Gauging WZ terms in σ-models

The nonlinear σ model describes harmonic maps

$$\varphi : \Sigma \longrightarrow M$$

where (Σ, h) & (M, g) are pseudoriemannian and Σ oriented. The action functional is given by

$$S_\sigma[\varphi] = \frac{1}{2}\int_\Sigma g_{ij}(\varphi)\, d\varphi^i \wedge * d\varphi^j$$

↑ Hodge rel. h

If G is a connected Lie group acting by isometries on (M, g). Then it acts on maps $\Sigma \to M$ leaving S_σ invariant. It is enough to check infinitesimally.

So let $X \in \mathfrak{g}$, $\xi_X \in \mathfrak{X}(M)$ Killing vector. Then under

$$\delta_X \varphi^i = \xi_X^i(\varphi)$$

the action changes by 0 by Killing's equation

$$\delta_X S_\sigma[\varphi] = \frac{1}{2}\int_\Sigma (\mathcal{L}_{\xi_X} g)_{ij}\, d\varphi^i \wedge * d\varphi^j$$

Gauging the σ model means coupling it to gauge fields in such a way that $S_\sigma[g, A]$ is invariant under local transformations in $\mathrm{Map}(\Sigma, \mathfrak{g})$.
Let (e_a) be a basis for $\mathfrak{g}$ and $\xi_a := \xi_{e_a}$ the KV.
Let $\lambda : \Sigma \to \mathfrak{g}$ $\lambda = \lambda^a e_a$, $\lambda^a \in C^\infty(\Sigma)$ and define

$$\delta_\lambda \varphi^i = \lambda^a \xi_a^i(\varphi) \quad \text{\& } S_\sigma \text{ no longer invariant.}$$

So introduce $A \in \Omega^1(\Sigma; \mathfrak{g})$ and $\delta_\lambda A = d\lambda + [A, \lambda]$.

Define
$$\nabla\varphi^i = d\varphi^i - A^a \xi_a^i(\varphi)$$
"minimal coupling"

and
$$S_G[\varphi] = \frac{1}{2}\int_\Sigma g_{ij}(\varphi)\, \nabla\varphi^i \wedge * \nabla\varphi^j$$

Let $\dim\Sigma = d$ and let $\omega \in \Omega^{d+1}(M)$ be a closed form. Let B be a $(d+1)$-dim'l manifold with $\partial B = \Sigma$ and extend φ to B. We can add
$$S_{WZ}[\varphi] = \int_B \varphi^* \omega$$

Because $d\omega = 0$, the variation of this term is an exact $(d+1)$-form & hence by Stokes's it is an integral over Σ and hence the field equations do not depend on the extension.

Suppose ω is G-invariant. Hence so is S_{WZ}. Can one gauge this? We could try minimal coupling: that produces a gauge-invariant action but the resulting field equations are no longer local because the minimally coupled $\varphi^*\omega$ need not be closed.

eg: if $\omega = d\theta$ and θ is G-invariant, then we can minimally couple: $\int_B \varphi^*\omega = \int_\Sigma \varphi^*\theta$.

Answer: S_{WZ} can be gauged if ω admits an equivariant closed extension.

ie: $\exists\ \hat\omega = \omega + F^a \psi_a + F^a F^b \psi_{ab} + \cdots$ $\quad \exists\ \psi_a \in \Omega^{d-1}(M)$

such that $\hat\omega$ is (1) $\mathfrak{g}$-invariant $\quad \psi_{ab} \in \Omega^{d-3}(M)$

(2) $d_c \hat\omega = 0$ $\quad$ etc...

where $d_c(F^a) = 0$ and $d_c(\omega) = \omega - F^a \iota_a \omega$.

Examples

1. 1d σ-model with symplectic target

$G \curvearrowright (M,\omega)$ symplectically is a symmetry of

$$S_{WZ} = \int_{B^2} \varphi^* \omega$$

and it can be gauge if and only if the G-action is hamiltonian, so there exists an equivariant moment map $\mu: M \to \mathfrak{g}^*$.

2. 2d WZW model

M is a Lie group w/ bi-invariant metric and Lie algebra $\mathfrak{m}$ with ad-invariant inner product $\langle , \rangle$. Then

LI MC 1-form

$$\omega = \frac{1}{6} \langle \theta_L, [\theta_L, \theta_L] \rangle$$

Let $G \subset M \times M$ be a subgroup of isometries. It preserves ω and hence S_{WZ}. G can be gauged if and only if $\ell^* \langle , \rangle = r^* \langle , \rangle$

where $(\ell, r): \mathfrak{g} \to \mathfrak{m} \oplus \mathfrak{m}$ are the LA homomorphisms induced by the embedding $G \hookrightarrow M \times M$.

eg: diagonal subgroups can always be gauged.
(e.g. coset construction in CFT)

chiral subgroups for which $\mathfrak{g} \subset \mathfrak{m}$ is isotropic
(e.g. Drinfeld-Sokolov reduction)

③ σ-models with Lie group targets & transgressive WZ terms

Again M is a Lie group with Lie algebra $\mathfrak{m}$.

Let $P \in (\odot^n \mathfrak{m}^*)^{\mathfrak{m}}$ be an invariant polynomial.

Let $\omega := P(\theta_L, \underbrace{d\theta_L, \dots, d\theta_L}_{n-1 \text{ times}}) \in \Omega^{2n-1}(M)$

Then ω is invariant under $M \times M$.

Let $G \subset M \times M$ and $\mathfrak{g} \subset \mathfrak{m} \oplus \mathfrak{m}$ with inclusion homomorphisms $\ell \oplus r$.

Conjecture The G symmetry of S_{WZ} can be gauged if and only if $\ell^* P = r^* P$.