Classification results on supergravity vacua

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Based on work in collaboration with

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- George Papadopoulos (King's College, London)
  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170`

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  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170`
- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
  - ⋆ `in preparation`

A geometric motivation

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These are in one-to-one correspondence with parallel sections of the bundle

$$\mathcal{E}(M) = TM \oplus \Lambda^2 T^* M$$

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Only the spherical case is solved (culminating in the work of Wolf in the 1970s), but there are many partial results in other cases.

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In other words, we will classify vacua up to local isometry.

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA | IIB			I			
9	$N = 2$			$N = 1$			
8	$N = 2$			$N = 1$			
7	$N = 4$			$N = 2$			
6	$(2, 2)$ | $(3, 1)$ | $(4, 0)$	$(2, 1)$ | $(3, 0)$		$(1, 1)$ | $(2, 0)$		$(1, 0)$	
5	$N = 8$	$N = 6$		$N = 4$		$N = 2$	
4	$N = 8$	$N = 6$	$N = 5$	$N = 4$	$N = 3$	$N = 2$	$N = 1$

[Van Proeyen, hep-th/0301005]

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- investigate their possible Kaluza–Klein reductions.

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defined by the supersymmetric variation of the gravitino:

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where I is an index labeling the following elements

$$\Gamma_a \quad \Gamma_{ab} \quad \Gamma_{abc} \quad \Gamma_{abcd} \quad \Gamma_{abcde}$$

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$$g = 2dx^+dx^- - \tfrac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2\right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

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All vacua embed isometrically in $\mathbb{E}^{2,11}$ as the intersections of two quadrics.

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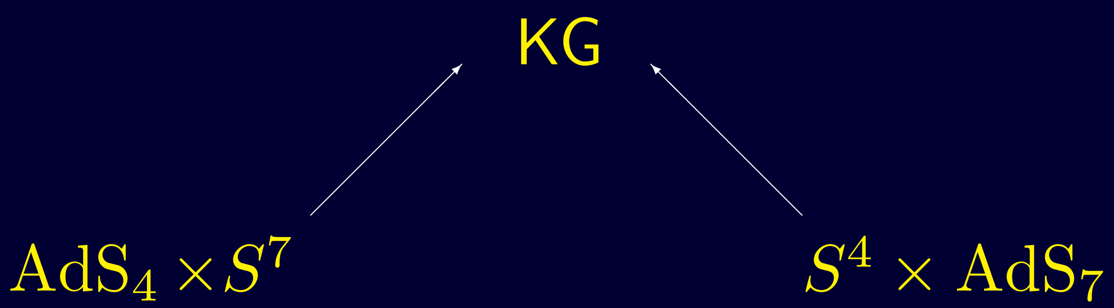

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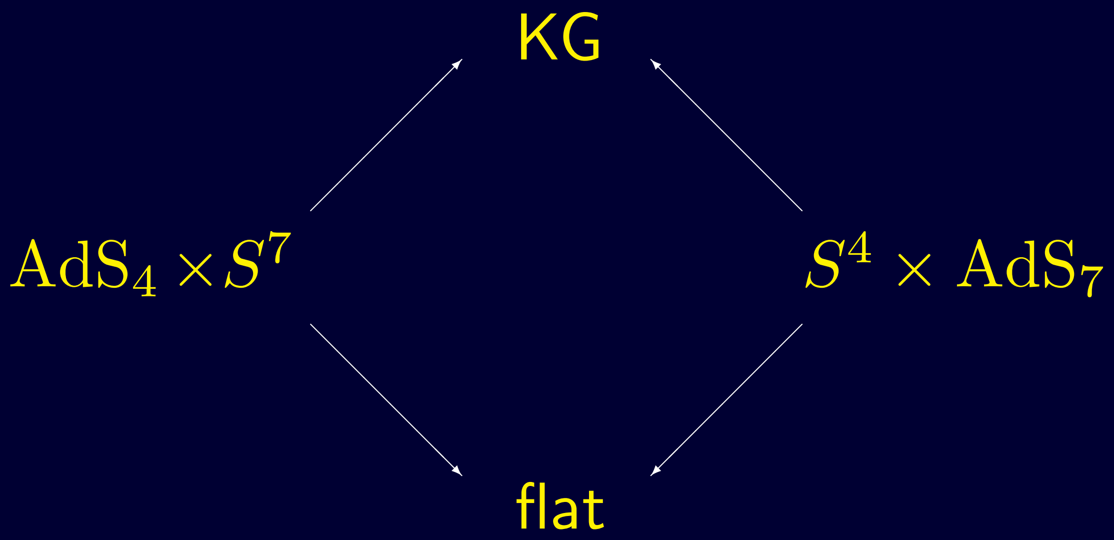

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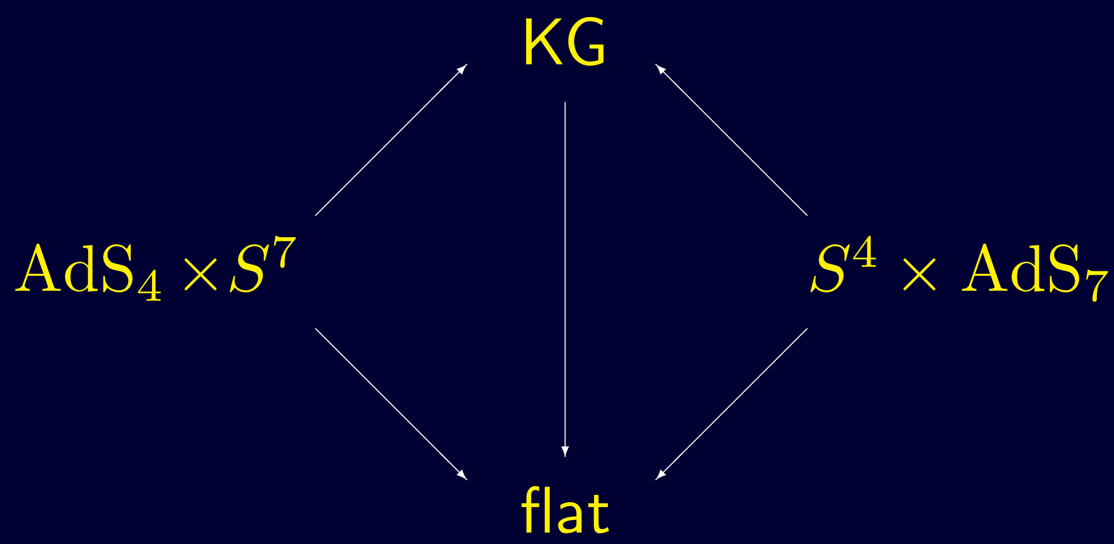

[Back]

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Maximal supersymmetry $\Longrightarrow$ D is flat.

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The solution to this problem is known.

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But there is a more general construction.

The double extension

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- since $\mathfrak{h}$ preserves the metric on $\mathfrak{g}$, there is a linear map

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- $\mathfrak{h}$ acts on $\mathfrak{g} \times_\omega \mathfrak{h}^*$ preserving the Lie bracket, so we can form the *double extension*

$$\mathfrak{d}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{h} \ltimes (\mathfrak{g} \times_\omega \mathfrak{h}^*)$$

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This construction is due to Medina and Revoy who proved an important structure theorem.

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Every metric Lie algebra is obtained as an orthogonal direct sum of indecomposables.

[See also FO–Stanciu hep-th/9506152]

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(Any semisimple factors in $\mathfrak{a}$ factor out of the double extension. [FO–Stanciu hep-th/9402035])

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- $\mathfrak{d}(\mathbb{R}^4, \mathbb{R})$, actually a family of Lie algebras parametrised by homomorphisms

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The third case is a six-dimensional version of the Nappi-Witten spacetime, NW_6, discovered by Meessen. [Meessen `hep-th/0111031`]

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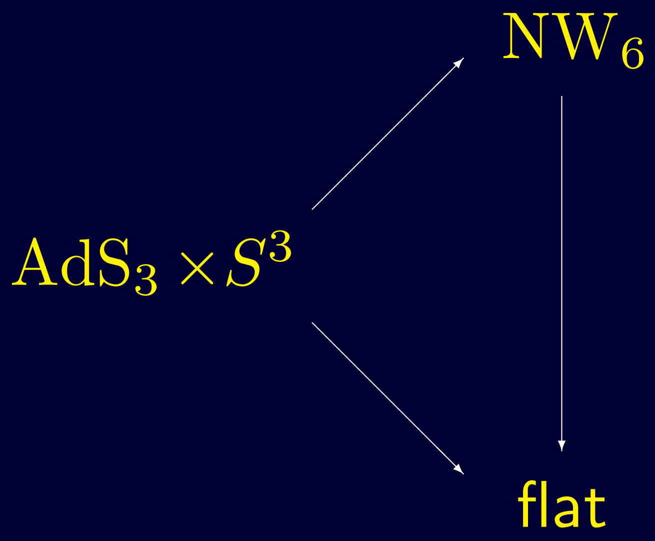

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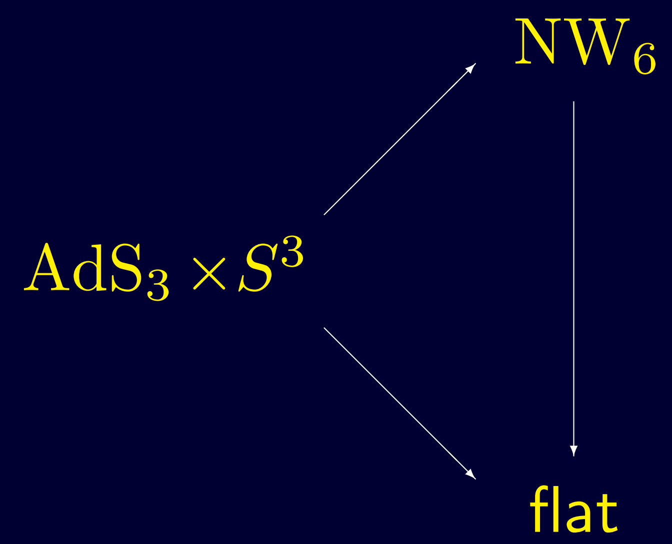

which in this case are *group contractions* à la Inönü–Wigner.

[Stanciu–FO `hep-th/0303212`]

[Back]

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(Unfortunate notation: a 2-Lie algebra is a Lie algebra.)

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(n-Lie algebras also appear naturally in the context of Nambu dynamics. [Nambu (1973)])

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[FO–Papadopoulos `math.AG/0211170`]

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[Blau–FO–Hull–Papadopoulos `hep-th/0110242`]

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Notice that g is a bi-invariant metric on a Lie group: a ten-dimensional version of the Nappi–Witten spacetime.

[Stanciu–FO `hep-th/0303212`]

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BFHP

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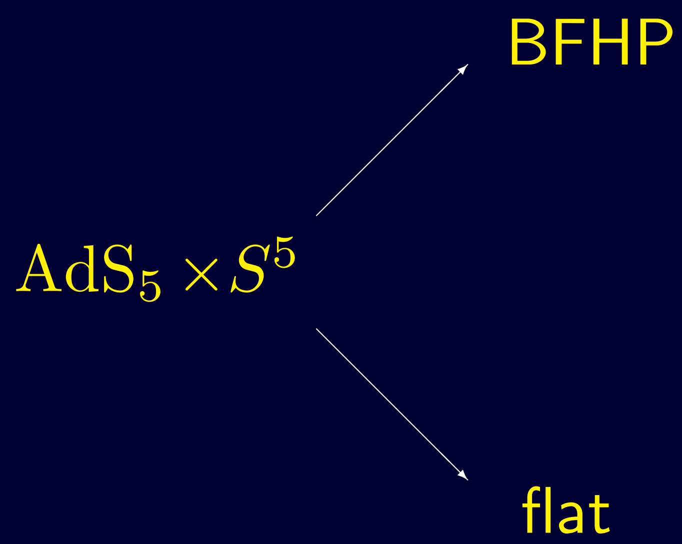

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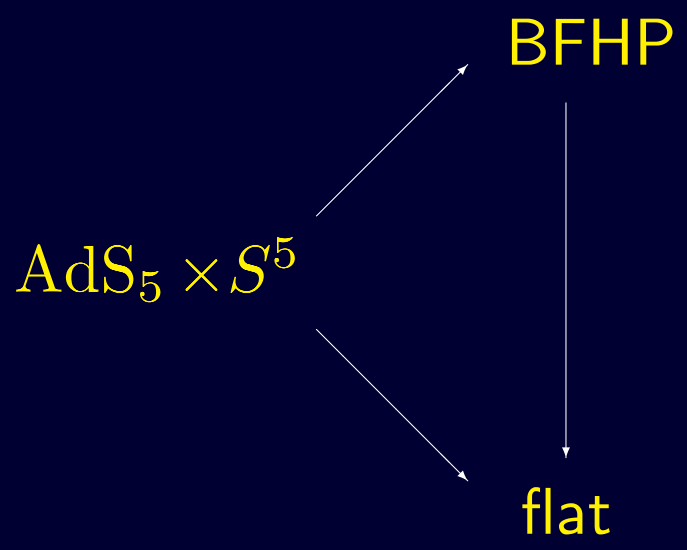

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[Gauntlett–Gutowsky–Hull–Pakis–Reall hep-th/0209114]

[Lozano-Tellechea–Meessen–Ortín hep-th/0206200]

Thank you.