Supersymmetry & Lie algebra deformations

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1511.08737	1605.00881	1804.00319
1511.09264	1608.05915	works in progress

Part I

Lie algebras of Killing vectors

Question What is the structure of the Lie algebra of isometries of (M,g)?

$$\mathfrak{g} = \{ \xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = 0 \}$$

$$\Updownarrow$$

$$\nabla \xi \in \underline{\mathfrak{so}}(TM) \qquad (\nabla_a \xi_b = -\nabla_b \xi_a)$$

$$(\nabla \xi)(Y) := \nabla_Y \xi$$

Notation $A_\xi = -\nabla \xi$

Killing identity $\nabla_X A_\xi = R(\xi, X) \Rightarrow \xi \in \mathfrak{g}$ determined by $\xi_p, (\nabla \xi)_p$

Killing transport [Kostant '55, Geroch '69]

$$E := TM \oplus \underline{so}(TM)$$

connection D $$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A + R(X,\xi) \end{pmatrix}$$

$$\boxed{D\begin{pmatrix} \xi \\ A \end{pmatrix} = 0 \iff \xi \text{ Killing}, \quad A = -\nabla\xi}$$

$p \in M$ $\quad \xi$ KV $\quad (v, A) \in E_p$

$\quad \eta$ KV $\quad (w, B)$

What about $[\xi, \eta]$? $\quad (Aw - Bv,\ AB - BA + R(v,w))$

$$\mathfrak{g} \subset E_p = T_pM \oplus \underline{so}(T_pM)$$

$$[(v,A),(w,B)] = (Aw - Bv,\ [A,B] + R(v,w))$$

Exercise: Prove directly the Jacobi identity.

(Hint: you will need both Bianchi identities.)

Grading: $\underset{-1}{T_pM} \oplus \underset{0}{\underline{so}(T_pM)}$ $\qquad \underset{2}{R} : \Lambda^2 T_pM \to \underline{so}(T_pM)$

$R = 0 \Rightarrow$ euclidean Lie algebra $\mathfrak{e}$ $\mathbb{Z}$-graded

$R \neq 0 \rightarrow$ isometry Lie algebra $\mathfrak{g}$ filtered

$$\mathfrak{g} = F^{-1}\mathfrak{g} \supset \underbrace{F^0\mathfrak{g}}_{\subset\, \underline{so}(T_pM)} \supset F^1\mathfrak{g} = 0 \quad \Rightarrow \quad gr(\mathfrak{g}) < \mathfrak{e}$$

Filtered deformations of graded Lie algebras

Lie algebra on V $\Leftrightarrow$ $\varphi \in \Lambda^2 V^* \otimes V$ $\quad [\![\varphi,\varphi]\!] = 0$

Nijenhuis-Richardson bracket

Deformations $\quad \Phi(t) = \varphi + \underbrace{\sum_{n \geq 1} t^n \varphi_n}_{\varphi_+(t)} \qquad \varphi_n \in \Lambda^2 V^* \otimes V$

$$0 = [\![\Phi,\Phi]\!] = \cancel{[\![\varphi,\varphi]\!]} + \underbrace{2[\![\varphi,\varphi_+]\!] + [\![\varphi_+,\varphi_+]\!]}_{\partial\varphi_+ + \frac{1}{2}[\![\varphi_+,\varphi_+]\!] = 0}$$

Maurer-Cartan

$\partial = [\![\varphi, -]\!]$ Chevalley-Eilenberg

$$\partial : \underset{\Lambda^p \mathfrak{g}^* \otimes \mathfrak{g}}{\underset{\parallel}{C^p(\mathfrak{g};\mathfrak{g})}} \longrightarrow C^{p+1}(\mathfrak{g};\mathfrak{g}) \qquad \partial^2 = 0$$

$$\partial\varphi_+ + \tfrac{1}{2}[\![\varphi_+,\varphi_+]\!] = 0 \qquad \varphi_+ = \sum_{n\geq 1} t^n \varphi_n$$

$O(t)$ $$\partial\varphi_1 = 0 \Rightarrow [\varphi_1] \in H^2(\mathfrak{g};\mathfrak{g})$$

$O(t^2)$ $$\partial\varphi_2 + \tfrac{1}{2}[\![\varphi_1,\varphi_1]\!] = 0 \Rightarrow \Big[[\![\varphi_1,\varphi_1]\!]\Big] = 0 \in H^3(\mathfrak{g};\mathfrak{g})$$

etc ...

$$\therefore \quad H^2(\mathfrak{g};\mathfrak{g}) = \{\text{infinitesimal deformations}\}$$

$$H^3(\mathfrak{g};\mathfrak{g}) \ni \{\text{obstructions}\}$$

Remark

$$\mathfrak{g} \ \ \mathbb{Z}\text{-graded} \Leftrightarrow \deg\varphi = 0 \Leftrightarrow \deg\partial = 0$$

Can refine by degree $\quad C^\bullet(\mathfrak{g};\mathfrak{g}) = \bigoplus_{\text{degree } d} C^{d,\bullet}(\mathfrak{g};\mathfrak{g})$

Filtered deformations $\equiv$ deformations in positive degree

e.g., $\quad \deg R = 2$ in LA of isometries

Fact $\quad$ Infinitesimal filtered deformations are governed by

$$H^{d,2}(\mathfrak{g}_-;\mathfrak{g}) \qquad d>0$$

negative degree subalgebra

This is called (generalised) Spencer cohomology

Example

$$\underset{-1}{\mathfrak{g} = V} \oplus \underset{0}{\mathfrak{so}(V)}$$

(V, η) inner product space

e.g., (T_pM, g_p)

$$\mathfrak{g}_- = V$$

degree 1

$$C^{1,1} = V^* \otimes \mathfrak{so}(V)$$
$$C^{1,2} = \Lambda^2 V^* \otimes V$$

$$\partial : C^{1,1} \xrightarrow{\cong} C^{1,2} \Rightarrow H^{1,2} = 0$$

degree 2

$$C^{2,2} = \Lambda^2 V^* \otimes \mathfrak{so}(V)$$
$$C^{2,3} = \Lambda^3 V^* \otimes V$$

$$R : \Lambda^2 V \longrightarrow \mathfrak{so}(V)$$

$$(\partial R)(u,v,w) = R(u,v)w + R(v,w)u + R(w,u)v = 0$$

algebraic Bianchi identity

$$\therefore \quad H^{2,2} = \{\text{algebraic curvature tensors}\}$$

Summary

(M, g)

1. Killing vectors $\longleftrightarrow$ parallel sections of $E = TM \oplus \mathfrak{so}(TM)$

2. Lie algebra of KVs is filtered and associated graded LA $< \mathfrak{e}$

   Lie algebra of KVs of flat space

   $\Rightarrow$ a filtered deformation

3. Infinitesimal filtered deformations of $\mathfrak{g}$ are governed by "Spencer cohomology" $H^{d,2}(\mathfrak{g}_- ; \mathfrak{g})$ $d > 0$

4. $H^{2,2}(\mathfrak{e}_- ; \mathfrak{e}) \cong \{$ algebraic curvature tensors $\}$

   (but Jacobi requires differential Bianchi identity)

Part II

Supersymmetry

A "square root" of Killing transport

(M,g) spin $\qquad \$ \longrightarrow M$ spinor bundle $\qquad (Cl(TM)\text{-modules})$

Promote $E = TM \oplus \underline{so}(TM)$ to a super VB

$$\mathcal{E} = \mathcal{E}_{\bar{0}} \oplus \mathcal{E}_{\bar{1}} = E \oplus \$$$

Extend the Killing transport connection to a superconnection

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A + R(X,\xi) \end{pmatrix} \qquad \sigma \in \Gamma(\$), \quad D_X \sigma = ?$$

Want: D-parallel sections of $\mathcal{E}$ define a Lie (super)algebra?

Lie superalgebras

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \qquad [-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \quad \text{(even)}$$

$$[X,Y] = -(-1)^{|X||Y|}[Y,X]$$

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]]$$

$[-,-]$ has three components:

$$\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}_{\bar{0}}$$

$$\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{1}}$$

$$\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$$

Jacobi has four components:

$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \Leftrightarrow \mathfrak{g}_{\bar{0}}$ is a LA

$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \Leftrightarrow \mathfrak{g}_{\bar{1}}$ is $\mathfrak{g}_{\bar{0}}$-module

$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \Leftrightarrow \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$ is $\mathfrak{g}_{\bar{0}}$-equivariant

$[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$

Example The Poincaré superalgebra

(V,η) lorentzian vector space

$\mathfrak{so}(V) \subset Cl(V)$ Clifford algebra $\quad v^2 = -\eta(v,v)\mathbb{1}$

S a Clifford module

$\langle -,- \rangle$ an inner product on S (could be symplectic)

$$\langle s_1, v\cdot s_2\rangle = \langle s_2, v\cdot s_1\rangle \quad \forall\, v\in V,\ s_1,s_2\in S$$

$\Longrightarrow$ $\kappa : S\times S \longrightarrow V$ Dirac current

$$\eta(\kappa(s_1,s_2),v) = \langle s_1, v\cdot s_2\rangle$$

$$\boxed{\cdot} = \underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V}$$

$\mathbb{Z}$-graded

$\boxed{\cdot}_{\bar 0} = \mathfrak{so}(V)\oplus V \qquad \boxed{\cdot}_{\bar 1} = S$

$s_1,s_2\in S \qquad [s_1,s_2] = \kappa(s_1,s_2)$

Killing superalgebras

(M, g) $\qquad \mathcal{E} = E \oplus \$$ $\qquad D$

Suppose

1) $\$$ has an inner product $\langle -,- \rangle$ satisfying

$$\langle \sigma_1, X\cdot\sigma_2 \rangle = \langle \sigma_2, X\cdot\sigma_1 \rangle \qquad \forall\, \sigma_1,\sigma_2 \in \Gamma(\$),\ X \in \mathfrak{X}(M)$$

2) the Dirac current $K : \$ \times \$ \to TM$ obeys

$$\underbrace{D\sigma_1 = D\sigma_2 = 0}_{\text{Killing spinors}} \Rightarrow K(\sigma_1,\sigma_2) \text{ is a KV}$$

3) $[\mathcal{L}_{K(\sigma_1,\sigma_2)}, D] = 0$

spinorial Lie derivative $\qquad \mathcal{L}_\xi \sigma = \nabla_\xi \sigma + \rho(A_\xi)\cdot\sigma$

$\rho : \mathfrak{so}(TM) \to \mathfrak{gl}(\$)$

Then let $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$

$$\mathfrak{k}_{\bar{0}} = \{ \xi \text{ KV} \mid [\mathcal{L}_\xi, D] = 0 \}$$
$$\mathfrak{k}_{\bar{1}} = \{ \sigma \in \Gamma(\$) \mid D\sigma = 0 \}$$

[-, -]

$\mathfrak{k}_{\bar{0}} \times \mathfrak{k}_{\bar{0}} \longrightarrow \mathfrak{k}_{\bar{0}}$ Lie bracket of vector fields

$\mathfrak{k}_{\bar{0}} \times \mathfrak{k}_{\bar{1}} \longrightarrow \mathfrak{k}_{\bar{1}}$ spinorial Lie derivative

$\mathfrak{k}_{\bar{1}} \times \mathfrak{k}_{\bar{1}} \longrightarrow \mathfrak{k}_{\bar{0}}$ Dirac current

Jacobi

$(\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{0}})$ ✓ Jacobi of Lie bracket of VFs

$[\mathcal{L}_\xi, \mathcal{L}_\zeta] = \mathcal{L}_{[\xi,\zeta]}$

$(\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{1}})$ ✓

$(\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}})$ ✓ Equivariance of Dirac current

$(\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}})$? Needs to be checked

If all Jacobis are satisfied, then

1. $\mathfrak{k}$ is a Lie superalgebra
2. $\mathfrak{k}$ is filtered
3. $\mathrm{gr}(\mathfrak{k}) < \boxdot$

ie: Killing superalgebras are filtered deformations of graded subalgebras of the Poincaré superalgebra

Infinitesimal deformations computable from the generalised Spencer cohomology

$$H^{2,2}(\mathfrak{g}_-;\mathfrak{g}) \qquad \mathfrak{g} < \boxdot$$

Part III

Applications & Calculations

Supersymmetric supergravity backgrounds

Fact Every supergravity theory gives D on $\$$ and every background has an associated Killing superalgebra $\mathfrak{k}$.

Example $D=11$ Supergravity [Nahm, Cremmer + Julia + Scherk '78]

$$D_X \sigma = \nabla_X \sigma + \frac{1}{8} F \cdot X \cdot \sigma + \frac{1}{24} X \cdot F \cdot \sigma \qquad F \in \Omega^4(M), \quad dF = 0$$

(Bosonic) field equations $\Leftrightarrow$ vanishing of Clifford trace of curvature of D

Theorem [JMF + Meessen + Philip '04]

(M, g) 11-dimensional spin $\quad F \in \Omega^4(M) \quad dF = 0$

$$\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1} \qquad \begin{aligned} \mathfrak{k}_{\bar 0} &= \{ \xi \text{ KV} \mid \mathcal{L}_\xi F = 0 \} \\ \mathfrak{k}_{\bar 1} &= \{ \sigma \in \Gamma(\$) \mid D\sigma = 0 \} \end{aligned}$$

is a Killing superalgebra.

Theorem [JMF + Hustler '13]

$$\dim \mathfrak{k}_{\bar 1} > \tfrac{1}{2} \operatorname{rank} \$ \implies [\mathfrak{k}_{\bar 1}, \mathfrak{k}_{\bar 1}] < \mathfrak{k}_{\bar 0} \text{ acts locally transitively on } M$$

(sharp)

A cohomological rederivation of D=11 SUGRA?

$\mathfrak{g} = \boxed{\cdot} = \mathfrak{so}(V) \oplus S \oplus V$

(V,η) 11-dim'l Lorentzian

S real, 32-dim'l

Theorem [JMF + Santi '16]

$$H^{2,2}(\mathfrak{g}_-;\mathfrak{g}) \cong \Lambda^4 V \qquad (\mathfrak{so}(V)\text{-mod})$$

$$[\beta + \gamma] \longleftarrow F$$

$\beta : V \longrightarrow \mathrm{End}(S)$

$\gamma : \odot^2 S \to \mathfrak{so}(V)$

$$\boxed{\beta_X s = \tfrac{1}{8} F\cdot X\cdot s + \tfrac{1}{24} X\cdot F\cdot s}$$

Fables of the Reconstuction

The Killing superalgebra (when $\dim \mathfrak{k}_{\bar{1}} > 16$) contains all all the information to reconstruct (M, g, F) ... locally.

Theorem [JMF + Santi '18]

If $\dim \mathfrak{k}_{\bar{1}} > \frac{1}{2} \operatorname{rank} \$$ $(=16)$, one can reconstruct a solution (M, g, F) of the supergravity field equations (up to local isometry) with Killing superalgebra $\mathfrak{k}$.

This allows a purely algebraic reformulation of the classification problem of "$>\frac{1}{2}$-BPS" supergravity backgrounds.

Supersymmetry in curved space

A Killing superalgebra $\mathfrak{k} = \{D\text{-parallel sections of } E \oplus \$\}$ defines a supersymmetry algebra on (M, g).

$\Rightarrow$ (rigidly) supersymmetric field theories on (M, g).

These are key ingredients in the application of supersymmetric localisation to computing exact quantities in QFT.

Spencer cohomology $H^{2,2}(\mathfrak{g}_- ; \mathfrak{g})$ for $\mathfrak{g} = \boxdot$ (in diverse dimensions, possibly extended) suggests connections D on $\$$ whose parallel spinors generate KSAs.

Supergravities

off-shell SUGRA

"conformal" SUGRA

Some calculations

$D=11$	$\Lambda^4 V$	[JMF + Santi '16]
$D=10$ (I)	$\Lambda^3 V$	[de Medeiros + JMF]
$D=6$ (1,0)	$\Lambda^3_+ V \oplus (V \otimes \mathfrak{sp}(1))$	[de Medeiros + JMF + Santi '18]
$D=6$ (1,0) + R	$\Lambda^3 V \oplus (V \otimes \mathfrak{sp}(1))$	[de Medeiros + JMF + Santi '18]
$D=5$	$\Lambda^2 V \oplus (V \otimes \mathfrak{sp}(1))$	[Beckett + JMF '20]
$D=4$ $N=1$	$\Lambda^0 V \oplus V \oplus \Lambda^4 V$	[de Medeiros + JMF + Santi '16]

Future & work in progress

Non-lorentzian : galilean + Bargmann ($D=3$)

Kaluza-Klein reduction

Extended $D=4$, $D=3$

Superconformal $D=3$

Exotic $D=6$