

Spinors and the Hopf fibrations

("Nachos" talk, 1/12/2019)

Recall that a finite-dimensional real vector space A is said to be an **algebra**, if there exists a bilinear operation $A \times A \rightarrow A$. A is said to be **unital**, if $\exists 1 \in A$ s.t. $1a = a1 = a \quad \forall a \in A$. $(a, b) \mapsto ab$ A is said to be **associative**, if $(ab)c = a(bc) \quad \forall a, b, c \in A$, and **commutative** if $ab = ba \quad \forall a, b \in A$. A is said to be a **division algebra** if $ab = 0 \Rightarrow a = 0$ or $b = 0$ (or both). An algebra A is said to be **normed** if $\exists$ a positive-definite symmetric bilinear form $\langle -, - \rangle : A \times A \rightarrow \mathbb{R}$ whose associated norm $\|a\| := \sqrt{\langle a, a \rangle}$ is multiplicative: $\|ab\| = \|a\| \|b\|$. In this talk we will concentrate on normed division algebras.

The ur-example is $\mathbb{R}$ itself, where $\|a\| = |a|$ the absolute value. $\mathbb{R}$ is associative, commutative ($ab = ba \quad \forall a, b \in \mathbb{R}$) and furthermore it is **ordered**: $\forall a < b$ then $a + c < b + c \quad \forall c$, and if $0 < a, 0 < b$ then $0 < ab$.

A celebrated theorem of Hurwitz's (published posthumously in 1923) says that any (finite-dim'l) real normed division algebra is isomorphic to one of four: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. One can construct $\mathbb{C}$ from $\mathbb{R}$, $\mathbb{H}$ from $\mathbb{C}$ and $\mathbb{O}$ from $\mathbb{H}$ via the Cayley-Dickson process. In so doing, one loses one property: $\mathbb{C}$ is not ordered, $\mathbb{H}$ is not commutative and $\mathbb{O}$ is not associative. (Applying the C-D process to $\mathbb{O}$ yields the "sedentions", which are not a division algebra.)

Closely related to the theorem of Hurwicz's, there is a theorem of Adams (1960, simplified considerably by Atiyah in 1966 using K-theory) the only pairs $(p, q) \in \mathbb{Z}^2$ ($p, q \geq 0$) for which a fibre bundle $S^p \rightarrow S^{p+q}$ exists are $(0, 1)$, $(1, 2)$, $(3, 4)$ and $(7, 8)$.

"Hopf-invariant" $\neq 1$

The "simplest" of these fibrations are the 'generalised' Hopf fibrations:

$$\begin{array}{ccc} S^0 & \rightarrow & S^1 \\ \cap & & \downarrow \\ \mathbb{R} & & S^1 \end{array}$$

$$\begin{array}{ccc} S^1 & \rightarrow & S^3 \\ \cap & & \downarrow \\ \mathbb{C} & & S^2 \end{array}$$

Hopf fibration (1931)

$$\begin{aligned} S^3 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \\ S^1 &= \{w \in \mathbb{C} \mid |w| = 1\} \end{aligned}$$

$$\begin{array}{ccc} S^3 & \rightarrow & S^7 \\ \cap & & \downarrow \\ \mathbb{H} & & S^4 \end{array}$$

$$\begin{array}{ccc} S^7 & \rightarrow & S^{15} \\ \cap & & \downarrow \\ \mathbb{O} & & S^8 \end{array}$$

$$S^3 \times S^1 \rightarrow S^3$$

$$((z_1, z_2), w) \mapsto (z_1 w, z_2 w)$$

free action with quotient $\mathbb{C}P^1 \cong S^2$.

It turns out that these Hopf fibrations are intimately linked to (lorentzian) spinors in dimensions 3, 4, 6 and 10. Curiously (also due to properties of spinors) these are precisely the dimensions on which we have simple supersymmetric Yang-Mills theories (i.e. with only gauge bosons and gauginos) and these are also the dimensions in which there exists a classical Green-Schwarz action for strings. This is also due to spinor identities. In today's talk, I would like to show how the Hopf fibration surjective submersions are essentially given by the "Dirac currents" which square spinors into vectors. This map plays a crucial role in the construction of Lie superalgebras out of spinor fields.

These fibrations all have "Hopf invariant" $\neq 1$: $S^n \rightarrow S^{2n-1}$
 $\downarrow \pi$
 S^n
 $\mu \in \Omega^n(S^n)$ $\pi^* \mu = d\omega$ $\exists \omega \in \Omega^{n-1}(S^{2n-1})$
 $\text{Hopf}(\pi) := \int_{S^{2n-1}} \omega \wedge d\omega$
 $\downarrow \pi$
 S^{2n-1}
 It is independent of ω & depends only on π up to homotopy and $\neq 1$ for the Hopf fibrations.

Real Hopf fibration

To get the ball rolling, we start with the simplest of the Hopf fibrations: $S^0 \xrightarrow{\mathbb{Z}^{\pm 1}} S^1 \rightarrow S^1$, which is associated with $\mathbb{R}$. We let $V := \mathbb{R}^{1,1} \oplus \mathbb{R}$ with inner product $\eta(v, v) = t^2 - x^2 - r^2$ for $v = (t, x, r) \in V$. Define $\gamma: V \rightarrow \mathbb{R}(2)$ by

$$\gamma(t, x, r) = \begin{pmatrix} r & x+t \\ x-t & -r \end{pmatrix}$$

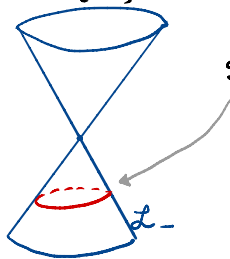
This is a "Clifford map": $\gamma(t, x, r)^2 = -\eta(v, v) \mathbb{1} \Rightarrow \gamma$ extends to a unique map $\mathcal{U}(V) \rightarrow \mathbb{R}(2)$.

Let $S = \mathbb{R}^2$ be the spinor representation. If $\epsilon_1, \epsilon_2 \in S$, we let $\langle \epsilon_1, \epsilon_2 \rangle := \epsilon_1^t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \epsilon_2$ denote a symplectic form on S . Then $\langle \gamma(v)\epsilon_1, \epsilon_2 \rangle = -\langle \epsilon_1, \gamma(v)\epsilon_2 \rangle$.

We define the **Dirac current** $K: S \rightarrow V$ by $\eta(K(\epsilon), v) = \langle \epsilon, \gamma(v)\epsilon \rangle$. If $\epsilon = \begin{pmatrix} a \\ b \end{pmatrix}$, then $K(\epsilon) = (-(a^2 + b^2), a^2 - b^2, -2ab)$. Notice that $K(\epsilon) \in \mathcal{L}_-$, the past-pointing light-cone.

Restricting K to the unit circle $S^1 = \{\epsilon \in S \mid \epsilon^t \epsilon = 1\} \subset S$, corresponding to $\epsilon = (\cos \theta, \sin \theta)$, we see $K(\epsilon) = (-1, \cos(2\theta), -\sin(2\theta))$ so that $K(\theta) = K(\theta + \pi)$, or $K(\epsilon) = K(-\epsilon)$. So we see

that $S^0 = \{\pm 1\} \rightarrow S^1(S)$
 $\downarrow K$
 $S^1(\mathcal{L}_-)$



The complex Hopf fibration

$$V = \mathbb{C} \oplus \mathbb{R}^{1,1} \ni (z, x, t) =: v, \quad \eta(v, v) = |z|^2 + x^2 - t^2$$

Define $\gamma: V \rightarrow \mathbb{C}(4)$ by

$$\gamma(z, x, t) := \left(\begin{array}{cc|cc} t & z & x & 0 \\ -\bar{z} & -t & 0 & x \\ \hline -x & 0 & -t & -z \\ 0 & -x & \bar{z} & t \end{array} \right) \Rightarrow \gamma(z, x, t)^2 = -(|z|^2 + x^2 - t^2) \mathbb{1}$$

$$\Rightarrow \gamma: \mathcal{Q}(V) \rightarrow \mathbb{C}(4)$$

Let $\text{vol} = \begin{pmatrix} 0 & i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}$, $\text{vol}^2 = -\mathbb{1}$ and let $S_{\pm} \subset S = \mathbb{C}^4$ denote the $(\pm i)$ -eigenspaces of vol .

On S we define the pseudo-hermitian inner product $\langle E_1, E_2 \rangle := E_1^\dagger \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E_2$, $I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$S_{\pm}$ are maximally isotropic. It follows that $\langle \gamma(v)E_1, E_2 \rangle = \langle E_1, \gamma(v)E_2 \rangle$.

We define $K: S_+ \rightarrow V$ by $\eta(K(E), v) = \langle E, \gamma(v)E \rangle$. Explicitly, if $E = \frac{1}{\sqrt{2}}(a, b, a, b) \in S_+$ then $K(E) = (2a\bar{b}, |a|^2 - |b|^2, -(|a|^2 + |b|^2))$, which belongs to $\mathcal{L}^- \subset V$. We may restrict K to

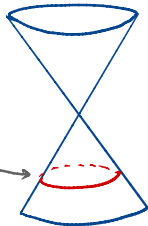
$S^3(S_+) = \{ E = \frac{1}{\sqrt{2}}(a, b, a, b) \mid |a|^2 + |b|^2 = 1 \}$ so that $K(E) = (2a\bar{b}, |a|^2 - |b|^2, -1) \in S^2(\mathcal{L}^-)$. What is the fibre?

$SL(2, \mathbb{C})$ acts transitively on $S^2(\mathcal{L}^-)$ and K is equivariant under the action of $\text{Spin}(3, 1) \cong SL(2, \mathbb{C})$. So we need only check the fibre at some point, e.g., $(0, 1, -1)$ and the fibre $K^{-1}(0, 1, -1) = \{ \frac{1}{\sqrt{2}}(a, 0, a, 0) \mid |a| = 1 \} \cong S^1$. So we get the complex Hopf fibration

$$S^1 \rightarrow S^3(S_+)$$

$$\downarrow \kappa$$

$$S^2(\mathcal{L}^-)$$



The quaternionic Hopf fibration

$$V = \mathbb{H} \oplus \mathbb{R}^{1,1} \ni v = (q, x, t) \quad , \quad \eta(v, v) = |q|^2 + x^2 - t^2$$

$$\gamma: V \rightarrow \mathbb{H}(4)$$

$$\gamma(q, x, t) = \left(\begin{array}{cc|cc} t & q & x & 0 \\ -\bar{q} & -t & 0 & x \\ \hline -x & 0 & -t & -\bar{q} \\ 0 & -x & \bar{q} & t \end{array} \right)$$

$$\Rightarrow \gamma(v)^2 = -\eta(v, v) \Rightarrow \gamma: \mathcal{U}(V) \rightarrow \mathbb{H}(4)$$

$$\text{vol} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad S_{\pm} \subset S = \mathbb{H}^4 \quad \text{the } \pm 1\text{-eigenspaces of vol.}$$

$$\mathbb{H}\text{-pseudo-hermitian inner product on } S: \quad \langle E_1, E_2 \rangle = E_1^T J E_2 \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

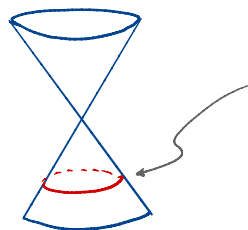
$$S_{\pm} \text{ are maximally isotropic and } \langle \gamma(v) E_1, E_2 \rangle = \langle E_1, \gamma(v) E_2 \rangle$$

$$\text{Let } K: S_+ \rightarrow V \text{ be defined by } \eta(K(E), v) = \langle E, \gamma(v) E \rangle. \quad \text{If } E = \frac{1}{\sqrt{2}}(a, b, a, b) \quad (a, b \in \mathbb{H})$$

$$K(E) = (2a\bar{b}, |a|^2 - |b|^2, -(|a|^2 + |b|^2)) \text{ and, as before, } K(E) \in \mathcal{L}_-.$$

$$S^7(S) = \left\{ \frac{1}{\sqrt{2}}(a, b, a, b) \mid |a|^2 + |b|^2 = 1 \right\}, \quad K(E) \in S^4(\mathcal{L}_-). \quad \text{The fibres are } \{(au, bu) \mid u \in \mathbb{S}p(1)\} \cong S^3.$$

$\therefore$ we get a fibration

$$\begin{array}{ccc} S^3 & \longrightarrow & S^7(S_+) \\ & & \downarrow \kappa \\ & & S^4(\mathcal{L}_-) \end{array}$$


The octonionic Hopf fibration

$$V = \mathbb{O} \oplus \mathbb{R}^{1,1} \ni v := (o, x, t) \quad \eta(v, v) = |o|^2 + x^2 - t^2 \quad \gamma: V \rightarrow \text{End}_{\mathbb{R}}(\mathbb{O}^4)$$

$$\gamma: \mathcal{U}(V) \rightarrow \text{End}_{\mathbb{R}}(\mathbb{O}^4) \Leftarrow \gamma(v)^2 = -\eta(v, v) \mathbb{1} \quad \Leftarrow \quad \gamma(o, x, t) := \left(\begin{array}{cc|cc} t & CR_0 & x & 0 \\ -CL_0 & -t & 0 & x \\ \hline -x & 0 & -t & -CR_0 \\ 0 & -x & CL_0 & t \end{array} \right)$$

On $S = \mathbb{O}^4$ we define a real symplectic structure

$$\langle e_1, e_2 \rangle = \text{Re}(e_1^\dagger J e_2)$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then, $\langle \gamma(v)e_1, e_2 \rangle = -\langle e_1, \gamma(v)e_2 \rangle$.

$$\text{vol} = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \text{vol}^2 = +\mathbb{1} \quad S_{\pm} \subset S \quad (\pm 1)\text{-eigenspaces} \quad S_- \cong \mathbb{O}^2 \quad e = \frac{1}{\sqrt{2}}(a, b, a, b) \quad (a, b \in \mathbb{O})$$

$$\text{Define } K: S_- \rightarrow V \text{ by } \eta(K(e), v) = \langle e, \gamma(v)e \rangle \Rightarrow K(e) = (-2 \overline{ab}, -(1a^2 - 1b^2), -(1a^2 + 1b^2)) \\ \Rightarrow K(e) \in \mathcal{L}_-$$

Restricting to $S^{15}(S_-) = \{ \frac{1}{\sqrt{2}}(a, b, a, b) \in \mathbb{O}^4 \mid |a|^2 + |b|^2 = 1 \}$, $K(e) \in S^8(\mathcal{L}_-)$.

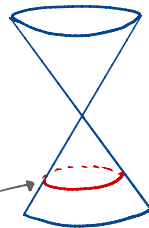
$\text{Spin}(9)$ preserves $S^8(\mathcal{L}_-)$ and acts transitively & K is equivariant, so enough to check one fibre: e.g., $K^{-1}(0, -1, -1) = \{ \frac{1}{\sqrt{2}}(a, 0, a, 0) \mid |a| = 1 \} \cong S^7$.

Therefore we get a Hopf fibration

$$S^7 \rightarrow S^{15}(S_-)$$

$$\downarrow K$$

$$S^8(\mathcal{L}_-)$$



To be honest, we still need to show that these fibrations are the Hopf fibrations. This follows from the fact that $\kappa: \mathbb{P}^1(S_1) \rightarrow \mathbb{P}^1(\mathcal{L})$ has degree 1, which can be shown.

Summary

Λ	$\gamma(v)$ (signature)	$\mathcal{U}(V, \eta)$	S	$\langle, \rangle$ (sign of γ)	$\mathcal{E}$	$\kappa(\mathcal{E})$
$\mathbb{R}$	$\begin{pmatrix} r & x+t \\ x-t & r \end{pmatrix}$ (1,2)	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}^2$	$\mathbb{R}$ -symplectic (-)	$\begin{pmatrix} a \\ b \end{pmatrix}$	$(-2ab, a^2-b^2, -(a^2+b^2))$
$\mathbb{C}$	$\begin{pmatrix} t & z & x & 0 \\ -\bar{z} & -t & 0 & x \\ -x & 0 & -t & -\bar{z} \\ 0 & -x & \bar{z} & t \end{pmatrix}$ (3,1)	$\mathbb{H}(2)$	$S_+ \subset \mathbb{C}^4$	$\mathbb{C}$ -hermitian (+)	$\frac{1}{\sqrt{2}} \begin{pmatrix} a \\ a \\ b \\ b \end{pmatrix}$	$(2a\bar{b}, a ^2- b ^2, -(a ^2+ b ^2))$
$\mathbb{H}$	$\begin{pmatrix} t & q & x & 0 \\ -\bar{q} & -t & 0 & x \\ -x & 0 & -t & -q \\ 0 & -x & \bar{q} & t \end{pmatrix}$	$\mathbb{H}(4)$	$S_+ \subset \mathbb{H}^4$	$\mathbb{H}$ -hermitian (+)	$\frac{1}{\sqrt{2}} \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix}$	$(2a\bar{b}, a ^2- b ^2, -(a ^2+ b ^2))$
$\mathbb{O}$	$\begin{pmatrix} t & CR_0 & x & 0 \\ -Cl_0 & -t & 0 & x \\ -x & 0 & -t & -CR_0 \\ 0 & -x & Cl_0 & t \end{pmatrix}$	$\text{End}_{\mathbb{R}}(\mathbb{O}^4)$	$S_- \subset \mathbb{O}^4$	$\mathbb{R}$ -symplectic (-)	$\frac{1}{\sqrt{2}} \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}$	$(-2a\bar{b}, - a ^2+ b ^2, -(a ^2+ b ^2))$