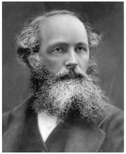
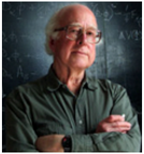
Homogeneous Supersymmetric Supergravity Backgrounds

José Miguel Figueroa O'Farrill





Field, Strings and Geometry Seminar
Surrey, 11 March 2014

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- $g, F, \ldots$ are subject to Einstein–Maxwell-like PDEs

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- Explicitly,

$$d \star F = \tfrac{1}{2} F \wedge F$$
$$\mathrm{Ric}(X, Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X, Y)|F|^2$$

together with $dF = 0$

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- It is convenient to organise this information according to how much "supersymmetry" the background preserves.

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- such spinor fields are called **Killing spinors**

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- a background is said to be ν-**BPS**, where $\nu = \frac{n}{32}$

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where the second row are now known to be homogeneous!

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- It is a very useful invariant of a supersymmetric supergravity background

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$$\begin{array}{ccc} H & \to & G \\ & & \downarrow \\ & & M \end{array}$$

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- This is the “right” working notion in supergravity

The homogeneity theorem

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

The homogeneity theorem

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In fact, vector fields in the Killing superalgebra already span the tangent spaces to every point of M

Generalisations

Theorem

Every ν-BPS background of type IIB supergravity with $\nu > \frac{1}{2}$ is homogeneous.

Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is homogeneous.

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The theorems actually prove the *strong* version of the conjecture: that the Killing superalgebra acts (locally) transitively.

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- Maximally supersymmetric $(2,0)$ backgrounds are also known to be homogeneous, but those with $\nu > \frac{1}{2}$ are not necessarily maximally supersymmetric.

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- By induction, proving the (strong) homogeneity theorem for those theories which are maximally 'oxidised' suffices.

HUSTLER (IN PROGRESS)

Poincaré supergravities

	32			24		20	16		12	8	4
11	M										
10	IIA	IIB					I				
9	$N = 2$						$N = 1$				
8	$N = 2$						$N = 1$				
7	$N = 4$						$N = 2$				
6	$(2, 2)$	$(3, 1)$	$(4, 0)$	$(2, 1)$	$(3, 0)$		$(1, 1)$	$(2, 0)$		$(1, 0)$	
5	$N = 8$			$N = 6$			$N = 4$			$N = 2$	
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This actually only shows local homogeneity.

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This is **good** because

- the supergravity field equations for homogeneous backgrounds are algebraic and hence simpler to solve than PDEs
- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

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$$\left\{ \begin{matrix} \mathrm{Ad}(H)\text{-invariant} \\ \text{tensors on } \mathfrak{m} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} H\text{-invariant} \\ \text{tensors on } T_p M \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} G\text{-invariant} \\ \text{tensor fields on } M \end{matrix} \right\}$$

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subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

Skip technical details

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We raise and lower indices with γ_{ij}.

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$$(\delta\varphi)_{ijk} = -\tfrac{3}{2} f_{m[i}{}^{n}\varphi^{m}{}_{jk]n} - 3U_{m[i}{}^{n}\varphi^{m}{}_{jk]n} - U_{m}{}^{mn}\varphi_{nijk}$$

where $U_{ijk} = f_{i(jk)}$

Homogeneous Ricci curvature

Finally, the Ricci tensor for a homogeneous (reductive) manifold is given by

$$R_{ij} = -\tfrac{1}{2} f_i{}^{k\ell} f_{jk\ell} - \tfrac{1}{2} f_{ik}{}^{\ell} f_{j\ell}{}^{k} + \tfrac{1}{2} f_{ik}{}^{a} f_{aj}{}^{k} + \tfrac{1}{2} f_{jk}{}^{a} f_{ai}{}^{k} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ij} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ji} + \tfrac{1}{4} f_{k\ell i} f^{k\ell}{}_{j}$$

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It is now a matter of assembling these ingredients to write down the supergravity field equations in a homogeneous Ansatz.

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- Solve the equations!

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Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper (i.e., inverse image of compact is compact). In particular, proper actions have compact stabilisers.

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This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

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- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$
  JMF+Ungureanu (in preparation)

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- In particular, we can "dial up" a semisimple G and hope to solve the homogeneous supergravity equations with symmetry G
- Checking supersymmetry is an additional problem, but there is an efficient algorithm which has already discarded many of the symmetric eleven-dimensional backgrounds.

  LISCHEWSKI (2014), HUSTLER+LISCHEWSKI (IN PROGRESS)