Maximally supersymmetric spacetimes

José Figueroa-O'Farrill

Edinburgh Mathematical Physics Group

School of Mathematics

Neve Shalom $\sim$ Wahat al-Salam, 27 May 2003

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$$D_\mu \begin{pmatrix} \xi_\nu \\ F_{\nu\rho} \end{pmatrix} = \begin{pmatrix} \nabla_\mu \xi_\nu - F_{\mu\nu} \\ \nabla_\mu F_{\nu\rho} - R_{\mu\nu\rho}{}^\sigma \xi_\sigma \end{pmatrix}$$

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The flat and spherical cases are completely solved (culminating in the work of Wolf in the 1970s), but the possible quotients of hyperbolic and lorentzian cases are still unclassified.

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In this talk I will report on progress towards answering this question.

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- George Papadopoulos (King's College, London)
  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170` (*J Geom Phys* to appear)

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- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
  - ⋆ `in preparation`

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA / IIB			I			
9	$N=2$			$N=1$			
8	$N=2$			$N=1$			
7	$N=4$			$N=2$			
6	$(2,2)$ / (3,1) / (4,0)	$(2,1)$ / (3,0)		$(1,1)$ / $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
4	$N=8$	$N=6$	$N=5$	$N=4$	$N=3$	$N=2$	$N=1$

[Van Proeyen, `hep-th/0301005`]

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defined by the supersymmetric variation of the gravitino:

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where I is an index labeling the following elements

$$\Gamma_a \quad \Gamma_{ab} \quad \Gamma_{abc} \quad \Gamma_{abcd} \quad \Gamma_{abcde}$$

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- if the plane is lorentzian, we can choose a pseudo-orthonormal frame in which the only nonzero component of F is F_{0123}
- if the plane is null, we can choose a pseudo-orthonormal frame in which the only nonzero component of F is F_{-123}

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- F null: a one parameter $\mu \in \mathbb{R}$ family of *symmetric plane waves*:

$$g = 2dx^+dx^- - \tfrac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2 \right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

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All vacua embed isometrically in $\mathbb{E}^{2,11}$ as the intersections of two quadrics.

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KG

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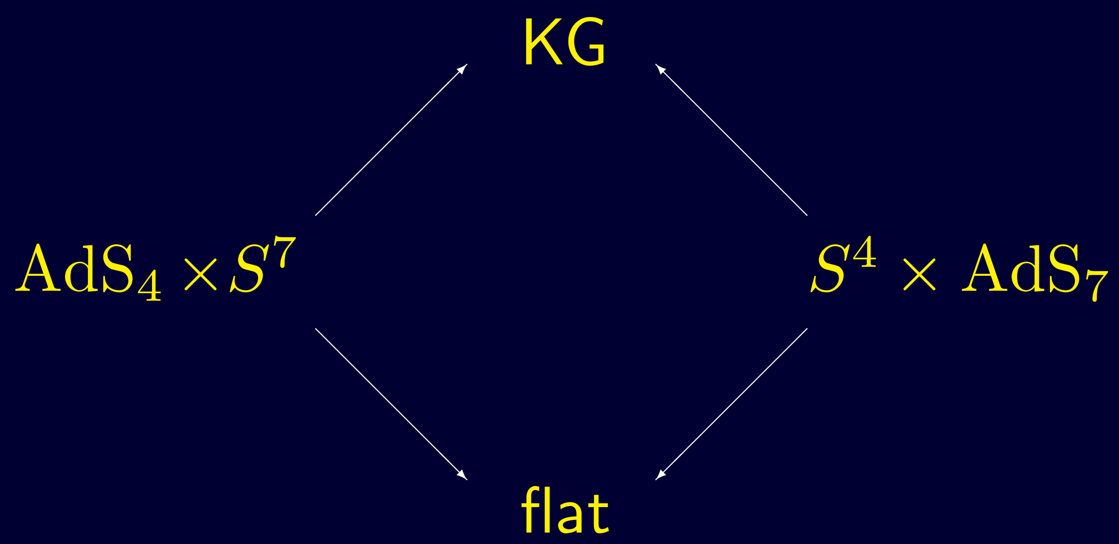

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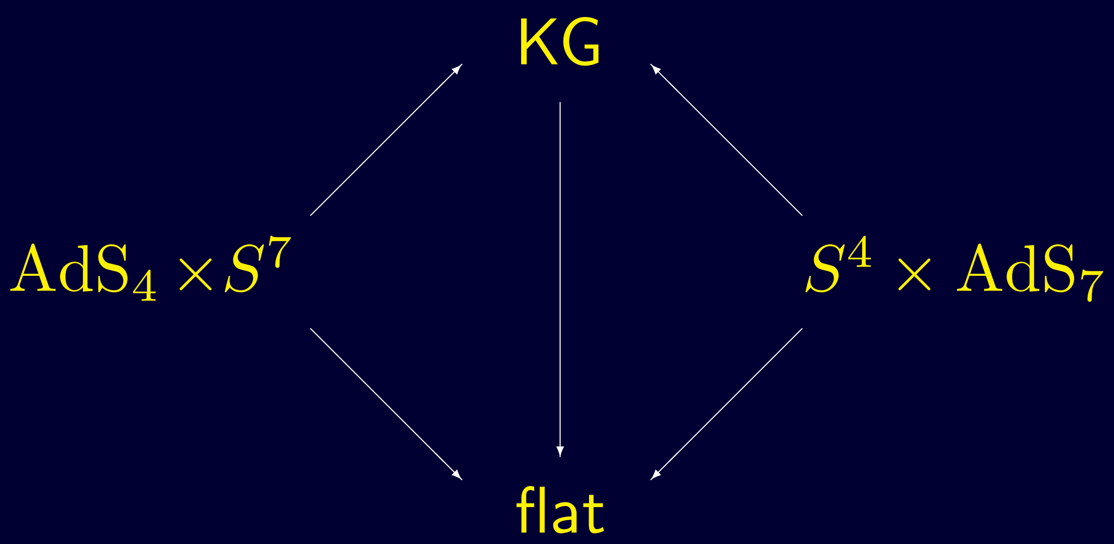

[Back]

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The solution to this problem is known.

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But there is a more general construction.

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- since $\mathfrak{h}$ preserves the metric on $\mathfrak{g}$, there is a linear map

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- $\mathfrak{h}$ acts on $\mathfrak{g} \times_\omega \mathfrak{h}^*$ preserving the Lie bracket, so we can form the *double extension*

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This construction is due to Medina and Revoy who proved an important structure theorem.

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(Any semisimple factors in $\mathfrak{a}$ factor out of the double extension.
[FO–Stanciu hep-th/9402035])

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The first case corresponds to the flat vacuum. The second case corresponds to $\mathrm{AdS}_3 \times S^3$ with equal radii of curvature and

$$F \propto \mathrm{dvol}(\mathrm{AdS}_3) + \mathrm{dvol}(S^3)$$

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The first case corresponds to the flat vacuum. The second case corresponds to $\mathrm{AdS}_3 \times S^3$ with equal radii of curvature and

$$F \propto \mathrm{dvol}(\mathrm{AdS}_3) + \mathrm{dvol}(S^3)$$

The third case is a six-dimensional version of the Nappi-Witten spacetime, NW_6, discovered by Meessen. [Meessen `hep-th/0111031`]

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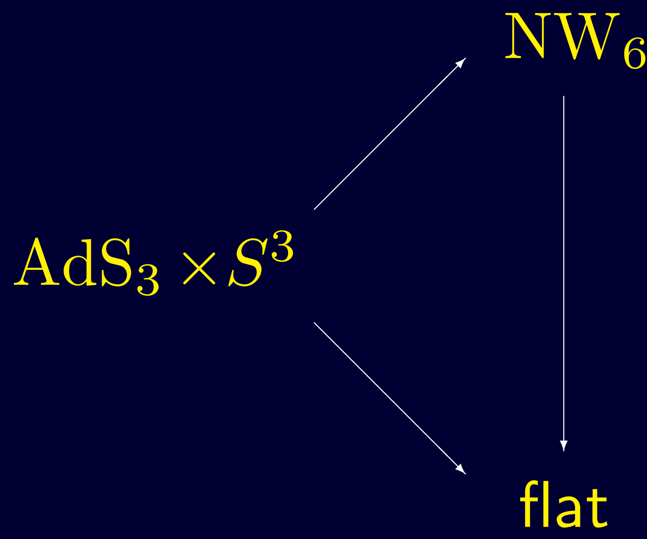

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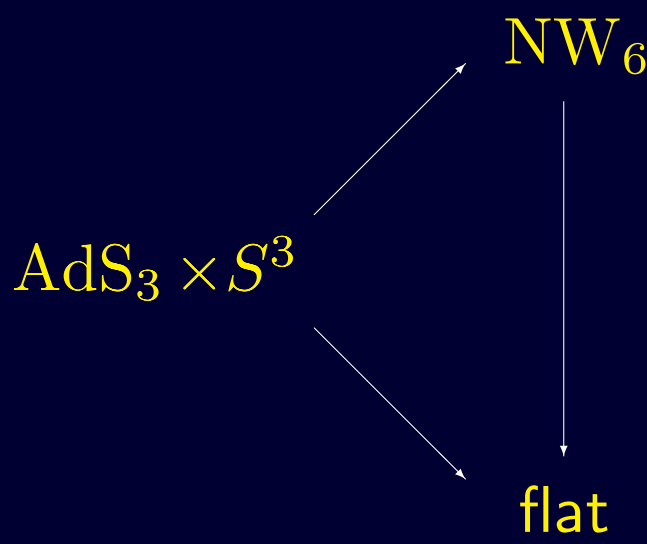

which in this case are *group contractions* à la Inönü–Wigner.

[Stanciu–FO `hep-th/0303212`, Olive–Rabinovici–Schwimmer `hep-th/9311081`]

[Back]

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(Unfortunate notation: a 2-Lie algebra is a Lie algebra.)

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(n-Lie algebras also appear naturally in the context of Nambu dynamics. [Nambu (1973)])

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[FO–Papadopoulos `math.AG/0211170`]

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[Blau–FO–Hull–Papadopoulos `hep-th/0110242`]

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Notice that g is a bi-invariant metric on a Lie group: a ten-dimensional version of the Nappi–Witten spacetime.

[Stanciu–FO `hep-th/0303212`]

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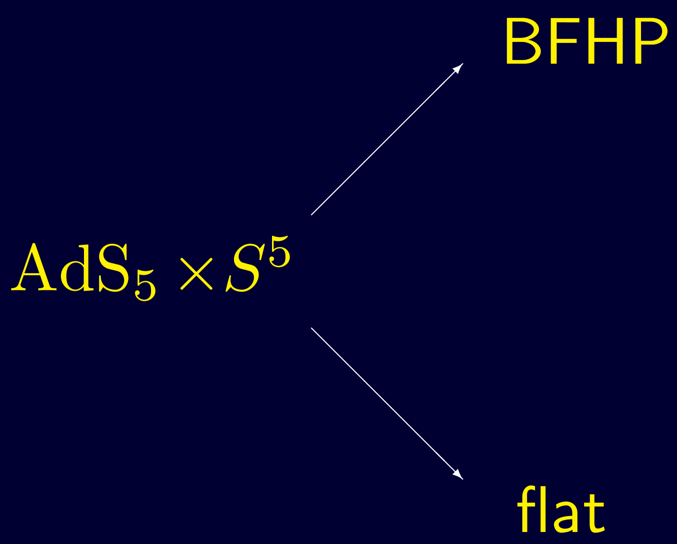

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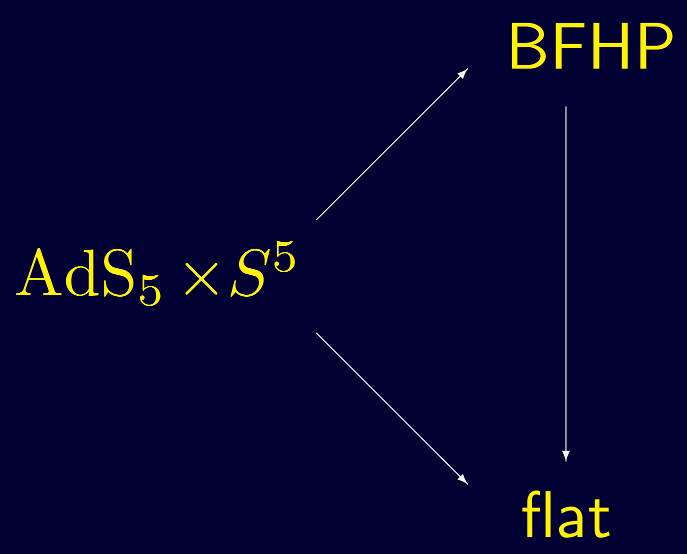

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[Gauntlett–Gutowsky–Hull–Pakis–Reall `hep-th/0209114`]

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This now has a natural explanation.

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Thank you.