KILLING SUPERALGEBRAS AND THEIR USES

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Abstract. This is the written version of a seminar I gave at the Rencontres Théoriciennes at the Institut Henri Poincaré in Paris on 18 February 2016. It is based on joint work with Paul de Medeiros and Andrea Santi.

Introduction

I have a dream: to classify $> \frac{1}{2}$-BPS backgrounds of supergravity theories with 32 supercharges.

The problem is still largely open. We have a classification of maximally supersymmetric backgrounds [1] and non-existence results for backgrounds with 31 [2, 3] and 30 [4] supersymmetries, but little in the way of a classification below that. Nevertheless we now know that they are (locally) homogeneous, a conjecture of Patrick Meessen which Noel Hustler and I proved a few years ago [6], and that gives an approach, albeit not a leisurely one, to classifying backgrounds with prescribed (semisimple) symmetries, as in the recent [5]. A key ingredient in the proof of homogeneity is the construction of the **Killing superalgebra**: the Lie superalgebra generated by the Killing spinors, which had been obtained with Patrick Meessen and Simon Philip for d=11 supergravity [7] and with Emily Hackett-Jones and George Moutsopoulos for the ten-dimensional theories [8].

It was observed recently in joint work with Andrea Santi, that the Killing superalgebra is a *filtered deformation* of a subalgebra of the Poincaré superalgebra. The word "deformation" should alert us to the existence of a cohomology theory controlling (at least the infinitesimal) such deformations. This is *Spencer cohomology*, which is a refinement of the Chevalley–Eilenberg cohomology of a Lie superalgebra with values in a module. This fact gives us a chance to use cohomological techniques (some extant [9, 10], some under development) in the classification problem.

In determining these Spencer cohomology groups, we noticed that these groups encode a lot of information about the supergravity theory and in particular about the possible Killing spinor equations on which to build rigidly supersymmetric theories in curved spaces. It is very gratifying when a technique solves a problem for which it was not designed and I'd like to tell you about it.

The outline of this talk is the following:

- What are filtered deformations?
- Spencer cohomology and Killing spinors
- Some classifications (d=4 and d=11 thus far)
- Outlook

Filtered deformations

Let (V, η) be a real d-dimensional lorentzian vector space (mostly minus). In this talk, $d = 4$ or $d = 11$. Let $\mathfrak{so}(V)$ be the corresponding Lorentz Lie algebra and let $C\ell(V)$ be the corresponding Clifford algebra. Let S be an irreducible Clifford module. (This is unique in $d = 4$, but requires a choice in $d = 11$.) The **Poincaré superalgebra** is the $\mathbb{Z}$-graded Lie superalgebra with underlying vector space

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2} = \mathfrak{so}(V) \oplus S \oplus V \ .$$

On S we have an $\mathfrak{so}(V)$-invariant symplectic structure $\langle -, - \rangle$ obeying

$$\langle v \cdot s_1, s_2 \rangle = - \langle s_1, v \cdot s_2 \rangle \ ,$$

for all $s_1, s_2 \in S$ and $v \in V \subset C\ell(V)$. The transpose of Clifford action $V \otimes S \to S$ gives a way to square spinors: namely, a map $\kappa : \odot^2 S \to V$ known as the **Dirac current**:

$$\eta(\kappa(s,s), v) = \langle s, v \cdot s \rangle \ ,$$

for all $v \in V$ and $s \in S$. The nonzero Lie brackets of the Poincaré superalgebra are

$$[A, B] = AB - BA \qquad [A, s] = As \qquad [A, v] = Av \qquad [s, s] = \kappa(s, s) \ ,$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$ and $v \in V$. It is a Lie superalgebra with $\mathfrak{p}_{\bar 0} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-2}$ and $\mathfrak{p}_{\bar 1} = \mathfrak{p}_{-1}$. We are also interested in $\mathbb{Z}$-graded subalgebras $\mathfrak{a} \subset \mathfrak{p}$, where

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} = \mathfrak{h} \oplus S' \oplus V' ,$$

where $\mathfrak{h} \subset \mathfrak{so}(V)$, $S' \subset S$ and $V' \subset V$. It follows that if $\dim S' > \frac{1}{2}\dim S$, then $V' = V$. This is the linear algebraic fact underlying the homogeneity of $> \frac{1}{2}$-BPS backgrounds.

By a **filtered deformation** of $\mathfrak{a}$ we mean a filtered Lie superalgebra $\mathfrak{g}$ whose associated graded Lie superalgebra is isomorphic to $\mathfrak{a}$.

Let's unpack this definition. The Lie superalgebra $\mathfrak{g}$ has the same underlying vector space as $\mathfrak{a}$, but the Lie brackets of $\mathfrak{g}$ are obtained by adding to those of $\mathfrak{a}$ terms with positive filtration degree. Since the $\mathbb{Z}$-grading is compatible with the $\mathbb{Z}_2$-grading, the filtration degree is always even. Explicitly, the most general filtered deformation $\mathfrak{g}$ of $\mathfrak{a}$ is given by

$$\begin{aligned}
[A, B] &= AB - BA\\
[A, s] &= As\\
[A, v] &= Av + t\alpha(A, v)\\
[s, s] &= \kappa(s, s) + t\gamma(s, s)\\
[v, s] &= t\beta(v, s)\\
[v, w] &= t\tau(v, w) + t^2\rho(v, w)
\end{aligned}$$

where $\alpha : \mathfrak{h} \otimes V' \to \mathfrak{h}$, $\gamma : \odot^2 S' \to \mathfrak{h}$, $\beta : V' \otimes S' \to S'$, $\tau : \Lambda^2 V' \to V'$ and $\rho : \Lambda^2 V \to \mathfrak{h}$ are subject to the Jacobi identities for all values of the parameter t, which has been introduced to keep track of the filtration degree. When $t = 0$, the Lie brackets are those of $\mathfrak{a}$ and when $t = 1$, those of $\mathfrak{g}$.

Spencer cohomology and Killing spinors

Infinitesimal filtered deformations of $\mathfrak{a}$ are governed by a Spencer cohomology group $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$. It is the degree-2 part of the Chevalley–Eilenberg cohomology of the Lie superalgebra $\mathfrak{a}_- = \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2}$ relative to the representation $\mathfrak{a}$. To compute this groups one bootstraps the calculation of $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$.

For $d = 11$, one finds [11]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^4 V$$

and for $d = 4$ one finds [12]

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \Lambda^0 V \oplus V \oplus \Lambda^4 V .$$

I trust this is familiar: it's the 4-form of eleven-dimensional supergravity and the scalar, vector and pseudoscalar auxiliary fields in the minimal off-shell formulation of N=1 d=4 supergravity (see, e.g., [13]).

But that's not all! The component $\beta : V \otimes S \to S$ of the Spencer cocycle is given for $d = 11$ by

$$\beta(v, s) = v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s \qquad \exists \varphi \in \Lambda^4 V ,$$

and in d=4 by

$$\beta(v, s) = v \cdot (A + B\gamma_5) \cdot s - C \cdot v \cdot \gamma_5 \cdot s + \eta(C, v)\gamma_5 \cdot s \qquad \exists A, B \in \mathbb{R}, \quad v \in V ,$$

which we recognise as the degree-0 piece of the gravitino variation in the respective supergravity theory. In other words, they indicate what the relevant Killing spinor equations are: namely,

$$\nabla_X \varepsilon = \tfrac{1}{24} X \cdot F \cdot \varepsilon - \tfrac{1}{8} F \cdot X \cdot \varepsilon$$

in $d = 11$ (with $\varphi = \frac{1}{24}F$) and

$$\nabla_X \varepsilon = X \cdot (A + B\gamma_5) \cdot \varepsilon - C \cdot X \cdot \gamma_5 \cdot \varepsilon + \eta(C, X)\gamma_5 \cdot \varepsilon$$

in $d = 4$.

It is worth remarking that at least in d=11 supergravity, the Killing spinor equation encodes *all* the information about bosonic backgrounds. Indeed, writing the Killing spinor equation as $D\varepsilon = 0$, then the Clifford trace of the curvature of D vanishes if and only if $dF = 0$ and the Einstein and Maxwell equations are satisfied [14]. So in a very real sense, the Spencer cohomology knows about d=11 supergravity and also knows about the off-shell formulation of N=1 d=4 supergravity.

Classifications

Now we come to some classifications: the results are not new, but the derivations are. We classify filtered deformations of subalgebras $\mathfrak{a}$ of the Poincaré superalgebra of the form $\mathfrak{h} \oplus S \oplus V$. The fact that $S' = S$ means we have maximal supersymmetry, whereas the fact that $V' = V$ (which is forced) means we are describing (locally) homogeneous geometries.

For $d{=}11$ we recover the Killing superalgebras of the maximally supersymmetric backgrounds:

- $\mathfrak{p}$ itself for Minkowski spacetime,
- $\mathfrak{osp}(8|4)$ for $\mathrm{AdS}_4 \times S^7$ [15],
- $\mathfrak{osp}(2,6|4)$ for $S^4 \times \mathrm{AdS}_7$ [16], and
- the Killing superalgebra of the maximally supersymmetric pp-wave [17, 18].

It is worth pausing to recall the usual path to this result starting also from the Poincaré superalgebra. Nahm [19] noticed that the smallest massless unitary irreducible representation of the Poincaré superalgebra is induced from the following unitary representation of the little group $\mathrm{Spin}(9)$:

$$\odot^2_0 W \oplus \Lambda^3 W \oplus (W \otimes \Sigma)_0 ,$$

where W is the vector representation (real and 9-dimensional), Σ is the spinor representation (real and 16-dimensional) and $(W \otimes \Sigma)_0$ is the kernel of Clifford multiplication $W \otimes \Sigma \to \Sigma$. In terms of Poincaré algebra representations, one recognises a metric, a 3-form potential and a gravitino. It was Cremmer, Julia and Scherk [20] who constructed the supergravity theory with this field content. Many years later, George Papadopoulos and I [1] solved the D-flatness equations to recover the known maximally supersymmetric backgrounds and to show that there were no others.

The $d{=}4$ filtered deformations are the following Killing superalgebras:

- $\mathfrak{p}$ itself for Minkowski spacetime,
- $C = 0$ and $(A, B) \neq 0$ for AdS_4,
- $A = B = 0$ and $C \neq 0$, a Lie group with bi-invariant metric and structure constants given in a pseudo-orthonormal basis by $\star C \in \Lambda^3 V$:
  - C spacelike: $\mathrm{AdS}_3 \times \mathbb{R}$,
  - C timelike: $\mathbb{R} \times S^3$, and
  - C lightlike: the Nappi–Witten group [21].

Only Minkowski spacetime is a maximally supersymmetric vacuum of $d{=}4$ $N{=}1$ supergravity, but all of them can be used as underlying geometries for rigidly supersymmetric field theories as in the approach of Festuccia and Seiberg [13].

Outlook

What's in the horizon? With Andrea Santi we are trying to extend the $d{=}11$ classification to the case of subalgebras $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$, where $\dim S > \dim S' > \frac{1}{2} \dim S$, and with the help of Paul de Medeiros we are computing Spencer cohomology in other dimensions to arrive at possible Killing spinor equations and geometries on which we can define rigidly supersymmetric field theories in dimension > 4.

Watch this (super)space!

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