Exceptional spheres and supergravity

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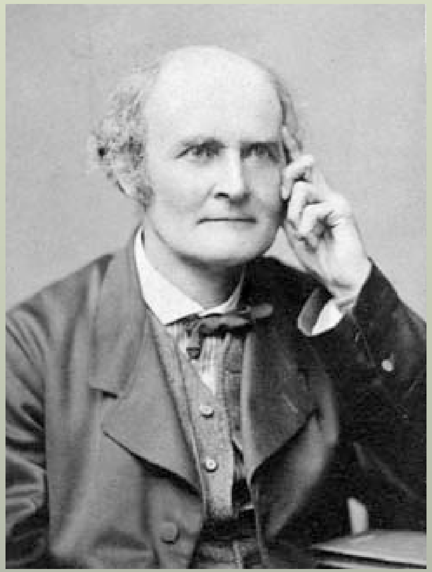
Introduction

Hamilton



Cayley

Lie

Killing

É. Cartan

Hurwitz

Hopf

J.F. Adams

This talk is about a relation between **exceptional** objects:

- **Hopf bundles**
- exceptional **Lie algebras**

using a **geometric** construction familiar from **supergravity**: the **Killing (super)algebra**.

Real division algebras

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\geq$			
$ab = ba$	$ab = ba$	$ab \neq ba$	
$(ab)c = a(bc)$	$(ab)c = a(bc)$	$(ab)c = a(bc)$	$(ab)c \neq a(bc)$

These are all the euclidean normed real division algebras. [Hurwitz]

Hopf fibrations

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array} \qquad \begin{array}{ccc} S^3 & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array} \qquad \begin{array}{ccc} S^7 & \longrightarrow & S^{15} \\ & & \downarrow \\ & & S^8 \end{array}$$

$S^1 \subset \mathbb{C}$	$S^3 \subset \mathbb{H}$	$S^7 \subset \mathbb{O}$
$S^3 \subset \mathbb{C}^2$	$S^7 \subset \mathbb{H}^2$	$S^{15} \subset \mathbb{O}^2$
$S^2 \cong \mathbb{CP}_1$	$S^4 \cong \mathbb{HP}_1$	$S^8 \cong \mathbb{OP}_1$

These are the only examples of fibre bundles where all three spaces are spheres. [Adams]

Simple Lie algebras

(over $\mathbb{C}$)

4 classical series:

$A_{n\geq 1}$	$SU(n+1)$
$B_{n\geq 2}$	$SO(2n+1)$
$C_{n\geq 3}$	$Sp(n)$
$D_{n\geq 4}$	$SO(2n)$

[Lie]

5 exceptions:

G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

[Killing, Cartan]

Mathematical hype?

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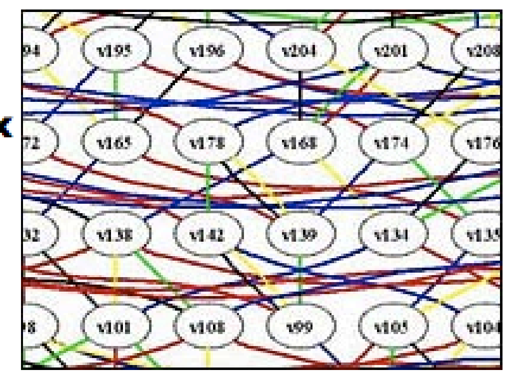
248-dimension maths puzzle solved

An international team of mathematicians has detailed a vast complex numerical "structure" which was described more than a century ago.



The structure is described in the form of a vast matrix

Mapping the 248-dimensional structure, called E8, took four years of work and produced more data than the Human Genome Project, researchers said.

E8 is a "Lie group", a means of describing symmetrical objects.

The team said their findings may assist fields of physics which use more than four dimensions, such as string theory.

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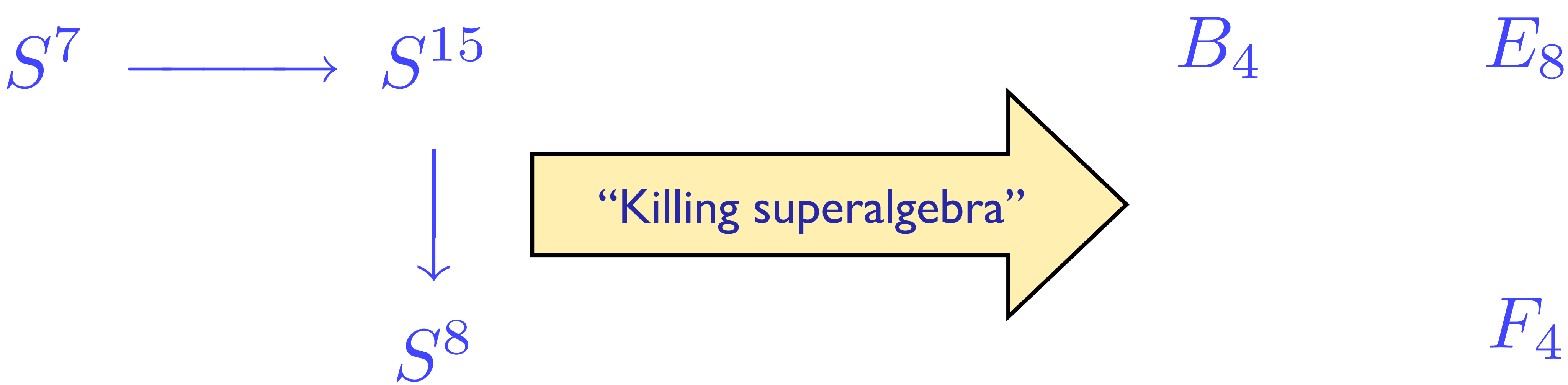
Supergravity

A supergravity background consists of a **lorentzian spin manifold** with additional geometric data, together with a notion of **Killing spinor**.

These spinors together with the infinitesimal automorphisms of the geometry generate the **Killing superalgebra**.

This is a **useful invariant** of the background.

Applying the Killing superalgebra construction to the **exceptional Hopf fibration**, one obtains a triple of **exceptional Lie algebras**:



Spinors

plane of numbers.

Rules of Multiplication in an Algebra of n units.

In general, if we consider an algebra of n units, $\iota_1, \iota_2, \ldots \iota_n$, such th
$\iota_r^2 = -1$, $\iota_r \iota_s = -\iota_s \iota_r$, a product of m linear factors will contain terms whi
are all of even order if m is even, and all of odd order if m is odd; for t

plane of numbers.

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Clifford

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Clifford algebras

V^n $\langle -, - \rangle$ real euclidean vector space

$$C\ell(V) = \frac{\bigotimes V}{\langle \boldsymbol{v} \otimes \boldsymbol{v} + |\boldsymbol{v}|^2 \mathbf{1} \rangle}$$ filtered associative algebra

$C\ell(V) \cong \Lambda V$ (as vector spaces)

$$C\ell(V) = C\ell(V)_0 \oplus C\ell(V)_1$$

$C\ell(V)_0 \cong \Lambda^{\mathrm{even}} V$ $C\ell(V)_1 \cong \Lambda^{\mathrm{odd}} V$

orthonormal frame $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$

$$\boldsymbol{e}_i \boldsymbol{e}_j + \boldsymbol{e}_j \boldsymbol{e}_i = -2\delta_{ij}\mathbf{1}$$

$$C\ell\left(\mathbb{R}^n\right) =: C\ell_n$$

Examples:

$$C\ell_0 = \langle \mathbf{1} \rangle \cong \mathbb{R}$$

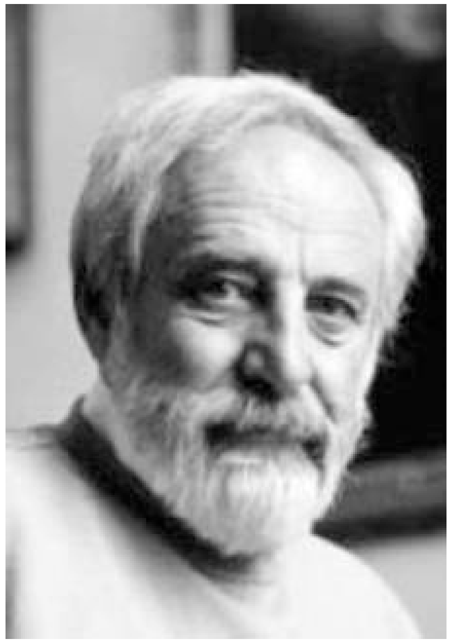
$$C\ell_1 = \left\langle \mathbf{1}, \boldsymbol{e}_1 \middle| \boldsymbol{e}_1^2 = -\mathbf{1} \right\rangle \cong \mathbb{C}$$

$$C\ell_2 = \left\langle \mathbf{1}, \boldsymbol{e}_1, \boldsymbol{e}_2 \middle| \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = -\mathbf{1}, \boldsymbol{e}_1 \boldsymbol{e}_2 = -\boldsymbol{e}_2 \boldsymbol{e}_1 \right\rangle \cong \mathbb{H}$$

Classification

n	$C\ell_n$
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$

Bott periodicity:



$$C\ell_{n+8} \cong C\ell_n \otimes \mathbb{R}(16)$$

e.g.,

$$C\ell_9 \cong \mathbb{C}(16)$$

$$C\ell_{16} \cong \mathbb{R}(256)$$

From this table one can read the type and dimension of the irreducible representations.

$C\ell_n$ has a **unique** irreducible representation if n is even and **two** if n is odd.

They are distinguished by the action of

$$\boldsymbol{e}_1 \boldsymbol{e}_2 \cdots \boldsymbol{e}_n$$

which is **central** for n odd.

Notation : $\mathfrak{M}_n$ or $\mathfrak{M}_n^{\pm}$ **Clifford modules**

$$\dim \mathfrak{M}_n = 2^{\lfloor n/2 \rfloor}$$

Spinor representatinos

$$\mathfrak{so}_n \to C\ell_n$$

$$\boldsymbol{e}_i \wedge \boldsymbol{e}_j \mapsto -\tfrac{1}{2}\boldsymbol{e}_i\boldsymbol{e}_j$$

exp

$$\mathrm{Spin}_n \subset C\ell_n$$

$$s \in \mathrm{Spin}_n, \quad \boldsymbol{v} \in \mathbb{R}^n \quad \Longrightarrow \quad s\boldsymbol{v}s^{-1} \in \mathbb{R}^n$$

which defines a 2-to-1 map $\mathrm{Spin}_n \to \mathrm{SO}_n$

with archetypical example $\mathrm{Spin}_3 \cong \mathrm{SU}_2 \subset \mathbb{H}$

$$\big\downarrow \ \textbf{\textit{2-1}}$$

$$\mathrm{SO}_3 \cong \mathrm{SO}(\mathrm{Im}\mathbb{H})$$

By restriction, every representation of $C\ell_n$ defines a representation of Spin_n :

$$C\ell_n \supset \mathrm{Spin}_n$$

$\mathfrak{M} = \Delta = \Delta_+ \oplus \Delta_-$	$\Delta_\pm$	**chiral spinors**
$\mathfrak{M}^\pm = \Delta$	Δ	**spinors**

One can read off the type of representation from

$$\mathrm{Spin}_n \subset (C\ell_n)_0 \cong C\ell_{n-1}$$

$$\dim \Delta = 2^{(n-1)/2} \qquad \dim \Delta_\pm = 2^{(n-2)/2}$$

Spinor inner product

$(-,-)$ bilinear form on Δ

$$(\varepsilon_1, \varepsilon_2) = \overline{(\varepsilon_2, \varepsilon_1)}$$

$$(\varepsilon_1, \boldsymbol{e}_i \cdot \varepsilon_2) = -(\boldsymbol{e}_i \cdot \varepsilon_1, \varepsilon_2) \qquad \forall \varepsilon_i \in \Delta$$

$$\implies (\varepsilon_1, \boldsymbol{e}_i \boldsymbol{e}_j \cdot \varepsilon_2) = -(\boldsymbol{e}_i \boldsymbol{e}_j \cdot \varepsilon_1, \varepsilon_2)$$

which allows us to define $[-,-] : \Lambda^2 \Delta \to \mathbb{R}^n$

$$\langle [\varepsilon_1, \varepsilon_2], \boldsymbol{e}_i \rangle = (\varepsilon_1, \boldsymbol{e}_i \cdot \varepsilon_2)$$

Spin geometry

Spin manifolds

M^n differentiable manifold, orientable, spin

g riemannian metric

$$
\begin{array}{ccccccc}
\mathrm{GL}(M) & \longleftarrow & \mathrm{O}(M) & \xleftarrow{w_1 = 0} & \mathrm{SO}(M) & \xleftarrow{w_2 = 0} & \mathrm{Spin}(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & & M & & M & & M
\end{array}
$$

$$
\mathrm{GL}_n \hookleftarrow \mathrm{O}_n \hookleftarrow \mathrm{SO}_n \twoheadleftarrow \mathrm{Spin}_n
$$

Possible $\mathrm{Spin}(M)$ are classified by $H^1(M;\mathbb{Z}/2)$.

e.g., $M = S^n \subset \mathbb{R}^{n+1}$

$$\mathrm{O}(M) = \mathrm{O}_{n+1}$$
$$\mathrm{SO}(M) = \mathrm{SO}_{n+1}$$
$$\mathrm{Spin}(M) = \mathrm{Spin}_{n+1}$$

$$S^n \cong \mathrm{O}_{n+1}/\mathrm{O}_n \cong \mathrm{SO}_{n+1}/\mathrm{SO}_n \cong \mathrm{Spin}_{n+1}/\mathrm{Spin}_n$$

$\pi_1(M) = \{1\} \implies$ unique spin structure

Spinor bundles

$$\begin{array}{c} C\ell(TM) \\ \downarrow \\ M \end{array}$$

Clifford bundle

$$C\ell(TM) \cong \Lambda TM$$

$$S(M) := \mathrm{Spin}(M) \times_{\mathrm{Spin}_n} \Delta$$

$$S(M)_\pm := \mathrm{Spin}(M) \times_{\mathrm{Spin}_n} \Delta_\pm$$

(chiral) spinor bundles

We will assume that $C\ell(TM)$ acts on $S(M)$

The Levi-Cività connection allows us to differentiate spinors

$$\nabla : S(M) \to T^*M \otimes S(M)$$

which in turn allows us to define

parallel spinor $\nabla \varepsilon = 0$

Killing spinor $\nabla_X \varepsilon = \lambda X \cdot \varepsilon$

Killing constant → λ

If (M,g) admits ...

parallel spinors $\Rightarrow$ (M,g) is **Ricci-flat**

Killing spinors $\Rightarrow$ (M,g) is **Einstein**

$$R = 4\lambda^2 n(n-1)$$

$$\Longrightarrow \qquad \lambda \in \mathbb{R} \cup i\mathbb{R}$$

Today we only consider **real** λ.

Killing spinors have their origin in **supergravity**.

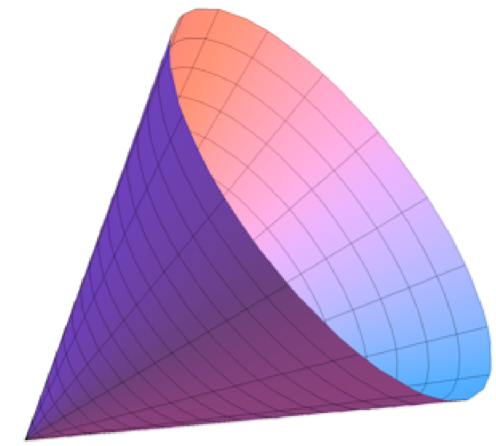
The name stems from the fact that they are "**square roots**" of Killing vectors.

$$\left(V \in \Gamma(TM) \quad \text{is **Killing** if} \quad \mathcal{L}_V g = 0 \right)$$

$\varepsilon_1, \varepsilon_2$ Killing ⇒ $[\varepsilon_1, \varepsilon_2]$ Killing

Which manifolds admit Killing spinors?

Ch. Bär



(M, g)

$(\overline{M}, \overline{g})$ **metric cone**

$\overline{M} = \mathbb{R}^+ \times M$ $\overline{g} = dr^2 + r^2 g$

Killing spinors in (M, g)

$(\lambda = \pm\frac{1}{2})$

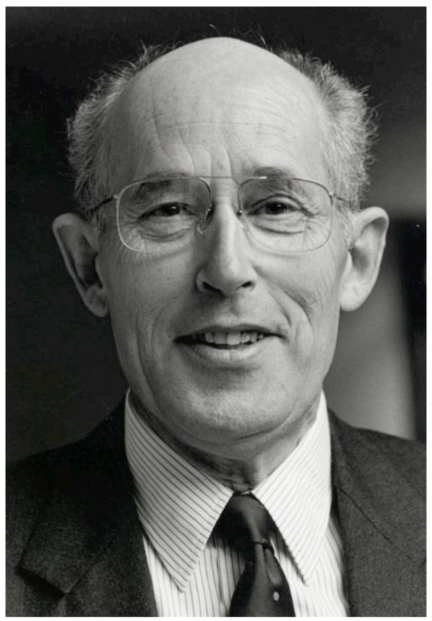
parallel spinors in the cone

More precisely...

If n is **odd**, Killing spinors are in one-to-one correspondence with **chiral** parallel spinors in the cone: the chirality is the **sign** of λ.

If n is **even**, Killing spinors with **both** signs of λ are in one-to-one correspondence with the parallel spinors in the cone, and the sign of λ enters in the relation between the Clifford bundles.

This reduces the problem to one (already solved) about the holonomy group of the cone.


M. Berger

n	Holonomy
n	SO_n
$2m$	U_m
$2m$	SU_m
$4m$	$\mathrm{Sp}_m \cdot \mathrm{Sp}_1$
$4m$	Sp_m
7	G_2
8	Spin_7


M. Wang

Or else the cone is flat and M is a sphere.

Killing superalgebra

Construction of the algebra

(M, g) riemannian spin manifold

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

$\mathfrak{k}_0 =$ {Killing vectors}

$\mathfrak{k}_1 =$ {Killing spinors}

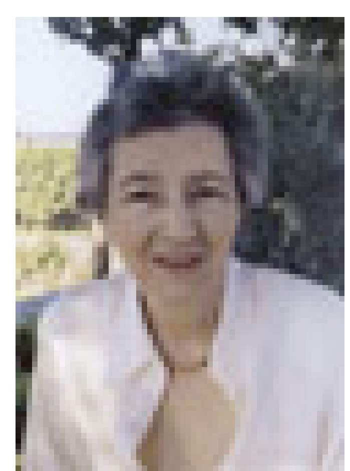
(with $\lambda = \frac{1}{2}$)

$$[-,-] : \Lambda^2 \mathfrak{k} \to \mathfrak{k} \; ?$$

$[-,-] : \Lambda^2 \mathfrak{k}_0 \to \mathfrak{k}_0$ ✓ $[-,-]$ of vector fields

$[-,-] : \Lambda^2 \mathfrak{k}_1 \to \mathfrak{k}_0$ ✓ $g([\varepsilon_1, \varepsilon_2], X) = (\varepsilon_1, X \cdot \varepsilon_2)$

$[-,-] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$? spinorial Lie derivative!



Kosmann

Lichnerowicz

$X \in \Gamma(TM)$ Killing $\Longleftrightarrow$ $\mathcal{L}_X g = 0$

$\Longleftrightarrow$ $A_X := Y \mapsto -\nabla_Y X \in \mathfrak{so}(TM)$

$\varrho : \mathfrak{so}(TM) \to \mathrm{End}\, S(M)$ spinor representation

$$\mathcal{L}_X := \nabla_X + \varrho(A_X)$$

spinorial Lie derivative

cf. $\mathfrak{L}_X Y = \nabla_X Y + A_X Y = \nabla_X Y - \nabla_Y X = [X, Y]$ ✓

Properties

$$\forall X, Y \in \mathfrak{k}_0, \quad Z \in \Gamma(TM), \quad \varepsilon \in \Gamma(S(M)), \quad f \in C^\infty(M)$$

$$\mathfrak{L}_X(Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathfrak{L}_X \varepsilon$$

$$\mathfrak{L}_X(f\varepsilon) = X(f)\varepsilon + f \mathfrak{L}_X \varepsilon$$

$$[\mathfrak{L}_X, \nabla_Z]\varepsilon = \nabla_{[X,Z]}\varepsilon$$

$$[\mathfrak{L}_X, \mathfrak{L}_Y]\varepsilon = \mathfrak{L}_{[X,Y]}\varepsilon$$

$$\forall \varepsilon \in \mathfrak{k}_1, X \in \mathfrak{k}_0$$

$$\mathcal{L}_X \varepsilon \in \mathfrak{k}_1$$

$$[-, -] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$$

$$[X, \varepsilon] := \mathcal{L}_X \varepsilon$$ ✓

The Jacobi identity

Jacobi: $\Lambda^3\mathfrak{k} \to \mathfrak{k}$

$$(X,Y,Z) \mapsto [X,[Y,Z]] - [[X,Y],Z] - [Y,[X,Z]]$$

4 components :

$\Lambda^3\mathfrak{k}_0 \to \mathfrak{k}_0$	✓	Jacobi for vector fields
$\Lambda^2\mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$	✓	$[\mathfrak{L}_X, \mathfrak{L}_Y]\varepsilon = \mathfrak{L}_{[X,Y]}\varepsilon$
$\mathfrak{k}_0 \otimes \Lambda^2\mathfrak{k}_1 \to \mathfrak{k}_0$	✓	$\mathfrak{L}_X(Z\cdot\varepsilon) = [X,Z]\cdot\varepsilon + Z\cdot\mathfrak{L}_X\varepsilon$
$\Lambda^3\mathfrak{k}_1 \to \mathfrak{k}_1$	?	but $\mathfrak{k}_0$ − equivariant

Some examples

$S^7 \subset \mathbb{R}^8$	$\mathfrak{k}_0 = \mathfrak{so}_8$	$\mathfrak{k}_1 = \Delta_+$	28 + 8 = 36	$\mathfrak{so}_9$
$S^8 \subset \mathbb{R}^9$	$\mathfrak{k}_0 = \mathfrak{so}_9$	$\mathfrak{k}_1 = \Delta$	36 +16 = 52	$\mathfrak{f}_4$
$S^{15} \subset \mathbb{R}^{16}$	$\mathfrak{k}_0 = \mathfrak{so}_{16}$	$\mathfrak{k}_1 = \Delta_+$	120+128 = 248	$\mathfrak{e}_8$

In all cases, the Jacobi identity follows from

$$\left(\mathfrak{k}_1 \otimes \Lambda^3 \mathfrak{k}_1^*\right)^{\mathfrak{k}_0} = \mathbf{0}$$

A sketch of the proof

Two observations:

1) The bijection between Killing spinors and parallel spinors in the cone is **equivariant** under the action of isometries.

➡ Use the cone to calculate $\mathcal{L}_X \varepsilon$.

2) In the cone, $\mathcal{L}_X \varepsilon = \varrho(A_X)\varepsilon$ and since X is **linear**, the endomorphism A_X is constant.

➡ It is the natural action on spinors.

We then compare with the known constructions.

Alternatively, we appeal to the classification of **riemannian symmetric spaces**.

These Lie algebras have the following form:

$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$

$\mathfrak{k}_0$	Lie algebra
$\mathfrak{k}_1$	$\mathfrak{k}_0$-representation
$(-,-)$	$\mathfrak{k}$-invariant inner product

⇒ K/K_0 symmetric space

Looking up the list, we find the following:

$$\mathrm{Spin}_9/\mathrm{Spin}_8$$

$$F_4/\mathrm{Spin}_9$$

$$E_8/\mathrm{Spin}_{16}$$

with the expected linear isotropy representations.

Open questions

- Other **exceptional** Lie algebras?
- Other **dimensions** and/or **signatures**?
- Are the Killing superalgebras of the Hopf spheres related?
- What structure in the 15-sphere has **E8** as **automorphisms**?

Thank you!