Deformation Theory of 3-Algebras

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Last October in Japan, one Taichi Takashita launched a campaign for the right to marry manga characters.

Telegraph.co.uk

Japanese launch campaign to marry comic book characters

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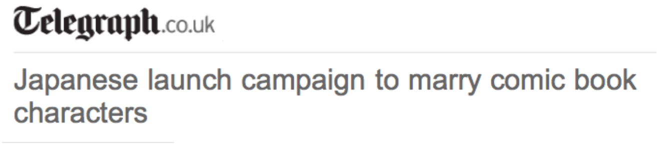
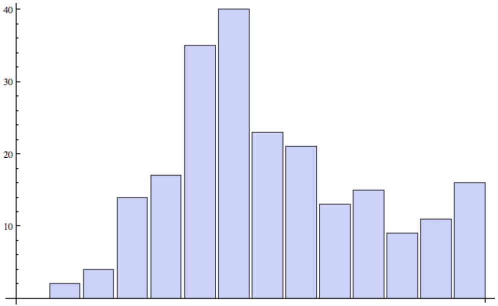

Telegraph.co.uk

Japanese launch campaign to marry comic book characters

His main reason:

"I am no longer interested in three dimensions."

For a while it seemed he was not alone:



Motivation

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Motivation

"One can learn a lot about a mathematical object by studying how it behaves under small perturbations."

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Metric 3-Lie algebras have entered the collective consciousness as a result of the work of Bagger–Lambert and Gustavsson.

They are a natural language in which to formulate superconformal Chern–Simons + matter theories in 3 dimensions.

This talk will be about 3-algebra **deformations** in the sense of Gerstenhaber.

Deformations are always controlled by a cohomology theory — in this case, that of a **Leibniz algebra**.

1. Metric 3-algebras
2. Cohomology of Leibniz algebras
3. Deformations of 3-Leibniz algebras
4. An explicit example

1. Metric 3-algebras
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4. An explicit example

Metric Lie algebras

Definition

A (real) **Lie algebra** is a real vector space $\mathfrak{g}$ together with a bilinear alternating bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the **Jacobi identity** for all $x, y, z \in \mathfrak{g}$:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \; .$$

It is said to be **metric** if $\mathfrak{g}$ possesses a symmetric inner product $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ obeying

$$\langle [x, y], z \rangle = - \langle y, [x, z] \rangle \; .$$

Metric 3-Lie algebras

Definition

A (real) **3-Lie algebra** is a real vector space V together with a trilinear alternating bracket $[-,-,-] : V \times V \times V \to V$ satisfying the **fundamental identity** for all $x, y, z_i \in V$:

$$[x, y, [z_1, z_2, z_3]] = [[x, y, z_1], z_2, z_3] + [z_1, [x, y, z_2], z_3] + [z_1, z_2, [x, y, z_3]] .$$

It is said to be **metric** if V possesses a symmetric inner product $\langle -,- \rangle : V \times V \to \mathbb{R}$ obeying

$$\langle [x, y, z_1], z_2 \rangle = - \langle z_1, [x, y, z_2] \rangle .$$

The fundamental identity

Define $D : \Lambda^2 V \to \operatorname{End} V$ by

$$D(x \wedge y)z = [x, y, z] .$$

The fundamental identity becomes

$$[D(X), D(Y)] = D(D(X) \cdot Y) ,$$

which says that $\operatorname{im} D < \mathfrak{gl}(V)$. But **why?**

cf. The adjoint representation

The Jacobi identity for $\mathfrak{g}$ may be written as

$$[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad}[x, y] \ ,$$

where $\operatorname{ad} : \mathfrak{g} \to \operatorname{End} \mathfrak{g}$ is defined by $\operatorname{ad}(x)y := [x, y]$.

ad is a Lie algebra homomorphism, hence $\operatorname{im} \operatorname{ad} < \mathfrak{gl}(\mathfrak{g})$.

Similarly, the most natural explanation for the image of D to be a Lie subalgebra would be for D to be a Lie algebra homomorphism.

Not quite a Lie bracket

It is therefore tempting to define a bracket on $\Lambda^2 V$ by

$$[X, Y] = D(X) \cdot Y ,$$

in terms of which the fundamental identity reads

$$[D(X), D(Y)] = D([X, Y]) .$$

However,

$$[X, Y] \neq -[Y, X] ;$$

although

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] .$$

In other words, $\Lambda^2 V$ becomes a **(left) Leibniz algebra** and the fundamental identity says that D is a Leibniz algebra homomorphism.

The associated metric Lie algebra

If V is a metric 3-Lie algebra, the map $D : \Lambda^2 V \to \mathfrak{so}(V)$. Its image is not just a Lie subalgebra of $\mathfrak{so}(V)$, but is actually **metric**, with inner product

$$(D(x \wedge y), D(z \wedge w)) = \langle [x, y, z], w \rangle \ .$$

In fact, associated to every metric Lie subalgebra of $\mathfrak{so}(V)$ there is a metric 3-algebra, whose bracket need not be alternating.

The Faulkner construction

Let $\mathfrak{g}$ be a metric Lie algebra with inner product $(-,-)$ and V a faithful representation. Transposing the $\mathfrak{g}$ action on V, we obtain a map

$$\mathscr{D} : V \times V^* \to \mathfrak{g}$$

defined by

$$(\mathscr{D}(v,w^*),X) = \langle w^*, X\cdot v\rangle \qquad \text{for all } X \in \mathfrak{g}.$$

This defines a trilinear map $[-,-,-] : V \times V^* \times V \to V$ by

$$[u,v^*,w] := \mathscr{D}(u,v^*)\cdot w\,.$$

The unitary case

An important special case is when V is a (real, complex or quaternionic) unitary representation of $\mathfrak{g}$. This identifies V^* with either V or $\bar{V}$ and hence the bracket defines a map

$$[-,-,-] : V \times \begin{matrix} V \\ \bar{V} \end{matrix} \times V \to V \qquad \begin{matrix} \mathbb{R}, \mathbb{H} \\ \mathbb{C} \end{matrix} .$$

The real case corresponds to the generalised 3-Lie algebras of Cherkis+Sämann (2008), whereas the complex case contains the hermitian 3-algebras of Bagger+Lambert (2008).
These 3-algebras share the fundamental identity as well as the metricity condition, suggesting the following definition.

Metric 3-Leibniz algebras

Definition

A (real, left) **3-Leibniz algebra** is a real vector space V together with a trilinear bracket $[-,-,-] : V \times V \times V \to V$ satisfying the **Leibniz identity** for all $x, y, z_i \in V$:

$$[x, y, [z_1, z_2, z_3]] = [[x, y, z_1], z_2, z_3] + [z_1, [x, y, z_2], z_3] + [z_1, z_2, [x, y, z_3]] \; .$$

It is said to be **metric** if V possesses a symmetric inner product $\langle -, - \rangle : V \times V \to \mathbb{R}$ obeying

$$\langle [x, y, z_1], z_2 \rangle = - \langle z_1, [x, y, z_2] \rangle \; .$$

The associated Leibniz algebra

Let now $D : V \otimes V \to \operatorname{End} V$ be defined by

$$D(x \otimes y)z = [x, y, z] \ .$$

In terms of D, the Leibniz identity becomes

$$[D(X), D(Y)] = D(D(X) \cdot Y) \ ,$$

for all $X, Y \in V \otimes V$.
We introduce on $L(V) := V \otimes V$ the bracket

$$[X, Y] = D(X) \cdot Y \ ,$$

which turns $L(V)$ into a left Leibniz algebra.

The Daletskii functor

The assignment $V \mapsto L(V)$ is a **covariant functor** from the category of 3-Leibniz algebras to the category of Leibniz algebras.
Indeed, if $\varphi : V \to W$ is a homomorphism of 3-Leibniz algebras, so that

$$\varphi[x, y, z] = [\varphi x, \varphi y, \varphi z] ,$$

then $L\varphi : L(V) \to L(W)$ defined by

$$L\varphi(x \otimes y) = \varphi x \otimes \varphi y ,$$

is a homomorphism of Leibniz algebras.

The metric version

Let V be a **metric** 3-Leibniz algebra. On $L(V) = V \otimes V$ we have a natural inner product:

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \ .$$

It follows that

$$\langle [X, Y], Z \rangle = - \langle Y, [X, Z] \rangle$$

making $L(V)$ into a **metric Leibniz algebra**.
$V \mapsto L(V)$ is also functorial in the metric subcategory.

Leibniz algebras

Definition

A (real) **Leibniz algebra** L is a real vector space with a bilinear bracket $[-,-] : L \times L \to L$ satisfying the **Leibniz identity**

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] ,$$

for all $X, Y, Z \in L$.

If in addition $[X,Y] = -[Y,X]$ then L is a Lie algebra.
(Strictly speaking the above defines a **left** Leibniz algebra.)
Leibniz algebras were introduced by Loday (1992).

The reduced Lie algebra

The subspace $K \subset L$ spanned by elements of the form $[X, X]$ is a 2-sided ideal:

$$[K, L] = 0 \qquad \text{and} \qquad [L, K] \subset K$$

Therefore $\mathfrak{g}_L := L/K$ is a Leibniz algebra which is Lie by construction. It is called the **reduced Lie algebra** of L.

Representations

The notion of representation of a Leibniz algebra is more subtle than for a Lie algebra.

As $\mathfrak{L}$ itself shows, representations admit both left and right actions, which must obey some compatibility conditions.

We follow the treatment in LODAY+PIRASHVILI (1993).

Another look at Lie algebra representations

A representation of a Lie algebra $\mathfrak{g}$ is the same thing as a short exact sequence of Lie algebras

$$0 \longrightarrow V \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

with V abelian.

The Lie bracket in $\tilde{\mathfrak{g}}$ gives an action of $\mathfrak{g}$ on V, by lifting $\mathfrak{g}$ to a *subspace* of $\tilde{\mathfrak{g}}$.

Abelian extensions of a Leibniz algebra

An **abelian extension** of a Leibniz algebra L is a short exact sequence of Leibniz algebras

$$0 \longrightarrow M \longrightarrow \tilde{L} \longrightarrow L \longrightarrow 0$$

with $[M, M] = 0$.
M admits both left and right actions of L:

$$L \times M \to M \qquad M \times L \to M$$
$$(X, m) \mapsto [X, m] \qquad (m, X) \mapsto [m, X]$$

subject to the Leibniz identity in $\tilde{L}$.

Representations

The Leibniz identity implies the following:

$$[[X, Y], m] = [X, [Y, m]] - [Y, [X, m]]$$
$$[[X, m], Y] = [X, [m, Y]] - [m, [X, Y]]$$
$$[[m, X], Y] = [m, [X, Y]] - [X, [m, Y]]$$

A **representation** of L is a vector space M admitting left and right actions of L subject to the above three conditions.

If $[m, X] = -[X, m]$, M is **symmetric** and only the first of the above conditions suffices. M is then induced from a representation of $\mathfrak{g}_L$.

The universal enveloping algebra

There is a categorical equivalence between representations of L and left modules over its **universal enveloping algebra** $UL(L)$, which is the quotient of the tensor algebra $T(L \oplus L)$ by the 2-sided ideal generated by

$$\ell_{[X,Y]} - \ell_X \otimes \ell_Y + \ell_Y \otimes \ell_X$$
$$r_{[X,Y]} - \ell_X \otimes r_Y - r_Y \otimes \ell_X$$
$$r_Y \otimes (\ell_X + r_X)$$

where $\ell_X = X \oplus 0$ and $r_X = 0 \oplus X$.

Cohomology

On the graded space

$$CL^\bullet(L,M) = \bigoplus_{p\geqslant 0} \mathrm{Hom}(L^{\otimes p}, M)$$

we define a differential $d : CL^p(L,M) \to CL^{p+1}(L,M)$ by

$$\begin{aligned}(d\varphi)(X_1,\ldots,X_{p+1}) &= \sum_{i=1}^{p} (-1)^{i-1}[X_i, \varphi(X_1,\ldots,\widehat{X_i},\ldots,X_{p+1})]\\ &+ (-1)^{p+1}[\varphi(X_1,\ldots,X_p), X_{p+1}]\\ &+ \sum_{1\leqslant i<j\leqslant p+1} (-1)^i \varphi(X_1,\ldots,\widehat{X_i},\ldots,[X_i,X_j],\ldots,X_{p+1})\end{aligned}$$

$d^2 = 0$ because M is a representation and L a Leibniz algebra.
The cohomology is denoted $HL^\bullet(L;M)$.

cf. Lie algebra cohomology

If M is a symmetric representation, then

$$
\begin{aligned}
(d\varphi)(X_1,\dots,X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} [X_i, \varphi(X_1,\dots,\widehat{X_i},\dots,X_{p+1})] \\
&+ \sum_{1\leqslant i<j\leqslant p+1} (-1)^i \varphi(X_1,\dots,\widehat{X_i},\dots,[X_i,X_j],\dots,X_{p+1})
\end{aligned}
$$

which is reminiscent of the Chevalley–Eilenberg differential computing Lie algebra cohomology.

The first few differentials

$$dm(X) = -[m, X]$$

$$d\varphi(X, Y) = [X, \varphi(Y)] + [\varphi(X), Y] - \varphi([X, Y])$$

$$\begin{aligned} d\theta(X, Y, Z) &= [X, \theta(Y, Z)] - [Y, \theta(X, Z)] - [\theta(X, Y), Z] \\ &\quad + \theta(X, [Y, Z]) - \theta(Y, [X, Z]) - \theta([X, Y], Z) \end{aligned}$$

Deformations

A complex governing infinitesimal deformations of an $\mathfrak{n}$-Lie algebra was successfully defined by GAUTHERON (1996) after an initial attempt by TAKHTAJAN (1994).

DALETSKII+TAKHTAJAN (1997) rewrote Gautheron's work in terms of a subcomplex of $CL^\bullet(L; L)$.

We will see that the proper complex to study deformations is $CL^\bullet(L; \operatorname{End} V)$.

The deformation complex

Let V be an 3-Leibniz algebra and $L := L(V)$ the associated Leibniz alebra. Both algebraic structures are determined by a map $D : L \to \mathfrak{gl}(V)$.

A **deformation** of V is an analytic one-parameter family of 3-Leibniz algebras on the same underlying vector space:

$$[x, y, z]_t := [x, y, z] + \sum_{k \geqslant 1} t^k \Phi_k(x, y, z) ,$$

or equivalent a family D_t of maps

$$D_t = D + \sum_{k \geqslant 1} t^k \varphi_k ,$$

where $\varphi_k : L \to \operatorname{End} V$ are defined by

$$\varphi_k(x, y) \cdot z = \Phi_k(x, y, z).$$

Infinitesimal deformations

The fundamental identity for D_t becomes an infinite number of equations for the φ_k, one per power of t.
The equation of order t^0 is the fundamental identity for D.
The equation of order t is a linear equation on φ_1:

$$[D(X), \varphi_1(Y)] + [\varphi_1(X), D(Y)] - D(\varphi_1(X) \cdot Y) - \varphi_1(D(X) \cdot Y) = 0,$$

which, comparing with the ▸ Leibniz differential is simply $d\varphi_1 = 0$ for $\varphi_1 \in CL^1(L; \mathrm{End}\, V)$ where

$$[X, \psi] = [D(X), \psi] \qquad \text{and} \qquad [\psi, X] = [\psi, D(X)] - D(\psi \cdot X) ,$$

for $\psi \in \mathrm{End}\, V$. (One checks that $\mathrm{End}\, V$ becomes a representation of L in this way.)

Trivial deformations

A deformation is **trivial** if it is the result of a one-parameter subgroup g_t of $\mathrm{GL}(V)$:

$$[x, y, z]_t = g_t^{-1}[g_t x, g_t y, g_t z]$$

or, equivalently,

$$D_t(X) = g_t^{-1} \circ D(g_t X) \circ g_t \ .$$

If $g_t(x) = x + t\gamma(x) + \dots$, then at order t^0 the equation is trivially satisfied, whereas at order t^1, one finds

$$\varphi_1(X) = -[\gamma, D(X)] + D(\gamma \cdot X) = -[\gamma, X] = d\gamma(X),$$

for $\gamma \in CL^0(L; \mathrm{End}\, V)$.

Deformations and Leibniz cohomology

Theorem

Isomorphism classes of infinitesimal deformations of a 3-Leibniz algebra V *are classified by* $HL^1(L; \operatorname{End} V)$*, with*

$$[X, \psi] = [D(X), \psi] \qquad \text{and} \qquad [\psi, X] = [\psi, D(X)] - D(\psi \cdot X)\ ,$$

for $\psi \in \operatorname{End} V$ *and* $X \in L$*.*

Obstructions

Similarly one can show that obstructions to integrating an infinitesimal deformation live in $\mathrm{HL}^2(\mathrm{L}; \mathrm{End}\, \mathrm{V})$. For example, to order t^2, the fundamental identity for D_t says that

$$
\begin{aligned}
&[D(X), \varphi_2(Y)] + [\varphi_2(X), D(Y)] + [\varphi_1(X), \varphi_1(Y)] \\
&\qquad = D(\varphi_2(X) \cdot Y) + \varphi_2(D(X) \cdot Y) + \varphi_1(\varphi_1(X) \cdot Y) ,
\end{aligned}
$$

which we recognise as

$$
d\varphi_2(X, Y) = \varphi_1(\varphi_1(X) \cdot Y) - [\varphi_1(X), \varphi_1(Y)] .
$$

The RHS is a cocycle in $\mathrm{CL}^2(\mathrm{L}; \mathrm{End}\, \mathrm{V})$ which for the deformation to integrate to second order needs to be a coboundary.

Deformations of V *vs.* deformations of L

Deformations of V induce deformations of L and vice versa.

However, trivial deformations differ: $\mathsf{GL}(V)$ *vs.* $\mathsf{GL}(L)$!

One can study deformations of V in terms of $CL^\bullet(L;L)$, as do Daletskii and Takhtajan, but they are forced to restrict to a subcomplex. Neither do they go beyond infinitesimal deformations.

To consider obstructions it is computationally convenient to exhibit the structure of a graded Lie algebra on the relevant complex. This follows from work of Balavoine (1995) for $CL^\bullet(L;L)$ or of Rotkiewicz (2005) for a complex isomorphic to $CL^\bullet(L;\mathsf{End}\,V)$ for V an n-Lie algebra.

The deformation complex as a graded Lie algebra

The following is analogous to the celebrated theorem of NIJENHUIS+RICHARDSON (1967) for Lie algebra cohomology.

Theorem (Rotkiewicz, 2005)

The complex $\mathrm{CL}^\bullet(\mathrm{L}; \operatorname{End} \mathrm{V})$ admits the structure of a graded Lie algebra in such a way that $\mathrm{d} = [\mathrm{D}, -]$ and the fundamental identity becomes $[\mathrm{D}, \mathrm{D}] = 0$.

The deformation equation

The fundamental identity for D_t is

$$[D_t, D_t] = 0 \; .$$

Suppose we have found a solution to order t^N, so that we have

$$D_N = D + \sum_{k=1}^{N} t^k \varphi_k$$

satisfying

$$[D_N, D_N] = t^{N+1}\xi + \dots$$

Then ξ is a cocycle:

$$[D, [D_N, D_N]] = [D - D_N, [D_N, D_N]] + [D_N, [D_N, D_N]] = O(t^{N+2})$$

Obstructions revisited

The deformation can be extended to D_{N+1} provided that $[\xi] = 0$ in $HL^2(L; \operatorname{End} V)$, so that $\xi = -2d\varphi_{N+1}$. This leads to an infinite number of obstructions in $HL^2(L; \operatorname{End} V)$.

Theorem

Infinitesimal deformations of a 3-Leibniz algebra V are classified by $HL^1(L; \operatorname{End} V)$ and the obstructions to integrating an infinitesimal deformation live in $HL^2(L; \operatorname{End} V)$.

3-Lie algebra deformations

If V is a 3-Lie algebra, $L(V) = \Lambda^2 V$ is the associated Leibniz algebra.

We again have $CL^\bullet(L; \operatorname{End} V)$,...

However not every cocycle in $CL^1(L; \operatorname{End} V)$ gives rise to a deformation of the 3-Lie algebra: the corresponding bracket need not be totally skewsymmetric.

We must restrict to a subcomplex $C^\bullet$ agreeing with $CL^\bullet(L; \operatorname{End} V)$ except in dimension 1, where $C^1 \subset CL^1(L; \operatorname{End} V)$ consists of all $\varphi \in CL^1(L; \operatorname{End} V)$ such that

$$\varphi(x, y) \cdot z = -\varphi(x, z) \cdot y \ .$$

$C^\bullet$ is indeed a subcomplex.

Theorem

Infinitesimal deformations of a 3-Lie algebra V *are classified by* $H^1(C^\bullet)$ *and the obstructions to integrating an infinitesimal deformation live in* $H^2(C^\bullet)$.

Metric 3-Leibniz algebra deformations

Here $\mathrm{D} : \mathrm{L} \to \mathfrak{so}(\mathrm{V})$, so we consider the subcomplex $\mathrm{CL}^{\bullet}(\mathrm{L}; \mathfrak{so}(\mathrm{V}))$, which is also a graded Lie subalgebra.

Theorem

Infinitesimal deformations of a metric 3-Leibniz algebra V *are classified by* $\mathrm{HL}^{1}(\mathrm{L}; \mathfrak{so}(\mathrm{V}))$ *and the obstructions to integrating an infinitesimal deformation live in* $\mathrm{HL}^{2}(\mathrm{L}; \mathfrak{so}(\mathrm{V}))$.

Metric 3-Lie algebra deformations

We restrict the complex $C^\bullet$ from $\operatorname{End} V$ to $\mathfrak{so}(V)$. This yields a subcomplex $\tilde{C}^\bullet$ which is also a graded Lie subalgebra.

Theorem

Infinitesimal deformations of a metric 3-Lie algebra V are classified by $H^1(\tilde{C}^\bullet)$ and the obstructions to integrating an infinitesimal deformation live in $H^2(\tilde{C}^\bullet)$.

The unique simple 3-Lie algebra

Let $V = \mathbb{R}^4$ with the standard euclidean inner product and elementary vectors e_i. Then FILIPPOV (1980) showed that

$$[e_i, e_k, e_k] = \varepsilon_{ijkl} e_l$$

defines a 3-Lie algebra, denoted S_4. Its complexification is the unique simple 3-Lie algebra, as shown by LING (1993). It is also the unique nonabelian positive-definite indecomposable metric 3-Lie algebra, as shown by NAGY (2007) and also by PAPADOPOULOS (2008) and GAUNTLETT–GUTOWSKI (2008). (This had been conjectured by FO+PAPADOPOULOS (2003).)

The Leibniz algebra

The associated Leibniz algebra is $\Lambda^2\mathbb{R}^4$ with basis $e_{ij} := e_i \wedge e_j$ and bracket

$$[e_{ij}, e_{kl}] = \varepsilon_{ijkm} e_{ml} + \varepsilon_{ijlm} e_{km} .$$

The image $\mathfrak{g}$ of D is all of $\mathfrak{so}(4)$, whence D is an isomorphism. The Leibniz algebra is Lie and isomorphic to $\mathfrak{so}(4)$, but **not** as metric Lie algebras! In $\mathfrak{g}$ the inner product is

$$\big(D(e_{ij}), D(e_{kl})\big) = \varepsilon_{ijkl},$$

whereas in L, it is

$$\langle e_{ij}, e_{kl} \rangle = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Deformations

It follows from an explicit computation that S_4 is **rigid** as a 3-Lie algebra, whereas it admits a one-parameter deformation as a 3-Leibniz algebra

$$[e_i, e_j, e_k]_t = \varepsilon_{ijkl} e_l + t\left(\delta_{jk} e_i - \delta_{ik} e_j\right)$$

which is metric and already of Cherkis–Sämann type, whence it can be understood from the Faulkner construction.

Interpretation

The metric Lie algebra in the Faulkner construction of S_4 is $\mathfrak{g} \cong \mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, which is metric relative to a one-parameter family of inner products, up to rescalings.

- $t = 0$ This corresponds to S_4 and to an inner product on $\mathfrak{g}$ which has split signature.
- $t = \pm 1$ This is "singular" from the Faulkner perspective: the inner product on $\mathfrak{g}$ becomes degenerate: it vanishes on one of the $\mathfrak{so}(3)$ factors. Actually the Faulkner Lie algebra should also be $\mathfrak{so}(3)$.
- $t > 1$ inner product is negative-definite
- $t < 1$ inner product is positive-definite
- $t^2 < 1$ inner product is split
- $t \to \pm\infty$ 3-algebra approoaches the metric Lie triple system associated to S^4 as the riemannian symmetric space $SO(5)/SO(4)$

Open questions

- We need more computable examples!
- How to compute Leibniz cohomology? Are there analogous results to the Whitehead lemmas and/or the Hochschild-Serre spectral sequence?
- Systematic interpretation of the deformations of metric 3-Leibniz algebras in terms of deformations of their Faulkner data.
- How do these deformations manifest themselves in the corresponding superconformal field theory?