Parallelisable string backgrounds

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- hep-th/0305079

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- `hep-th/0305079`, *Class. Quant. Grav.* **20** (2003) 3327-3340

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- `hep-th/0305079`, *Class. Quant. Grav.* **20** (2003) 3327-3340
- `hep-th/0308141`, *JHEP* **10** (2003) 012, with KAWANO Teruhiko and YAMAGUCHI Satoshi

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In this talk: type II string backgrounds from supergravity.

Supergravity backgrounds

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- extremals of the action (in string frame)

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$$\int_M e^{-2\phi}\left(R + 4|d\phi|^2 - \tfrac{1}{2}|H|^2\right) \mathrm{dvol}_g$$

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If M is simply-connected, this implies that M is parallelisable.

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These geometries are easily characterised.

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[Chamseddine–FO–Sabra hep-th/0306278]

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[Cartan–Schouten (1926), Wolf (1970)]

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Only the first two have $dH = 0$.

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Summary

(M, g) simply-connected indecomposable lorentzian parallelisable is a Lie group with a bi-invariant metric, whence $dH = 0$
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Summary: Parallelisable geometries with closed torsion 3-form

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Summary: Parallelisable geometries with closed torsion 3-form are locally isometric to Lie groups with bi-invariant metrics.

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- since $\mathfrak{h}$ preserves the metric on $\mathfrak{g}$, there is a linear map

$$\mathfrak{h} \longrightarrow \mathfrak{so}(\mathfrak{g}) \cong \Lambda^2\mathfrak{g}$$

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$$\omega : \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{h}^*$$

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[See also FO–Stanciu `hep-th/9506152`]

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(Any semisimple factors in $\mathfrak{a}$ factor out of the double extension. [FO–Stanciu hep-th/9402035])

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[Cahen–Wallach (1970); FO-Papadopoulos hep-th/0105308]

- we will call them $\mathrm{CW}(J)$

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$\mathbb{R}^{1,0} \times S^3 \times S^3 \times S^3$

$\mathbb{R}^{1,2} \times S^7$

$\mathbb{R}^{1,6} \times S^3$

$\mathrm{CW}_{10}(J)$

$\mathrm{CW}_6(J) \times S^3 \times \mathbb{R}$

$\mathrm{CW}_4(J) \times S^3 \times S^3$

$\mathrm{CW}_4(J) \times \mathbb{R}^6$

$\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$

$\mathrm{AdS}_3 \times \mathbb{R}^7$

$\mathbb{R}^{1,1} \times \mathrm{SU}(3)$

$\mathbb{R}^{1,3} \times S^3 \times S^3$

$\mathbb{R}^{1,9}$

$\mathrm{CW}_8(J) \times \mathbb{R}^2$

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The case of linear dilaton was analysed by Kawano and Yamaguchi.

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Spacetime	Supersymmetry
$\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$	16
$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	16
$\mathrm{CW}_{10}(J)$	16,18(A),20,22(A),24(B),28(B)
$\mathrm{CW}_8(J) \times \mathbb{R}^2$	16,20
$\mathrm{CW}_6(J) \times \mathbb{R}^4$	16,24
$\mathrm{CW}_4(J) \times \mathbb{R}^6$	16
$\mathbb{R}^{1,9}$	32

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[Kawano–Yamaguchi hep-th/0306038]

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[Kawano–Yamaguchi hep-th/0306038]

All these backgrounds are exact string backgrounds

Turning on the linear dilaton all backgrounds are now half-BPS:

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Non-simply connected backgrounds are obtained by orbifolding.

Parallelisable heterotic backgrounds

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$$\int_M e^{-2\phi} \left(R + 4|d\phi|^2 - \tfrac{1}{2}|H|^2 - \tfrac{N}{2}|F|^2\right) \mathrm{dvol}_g$$

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Linear dilaton: all backgrounds are $\frac{1}{2}$-BPS, whereas for constant dilaton one has:

Spacetime	Supersymmetry
$\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$	8
$\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$	8
$\mathrm{CW}_{10}(J)$	8,10,12,14
$\mathrm{CW}_8(J) \times \mathbb{R}^2$	8,10
$\mathrm{CW}_6(J) \times \mathbb{R}^4$	8,12
$\mathrm{CW}_4(J) \times \mathbb{R}^6$	8
$\mathbb{R}^{1,9}$	16

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- $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4 \rightsquigarrow \mathrm{CW}_6(J) \times \mathbb{R}^4$, and $\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R} \rightsquigarrow \mathrm{CW}_8(J) \times \mathbb{R}^2$

Some geometric limits

Some of these backgrounds are related by geometric limits:

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- $\mathrm{CW}_{2n}(J) \rightsquigarrow \mathrm{CW}_{2n-2}(J') \times \mathbb{R}^2$ by allowing J to degenerate
- $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4 \rightsquigarrow \mathrm{CW}_6(J) \times \mathbb{R}^4$, and $\mathrm{AdS}_3 \times S^3 \times S^3 \times \mathbb{R} \rightsquigarrow \mathrm{CW}_8(J) \times \mathbb{R}^2$ by taking a Penrose limit

Thank you.