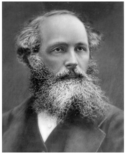
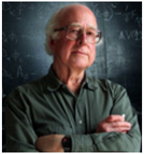
Hyperbolic monopoles and supersymmetry

José Miguel Figueroa O'Farrill





`hep-th` **seminar**

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Supersymmetry underlies any situation where

- 1st order PDE implies 2nd order PDE
- solutions of the 1st order PDE are **optimal** among all solutions of the 2nd order PDE

Well-known examples

- **calibrated geometry**

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- **monopoles**

  $$\text{Bogomol'nyi} \implies \text{Yang–Mills–Higgs}$$

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- Reformulation as BPS configurations in a supersymmetric Yang–Mills–Higgs theory on hyperbolic space
- Determination of the geometry of the monopole moduli space
- Based on joint work (1311.3588) with MOUSTAFA GHARAMTI

Outline of talk

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1. Hyperbolic monopoles
2. Supersymmetric Yang–Mills–Higgs in hyperbolic space

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2. Supersymmetric Yang–Mills–Higgs in hyperbolic space
3. The geometry of the monopole moduli space
4. Conclusions and future directions

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  - $d_A \phi = d\phi + [A, \phi]$
  - $\star$ is the Hodge star operator
- A pair (A, ϕ) satisfying the Bogomol'nyi equation is called a **euclidean monopole**

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- In other words, euclidean monopoles are translationally invariant instantons

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using polar coordinates in the (x_3, x_4) plane:

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- Instantons on $\mathbb{R}^4 \setminus \mathbb{R}^2$ invariant under rotations in the (x_3, x_4) plane give solutions of the Bogomol'nyi equation in H^3

$$d_A \phi = - \star F_A$$

where ϕ is the θ-component of $\mathscr{A}$

Mass and charge

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- We can rescale the mass to unity, but this changes the curvature of H^3 from -1 to $-1/m^2$
- a hyperbolic monopole extends to a rotationally invariant instanton on **all** of $\mathbb{R}^4$ if and only if $m \in \mathbb{Z}$

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- the metric for $k = 2$ is known explicitly, as is the metric for well-separated monopoles MANTON (1985)

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- Since $a_1, \ldots, a_k, b_1, \ldots, b_k$ are complex numbers, $\mathscr{M}_{k,m}$ is a real $4k$-dimensional manifold

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- It is still an open problem to relate the Hitchin family of metrics to the gauge theory

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- Pluricomplex manifolds have a 2-sphere worth of integrable complex structures, but no compatible metric
- Neither are they hypercomplex; although they can be characterised as admitting a complex-linear hypercomplex structure on the complexification of the tangent bundle
- In the euclidean limit, the pluricomplex structure gives rise to a hyperkähler structure BIELAWSKI+SCHWACHHÖFER (2012)

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- The lack of L^2 metric means that there is no effective action for the moduli
- But we can constrain the geometry by demanding the closure of the supersymmetry algebra
- This is reminiscent of 4d Wess–Zumino sigma models without actions, in which case the target space geometry need not be Kähler STELLE+VAN PROEYEN (2003)

Supersymmetric Yang–Mills theories

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- Deform the theory from $\mathbb{R}^3$ to H^3

The lagrangian

The lagrangian density is given by

$$\mathscr{L} = -i\chi^\dagger \not{\mathscr{D}}\psi - \chi^\dagger[\phi, \psi] - i\lambda\chi^\dagger\psi - \tfrac{1}{4}F^2 - \tfrac{1}{2}|\mathscr{D}\phi|^2 - \tfrac{1}{2}D^2$$

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- $\mathscr{D}$ is the fully covariant derivative: $\mathscr{D}_i = \nabla_i + [A_i, -]$
- and $-\lambda^2$ is proportional to the scalar curvature of H^3

Supersymmetry transformations (I)

$\mathscr{L}$ transforms as

$$\delta_L \mathscr{L} = \nabla_i \left(-i\chi^\dagger \left(\sigma^i D + \sigma_j F^{ij} - i\mathscr{D}^i \phi \right) \epsilon_L \right)$$

under

$$\begin{aligned} \delta_L A_i &= i\chi^\dagger \sigma_i \epsilon_L \\ \delta_L \phi &= \chi^\dagger \epsilon_L \\ \delta_L \chi^\dagger &= 0 \\ \delta_L \psi &= D\epsilon_L + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} - \mathscr{D}_k \phi)\sigma^k \epsilon_L \\ \delta_L D &= i\chi^\dagger \overleftarrow{\not{\mathscr{D}}} \epsilon_L + [\phi, \chi^\dagger]\epsilon_L - i\lambda\chi^\dagger \epsilon_L \end{aligned}$$

provided that

$$\nabla_i \epsilon_L = \lambda \sigma_i \epsilon_L$$

Supersymmetry transformations (II)

$\mathscr{L}$ also transforms as

$$\delta_R \mathscr{L} = \nabla_i \left(\varepsilon^{ijk} \epsilon_R^\dagger \left(-\tfrac{1}{2} F_{jk} + i \mathscr{D}_j \phi \sigma_k \right) \psi \right)$$

under

$$\begin{aligned}
\delta_R A_i &= -i \epsilon_R^\dagger \sigma_i \psi \\
\delta_R \phi &= -\epsilon_R^\dagger \psi \\
\delta_R \chi^\dagger &= -D \epsilon_R^\dagger - i (\tfrac{1}{2} \varepsilon_{ijk} F^{ij} + \mathscr{D}_k \phi) \epsilon_R^\dagger \sigma^k \\
\delta_R \psi &= 0 \\
\delta_R D &= i \epsilon_R^\dagger \not{\mathscr{D}} \psi + \epsilon_R^\dagger [\phi, \psi] + i \lambda \epsilon_R^\dagger \psi
\end{aligned}$$

provided that

$$\nabla_i \epsilon_R^\dagger = -\lambda \epsilon_R^\dagger \sigma_i$$

Closure

The above supersymmetry transformations obey the following superalgebra:

$$[\delta_L, \delta_L'] = 0 = [\delta_R, \delta_R'] \qquad [\delta_L, \delta_R] = \mathscr{L}_\xi + \delta_\Lambda^{\text{gauge}} + \delta_\varpi^R$$

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- δ_ϖ^R is an R-symmetry transformation:

$$\delta_\varpi^R \psi = i\varpi\psi \qquad \text{and} \qquad \delta_\varpi^R \chi^\dagger = -i\varpi\chi^\dagger$$

with $\varpi = -4\lambda\epsilon_R^\dagger \epsilon_L$, which is indeed constant

Some remarks

- All fields are **complex** and the lagrangian as written is not real
- The theory has **8 real supercharges**, because $\epsilon_{L,R}$ are Killing spinors on $\mathbb{H}^3$, which admits the maximum number of Killing spinors with either sign of the Killing constant
- Similar (but not identical) to supersymmetric theories in "Family A" in work of Blau (2000)

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- Similarly, bosonic configurations with $\mathscr{D}_k \phi = -\frac{1}{2}\varepsilon_{ijk}F^{ij}$ and $D = 0$ are precisely the ones which preserve the δ_R supersymmetries
- We will study the moduli space $\mathscr{M}$ of bosonic configurations preserving the δ_R supersymmetries

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- For hyperbolic monopoles there is no natural riemannian metric, so we will employ supersymmetry to define this complement

Fermionic zero modes

- A **fermionic zero mode** $\dot{\psi}$ is a solution of the (already linear) Dirac equation in the presence of the monopole $\mathscr{A}_0 = (A(0), \phi(0))$:

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- But we will instead use supersymmetry ZUMINO (1977)

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- The generalised Gauss Law is invariant under $\mathscr{G}$ and defines a complement to the tangent space to the gauge orbits

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- In summary, there are linear maps (parametrised by Killing spinors on H^3) mapping between bosonic and fermionic zero modes
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- But it is easier to see this in a four-dimensional formalism

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- The Killing spinor equations in H^3 become

$$\nabla_i \eta_R = -i\lambda \Gamma_i \Gamma_4 \eta_R \qquad \nabla_i \zeta_R^\dagger = -i\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i$$

in addition to $\nabla_4 \eta_R = 0$ and $\nabla_4 \zeta_R^\dagger = 0$

Zero modes in four-dimensional formalism

- In this formalism, a **fermionic zero mode** $\dot{\Psi}_L = \begin{pmatrix} \dot{\psi} \\ 0 \end{pmatrix}$ obeys

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- Of course, $\nabla_4\dot{\Psi}_L = 0$ and $\nabla_4\dot{A}_\mu = 0$

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- We have real bilinear maps

$$\begin{aligned} K^+ \times Z_0 &\to Z_1 \\ (\eta_R, \dot{A}_\mu) &\mapsto i\dot{A}_\mu \Gamma^\mu \eta_R \end{aligned} \quad \text{and} \quad \begin{aligned} K^- \times Z_1 &\to Z_0 \\ (\zeta_R, \dot{\Psi}_L) &\mapsto -i\zeta_R^\dagger \Gamma_\mu \dot{\Psi}_L \end{aligned}$$

Supersymmetry between zero modes (IV)

- We can compose them:

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- In particular, both maps are isomorphisms and hence $\dim Z_0 = \dim Z_1$

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- If we normalise $\zeta_R^\dagger \eta_R = 1$, then E is a complex structure

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- Since η_R and ζ_R are Killing spinors, $\nabla_4 E_{\mu\nu} = 0$ and

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- If $\dot{A}_{a\mu}$ denotes a basis for Z_0, then

$$E_\mu{}^\nu \dot{A}_{a\nu} = \mathscr{E}_a{}^b \dot{A}_{b\mu}$$

defines an almost complex structure $\mathscr{E}$ on $T_{\mathbb{C}}\mathscr{M}$

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- This implies that if $\dot{A}_\mu$ is a bosonic zero mode, so is $E_\mu{}^\nu \dot{A}_\nu$
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  defines an almost complex structure $\mathscr{E}$ on $T_{\mathbb{C}}\mathscr{M}$
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- Supersymmetry $\implies$ they are integrable

Linearising the supersymmetry transformations (I)

- In 4d-language, the supersymmetry transformation of A_μ is

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- but also $\delta_\epsilon \dot{A}_\mu = -i\epsilon_R^\dagger \Gamma_\mu \dot{\Psi}_{La}\theta^a = \dot{A}_{a\nu}\epsilon_R^\dagger \Gamma_\mu \Gamma^\nu \eta_R \theta^a$

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- Putting both together and using the Clifford relations

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- Since the $\dot{A}_{a\mu}$ are a basis,

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- In contrast with the case of euclidean monopoles, there is no action for this sigma model due to the lack of natural riemannian metric on $\mathscr{M}$
- Since hyperbolic monopoles are $\frac{1}{2}$-BPS, we expect that this sigma model should have **4 real supercharges**, although (in this talk) I work with two supercharges at a time

Closing the supersymmetry algebra (I)

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- The closure on the θ^{a} gives no further constraints

The pluricomplex structure

- We have shown that for all $\eta_R \in K^+$ and $\zeta_R \in K^-$ such that $\zeta_R^\dagger \eta_R = 1$, there is an integrable complex structure $\mathscr{E}$ on $T_{\mathbb{C}}\mathscr{M}$ acting complex linearly

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- This defines a **pluricomplex structure** on $\mathscr{M}$
- This means that the moduli X^a and θ^a belong to a multiplet of the $d = 1$ $N = 4$ **supersymmetry algebra**, as expected for $\frac{1}{2}$-BPS configurations

1. Hyperbolic monopoles
2. Supersymmetric Yang–Mills–Higgs in hyperbolic space
3. The geometry of the monopole moduli space
4. Conclusions and future directions

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- We have shown that there is a supersymmetry relating the bosonic and fermionic moduli
- Closing the algebra requires a pluricomplex structure on the moduli space

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- Can the pluricomplex structure be used to analyse the dynamics of hyperbolic monopoles?
- Pluricomplex manifolds have a unique torsion-free connection leaving the complex structures invariant. Are geodesics with respect to that connection perhaps the trajectories of low-energy hyperbolic monopoles?