WHEN NAHM MET TANAKA: A NOVEL APPROACH TO THE CLASSIFICATION OF SUPERGRAVITY BACKGROUNDS

JOSÉ FIGUEROA-O'FARRILL

Abstract. This is the written version of a Geometry Seminar I gave (in Spanish) at the Hitchin Lab of ICMAT in Madrid on 14 December 2015. It is based on joint work with Andrea Santi.

I would like to report on some recent and ongoing work with Andrea Santi in Edinburgh aimed at cracking the classification problem for (highly) supersymmetric supergravity backgrounds. There are two sources of inspiration for this work: the theory of Tanaka structures and the construction of a Lie superalgebra associated to each such background. But first, let us start with a classical analogy.

Given a (pseudo-) riemannian manifold (M, g), we may associate a Lie algebra with it by considering the "infinitesimal isometries": vector fields $\xi \in \mathscr{X}(M)$ safisfying

$$\mathscr{L}_\xi g = 0 .$$

We call them **Killing vectors**. Equivalently, a vector field ξ is Killing if the endomorphism A_ξ of TM defined by

$$A_\xi(X) = -\nabla_X \xi$$

is skew-symmetric relative to g; that is, $g(A_\xi(X), X) = 0$ for all vector fields X. We say $A_\xi \in \mathfrak{so}(TM)$.

A result of Kostant from the 1950s [1] identifies Killing vectors with parallel sections of

$$\mathscr{E} = TM \oplus \mathfrak{so}(TM) \cong TM \oplus \Lambda^2 TM ,$$

relative to a certain connection we now describe. This result follows from the celebrated **Killing identity**

$$\nabla_X A_\xi = R(X, \xi) ,$$

which essentially says that if a Killing vector and its derivative vanish at a point, then it is identically zero. If $(\xi, A) \in \Gamma(TM \oplus \mathfrak{so}(TM))$, then its covariant derivative is defined by

$$\mathscr{D}_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A - R(X, \xi) \end{pmatrix} . \tag{1}$$

It is clear that a section (ξ, A) is parallel precisely if ξ is a Killing vector and $A = A_\xi$. This allows us to localise at any $p \in M$. We let $V = T_pM$, $\langle -, - \rangle = g_p$, so that the fibre $\mathscr{E}_p = V \oplus \mathfrak{so}(V)$. The Lie algebra of isometries can be written in this language as an algebraic bracket at $p \in M$:

$$[(v, A), (w, B)] = (Aw - Bv, AB - BA + R(v, w)) .$$

There are a number of observations to be made:

(1) $V \oplus \mathfrak{so}(V)$ is the underlying vector space of the euclidean Lie algebra $\mathfrak{e}(V)$ with brackets

$$[(v, A), (w, B)] = (Aw - Bv, AB - BA) .$$

We see that $\mathfrak{e}(V)$ is isomorphic to the Lie algebra of isometries of flat space ($R = 0$).

(2) $\mathfrak{e}(V)$ is a $\mathbb{Z}$-graded Lie algebra with V in degree -1 and $\mathfrak{so}(V)$ in degree 0, and hence it is also canonically filtered:

$$\cdots \supset F^{-1} \supset F^0 \supset F^1 \supset \cdots$$

where $F^j = \mathfrak{e}(V)$ for $j \leqslant -1$, $F^0 = \mathfrak{so}(V)$ and $F^j = 0$ for $j \geqslant 1$. Every filtered Lie algebra has an associated graded Lie algebra and in this case, it is again isomorphic to $\mathfrak{e}(V)$. A filtered Lie algebra $\mathfrak{g}$ is said to be a **filtered deformation** of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{h}$, if its associated graded Lie algebra $\operatorname{gr}\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ as $\mathbb{Z}$-graded Lie algebras.

(3) The Lie algebra $\mathfrak{g}$ of isometries of (M, g) is a filtered deformation of a Lie subalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_{-1}$ of $\mathfrak{e}(V)$, where the vector subspaces $\mathfrak{h}_0 \subset \mathfrak{so}(V)$ and $\mathfrak{h}_{-1} \subset V$ are defined by

$$0 \longrightarrow \mathfrak{h}_0 \longrightarrow \mathfrak{g} \xrightarrow{\mathrm{ev}_p} \mathfrak{h}_1 \longrightarrow 0 ,$$

where ev_p is the evaluation at p.

We would like to "superise" this construction. There exist many different superisations: basically one for each supergravity theory. In this talk I will concentrate on eleven-dimensional supergravity.

So we let $(V, \langle -, - \rangle)$ denote an eleven-dimensional lorentzian vector space and let $C\ell(V)$ denote the associated Clifford algebra, defined by

$$v^2 = -\langle v, v \rangle \mathbb{1} .$$

As a real, unital, associative algebra

$$C\ell(V) \cong \operatorname{End}(S_+) \oplus \operatorname{End}(S_-) ,$$

where $S_\pm$ are irreducible Clifford modules of real dimension 32, distinguished by the action of the centre of $C\ell(V)$: trivial action on S_+ and nontrivial action on S_-. We take $S = S_-$ in this talk. (The choice is actually immaterial.)

On S we have a symplectic structure ω which obeys

$$\omega(v \cdot s_1, s_2) = -\omega(s_1, v \cdot s_2) \qquad \forall\, v \in V\,, s_1, s_2 \in S\,.$$

The spin group $\mathrm{Spin}(V) \subset C\ell(V)$ and ω is $\mathrm{Spin}(V)_0$-invariant, where $\mathrm{Spin}(V)_0$ is the identity component.

We consider the following $\mathbb{Z}$-graded vector space

$$\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V) ,$$

where we now put V in degree -2, S in degree -1 and $\mathfrak{so}(V)$ again in degree 0. We can turn $\mathfrak{p}$ into a $\mathbb{Z}$-graded Lie superalgebra by extending $\mathfrak{e}(V)$ with the following additional (nonzero) Lie brackets, for $A \in \mathfrak{so}(V)$ and $s \in S$,

$$[A, s] = As \qquad \text{and} \qquad [s, s] = \kappa(s, s) \in V ,$$

where As is the action of $\mathfrak{so}(V)$ on S and $\kappa(s, s)$ is the **Dirac current** of s, defined by

$$\langle \kappa(s, s), v \rangle = \omega(v \cdot s, s) \qquad \forall\, v \in V\,, s \in S\,.$$

The resulting superalgebra is called the **Poincaré superalgebra**.

In 1978 Nahm [2] studied unitary irreducible representations of $\mathfrak{p}$. The smallest nontrivial such representation is induced from the following representation of $\mathrm{Spin}(9)$:

$$\odot^2_0 W \oplus (W \otimes \Sigma)_0 \oplus \Lambda^3 W ,$$

where W is the real 9-dimensional vector representation and Σ the real 16-dimensional spinor representation of $\mathrm{Spin}(9)$, both of which are orthogonal representations, $\odot^2_0$ means traceless symmetric and $(W \otimes \Sigma)_0$ is the kernel of the Clifford action of W on Σ; that is,

$$0 \longrightarrow (W \otimes \Sigma_0) \longrightarrow W \otimes \Sigma \xrightarrow{\ \mathrm{cl}\ } \Sigma \longrightarrow 0 .$$

We recognise these representations as the physical degrees of freedom of a metric g, a gravitino Ψ and a 3-form potential A. Notice that

$$\dim_{\mathbb{R}} \odot^2_0 W - \dim_{\mathbb{R}} (W \otimes \Sigma)_0 + \dim_{\mathbb{R}} \Lambda^3 W = 44 - 128 + 84 = 0 ,$$

which is the balance of bosonic and fermionic physical degrees of freedom we expect from supersymmetry.

Shortly after Nahm's paper, Cremmer, Julia and Scherk [3] constructed a supergravity theory with the field content (g, Ψ, A) and all the trimmings: action, field equations, supersymmetry transformations,... A **(bosonic) background** of eleven-dimensional supergravity is an eleven-dimensional lorentzian spin manifold (M, g) with spinor bundle $\$ \to M$ and a closed 4-form F satisfying two coupled partial differential equations:

$$\begin{aligned} &\mathrm{Ric} \sim F^2 && \text{(Einstein)}\\ &d \star F = \tfrac{1}{2} F \wedge F && \text{(Maxwell)} \end{aligned}$$

These equations, together with the **Bianchi identity** $dF = 0$, are equivalent to the vanishing of the Clifford trace of the curvature of a connection $\mathscr{D}$ on $\$$, defined by

$$\mathscr{D}_X \varepsilon = \nabla_X \varepsilon + \tfrac{1}{8} F \cdot X \cdot \varepsilon + \tfrac{1}{24} X \cdot F \cdot \varepsilon . \tag{2}$$

Indeed, its curvature $\mathscr{R} : \Lambda^2 TM \to \operatorname{End}(\$)$ and its Clifford trace is

$$\sum_i e^i \cdot \mathscr{R}(e_i, -) : TM \to \operatorname{End}(\$) ,$$

where (e_i) is a pseudo orthonormal frame and (e^i) the metrically dual frame: $g(e^i, e_j) = \delta^i_j$. Then one has [4] that

$$\sum_i e^i \cdot \mathscr{R}(e_i, X) = 0 \quad \forall\, X \iff \begin{cases} \text{Einstein} \\ \text{Maxwell} \\ \text{Bianchi} \end{cases}$$

A background (M, g, F) is **supersymmetric** if there exists a nonzero $\mathscr{D}$-parallel spinor field $\varepsilon \in \Gamma(\$)$. Such spinor fields are called **Killing spinors**. The name reflects the fact that the Dirac current of a Killing spinor is a Killing vector which, in addition, preserves F [4].

The Killing spinors and the F-preserving Killing vectors define a Lie superalgebra $\mathfrak{k}$, called the **Killing superalgebra** [5]; although sometimes this term is reserved for the ideal generated by the Killing spinors. They are parallel sections of a supervector bundle $\mathscr{S} = \mathscr{S}_{\bar{0}} \oplus \mathscr{S}_{\bar{1}}$, where

$$\mathscr{S}_{\bar{0}} = TM \oplus \mathfrak{so}(TM) \qquad \text{and} \qquad \mathscr{S}_{\bar{1}} = \$,$$

with connection $\mathscr{D}$ given by equation (1) on $\mathscr{S}_{\bar{0}} = \mathscr{E}$ and by equation (2) on $\mathscr{S}_{\bar{1}}$. Of course $\mathfrak{k}_{\bar{0}}$ is the subspace of the parallel sections of $\mathscr{S}_{\bar{0}}$ which preserve F.

We make a couple of observations:

(1) the homogeneity conjecture, now a theorem [6], says that if $\dim_{\mathbb{R}} \mathfrak{k}_{\bar{1}} > 16$ then (M, g, F) is (locally) homogeneous; and
(2) $\mathfrak{k}$ is a filtered deformation of a $\mathbb{Z}$-graded subalgebra of $\mathfrak{p}$.

The problem of classifying filtered deformations of $\mathbb{Z}$-graded Lie (super)algebras is well-defined mathematically and has been the subject of a number of recent papers starting with [7, 8].

The problem Andrea Santi and I set ourselves was to calculate filtered deformations of $\mathbb{Z}$-graded subalgebras $\mathfrak{h}$ of $\mathfrak{p}$ which differ from $\mathfrak{p}$ only in degree 0; that is, $\mathbb{Z}$-graded Lie superalgebras of the form

$$\mathfrak{h} = V \oplus S \oplus \mathfrak{h}_0 \qquad \text{with} \qquad \mathfrak{h}_0 < \mathfrak{so}(V) . \tag{3}$$

As with all deformation theories, there is an underlying cohomology theory governing the first-order deformations (and the obstruction to their integrability). Here it is **Spencer cohomology**, a graded refinement of the Chevalley–Eilenberg cohomology of the nilpotent Lie superalgebra $\mathfrak{h}_- = V \oplus S$ with values in the module $\mathfrak{h}$. The relevant cohomology group classifying the first-order deformations is $H^{2,2}(\mathfrak{h}_-, \mathfrak{h})$ consistent of degree-2 maps $\Lambda^2\mathfrak{h}_- \to \mathfrak{h}$, where Λ^2 is meant in the super sense. We bootstrap this calculation from that of $H^{2,2}(\mathfrak{h}_-, \mathfrak{p})$ and here we find the first surprise

$$H^{2,2}(\mathfrak{h}_-; \mathfrak{p}) \cong \Lambda^4 V .$$

In other words, the 4-form in eleven-dimensional supergravity arises from the calculation! Moreover the component

$$\beta : V \otimes S \to S ,$$

of the cocycle representative of the class in $H^{2,2}(\mathfrak{h}_-, \mathfrak{p})$ corresponding to the four-form $F \in \Lambda^4 V$ is given by

$$\beta(v, s) = \tfrac{1}{8} F \cdot v \cdot s + \tfrac{1}{24} v \cdot F \cdot s .$$

This is precisely the zeroth order term in the connection $\mathscr{D}$ in equation (2). Therefore we recover the connection $\mathscr{D}$ and hence the notion of a supersymmetric supergravity background.

In [9] we classified the filtered deformations of Lie superalgebras $\mathfrak{h}$ of the form in (3). We find the Killing superalgebras of the maximally supersymmetric backgrounds classified in [10] and, surprisingly, nothing else.

Where to from here? We have two avenues of research we are actively pursuing: namely,

(1) the classification of filtered deformations of $\mathbb{Z}$-graded subalgebras of $\mathfrak{p}$ of the form

$$V \oplus S' \oplus \mathfrak{h}_0 \qquad \text{with} \qquad \mathfrak{h}_0 < \mathfrak{so}(V) ;$$

(2) and, in collaboration with Paul de Medeiros, the extension of this work to other supergravity theories.

This latter avenue has already yielded some interesting results: namely, the classification of possible Killing spinor equations and hence possible geometries in which we may define rigidly supersymmetric theories. In 4-dimensions we recover (and extend) results of Fetuccia and Seiberg [11].

Acknowledgments

The research of JMF is supported in part by the grant ST/J000329/1 "Particle Theory at the Tait Institute" from the UK Science and Technology Facilities Council. This note was written while JMF was a guest of the Grupo de Investigación "Geometría diferencial y convexa" of the Universidad de Murcia, and he wishes to thank Ángel Ferrández Izquierdo for the invitation and for making this visit possible. Finally, my thanks to Óscar García-Prada for the invitation to visit the ICMAT and my sincere hope that the force field preventing the physicists from crossing over to their side of the building is eventually overcome.

References

[1] B. Kostant, "Holonomy and the Lie algebra of infinitesimal motions of a riemannian manifold," *Trans. Am. Math. Soc.* **80** (1955) 528–542.

[2] W. Nahm, "Supersymmetries and their representations," *Nucl. Phys.* **B135** (1978) 149–166.

[3] E. Cremmer, B. Julia, and J. Scherk, "Supergravity in eleven dimensions," *Phys. Lett.* **76B** (1978) 409–412.

[4] J. P. Gauntlett and S. Pakis, "The geometry of $D = 11$ Killing spinors," *J. High Energy Phys.* **04** (2003) 039, `arXiv:hep-th/0212008`.

[5] J. M. Figueroa-O'Farrill, P. Meessen, and S. Philip, "Supersymmetry and homogeneity of M-theory backgrounds," *Class. Quant. Grav.* **22** (2005) 207–226, `arXiv:hep-th/0409170`.

[6] J. Figueroa-O'Farrill and N. Hustler, "The homogeneity theorem for supergravity backgrounds," *JHEP* **1210** (2012) 014, `arXiv:1208.0553 [hep-th]`.

[7] S.-J. Cheng and V. G. Kac, "Generalized Spencer cohomology and filtered deformations of **Z**-graded Lie superalgebras," *Adv. Theor. Math. Phys.* **2** (1998), no. 5, 1141–1182.

[8] S.-J. Cheng and V. Kac, "Addendum: "Generalized Spencer cohomology and filtered deformations of $\mathbb{Z}$-graded Lie superalgebras" [Adv. Theor. Math. Phys. **2** (1998), no. 5, 1141–1182; mr1688484]," *Adv. Theor. Math. Phys.* **8** (2004), no. 4, 697–709.

[9] J. Figueroa-O'Farrill and A. Santi, "Spencer cohomology and eleven-dimensional supergravity," `arXiv:1511.08737 [hep-th]`.

[10] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Maximally supersymmetric solutions of ten- and eleven-dimensional supergravity," *J. High Energy Phys.* **03** (2003) 048, `arXiv:hep-th/0211089`.

[11] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," *JHEP* **06** (2011) 114, `arXiv:1105.0689 [hep-th]`.

Maxwell Institute and School of Mathematics, The University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom