the space G/G_m of right G_m cosets. If $G \cdot m = M$ the action is said to be transitive and then $M = G/G_m$ itself is a coset space.

Let G be a Lie group and H a closed subgroup. Then $M := G/H$ is a homogeneous G-space. If G and H are connected, M is simply-connected. Analogously to the theorem that says that there is a one-to-one correspondence between (iso classes of) Lie algebras and (iso classes of) simply-connected Lie groups, there is a theorem that says that there exists a one-to-one correspondence between simply-connected homogeneous spaces and Lie pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a LA and $\mathfrak{h}$ a Lie subalgebra which are (1) effective ($\mathfrak{h}$ does not contain any nonzero ideal of $\mathfrak{g}$) and (2) geometrically realisable, so that there is a Lie group with LA $\mathfrak{g}$ s.t. that Lie subgroup corresponding to $\mathfrak{h}$ is closed. This result reduces the problem of classifying simply-connected homogeneous spaces to the algebraic problem of classifying (effective, geom. realisable) Lie pairs.

Let $M = G/H$ be described by a Lie pair $(\mathfrak{g}, \mathfrak{h})$. We have a canonical sequence of H-reps:

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

If this sequence splits, so that there exists an H-subrep $\mathfrak{m}$ of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we say that $(\mathfrak{g}, \mathfrak{h})$ is reductive. By abuse of language we say M is reductive, even though reductivity is a property of $(\mathfrak{g}, \mathfrak{h})$ and not an intrinsic property of M. There are M which can be described as G/H or G'/H' and $(\mathfrak{g}, \mathfrak{h})$ is reductive but $(\mathfrak{g}', \mathfrak{h}')$ is not.

If $(\mathfrak{g}, \mathfrak{h})$ is reductive, then if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ we say that $(\mathfrak{g}, \mathfrak{h})$ is symmetric. Reductive homogeneous spaces have a canonical invariant affine connection whose torsion vanishes in the symmetric case. If G/H is symmetric and has a G-invariant metric, the canonical connection is the Levi-Civita connection. Not all symmetric homogeneous spaces admit an invariant metric.

Let $M = G/H$ and let $o \in M$ be a point with $G_o = H$. Then for any $h \in H$ $h_* : T_oM \to T_oM$, and the chain rule says that T_oM is a representation of H: the linear isotropy representation.

One of the most useful theorems in this topic says that there is a one-to-one correspondence between G-invariant tensor fields on M and H-invariant tensors at $o \in M$. And if H is connected, this is is the same as $\mathfrak{h}$-invariant tensors at $o \in M$. In homogeneous geometry we can "localize" many calculations at $o \in M$.
It is often convenient to think of a homogeneous manifold G/H as the base of a principal H-bundle $H \to G \to G/H$. Then any representation of H gives rise to a homogeneous vector bundle associated to it. For example the homog. VB associated to the linear isotropy rep is TM. In the reductive case, the canonical connection acts naturally on sections of any homogeneous VBs and we may use it in order to define invariant PDEs. These PDEs localize at $o \in M$ to algebraic equations.

Some famous examples

Maximally symmetric riemannian manifolds:

symmetric spaces

- Euclidean space : $\mathfrak{g}$ = euclidean group $\underline{\mathfrak{so}}(D) \ltimes \mathbb{R}^D$, $\mathfrak{h} = \underline{\mathfrak{so}}(D)$
- Sphere : $\mathfrak{g} = \underline{\mathfrak{so}}(D+1)$, $\mathfrak{h} = \underline{\mathfrak{so}}(D)$
- Hyperbolic space : $\mathfrak{g} = \underline{\mathfrak{so}}(D,1)$, $\mathfrak{h} = \underline{\mathfrak{so}}(D)$

Maximally symmetric lorentzian manifolds:

symmetric spaces

- Minkowski spacetime : $\mathfrak{g}$ = poincaré $\underline{\mathfrak{so}}(D-1,1) \ltimes \mathbb{R}^D$, $\mathfrak{h} = \underline{\mathfrak{so}}(D-1,1)$
- de Sitter spacetime : $\mathfrak{g} = \underline{\mathfrak{so}}(D,1)$, $\mathfrak{h} = \underline{\mathfrak{so}}(D-1,1)$
- Anti de Sitter spacetime : $\mathfrak{g} = \underline{\mathfrak{so}}(D-1,2)$, $\mathfrak{h} = \underline{\mathfrak{so}}(D-1,1)$

The conformal group, with LA $\underline{\mathfrak{so}}(D,2)$, of Minkowski spacetime acts transitively on (the conformal compactification of) Minkowski spacetime. But as a homogeneous space of the conformal group, it is not reductive.

In the seminar we will see a classification of spatially isotropic homogeneous spacetimes which extend the above examples into the realm of non-riemannian/non-lorentzian geometry.