THE EQUIVALENCE BETWEEN THE GAUGED WZNW AND GKO CONFORMAL FIELD THEORIES

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ABSTRACT

We quantize the gauged Wess-Zumino-Novikov-Witten model *à la* BRST and prove that the conformal field theory it induces on the physical space is equivalent to the G/H coset construction of Goddard-Kent-Olive. This extends to general H the result obtained by Karabali and Schnitzer for H abelian; although our proof is more direct.

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Introduction

The coset construction of Goddard, Kent, and Olive[1] (GKO) is one of the most important results in two dimensional conformal quantum field theory (CFT). It gives a constructive proof of the unitarity of the CFTs corresponding to the discrete series of representations of $c < 1$ found by Friedan, Qiu, and Shenker[2]. Is is therefore important to have a field theoretic description of such theories.

In [**3**] the authors investigated the path integral quantization of a Wess-Zumino-Novikov-Witten (WZNW) model[4] in which an anomaly free subgroup H of the original $G \times G$ symmetry group had been gauged. The theory which results after gauge fixing and the introduction of the Faddeev-Popov ghosts turns out to be conformal with central charge equal to that of the GKO construction of a G/H coset CFT. However the energy momentum tensor obtained in [**3**] does not agree (as an operator) with the one given by the GKO construction. This is, of course, to be expected since the quantization of the gauged WZNW model contains more degrees of freedom (*e.g.* ghosts) than the ones appearing in the GKO construction. Under these circumstances the best that one can hope is for the difference between the two energy momentum tensors to be zero acting on physical states. It was shown in [**3**] that this is indeed the case when sandwiched between physical states which are also ghost free. This, of course, is not a proof but it lends validity to the conjecture given that from our experience with BRST quantization in string theory we know one can usually find representatives for the physical states which are ghost free.

Later, in [**5**] , Karabali and Schnitzer proposed a method of proof based on the following fact—proven, for example, in [**6**] —: any unitary representation of the Virasoro algebra with vanishing central charge and where L_0 is bounded below is necessarily trivial. Since the gauged WZNW and GKO CFTs have the same conformal central charge, the difference between their energy momentum tensors has zero central charge. Hence all one has to show is the unitarity of the physical states; since it follows by construction that L_0 is bounded below.

Therefore, in this context, the proof of the operator equivalence between the two CFTs boils down to a "no-ghost" theorem. This was proven in [5] for the special case of H abelian using the quartet mechanism of Kugo and Ojima[7]; and in [8] using homological methods. The quartet mechanism consists roughly of explicitly building projectors onto the non-trivial BRST cocycles. This procedure is only effective when the constraint algebra is abelian so that the BRST operator is linear in the ghosts. In cases when the constraints arise from a Lie algebra, however, the resulting BRST operator is none other than the operator which computes the semi-infinite cohomology of Feigin[9]. For this cohomology theory, Frenkel, Garland, and Zuckerman[10],[11] have developed some methods, based on homological algebra and representation theory, which have been used successfully to give the simplest and most enlightening proofs of the "no-ghost" theorems in string theory[10],[11],[12],[13]. However[8] if one tries to apply these methods to the present case, one runs into technical difficulties which have only been resolved, interestingly enough, for the abelian case.

In this paper we follow a more direct approach to prove the equivalence between the gauged WZNW and GKO CFTs. In particular, we do not need the "no-ghost" theorem; which is not to say that proving the "no-ghost" theorem is not an interesting result in its own right. Our method consists of explicitly building a chain homotopy connecting the difference of the energy momentum tensors with the zero operator; that is, we construct an operator whose anticommutator with the BRST operator yields the difference between the energy momentum tensors of the two CFTs whose equivalence we want to prove. Conceptually, therefore, the proof is extremely simple; albeit not very enlightening.

In the next section we review the results of Karabali and Schnitzer concerning the CFT arising from the gauged WZNW model. Then we discuss the BRST quantization of the gauged WZNW theory, explain the idea behind the equivalence between the two CFTs, and prove the equivalence of the two CFTs by constructing the chain homotopy explicitly. We then conclude with some open problems.

The Gauged WZNW Model as a CFT

We start by describing the CFT obtained from the gauged WZNW model. In order to minimize the overlap between this paper and [**5**] we will not rederive the CFT from the path integral quantization of a gauged WZNW action. This is done lucidly in [**3**] and [**5**] and so we shall take the CFT as our starting point.

The CFT of the gauged WZNW model consists of the following ingredients. First we have a WZNW CFT with group G and level k. This is described by the current algebra corresponding to the affine Lie algebra $\widehat{\mathfrak{g}}$ at level k; where $\mathfrak{g}$ is the Lie algebra of G. That is we have a set of currents[1] $J^a(z)$ obeying the following operator product expansion (OPE)

$$J^a(z)J^b(w) = \frac{k\gamma^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c}{z-w}J^c(w) + \text{reg} \ , \tag{1}$$

where $a, b = 1, \ldots, d_{\mathfrak{g}} = \dim \mathfrak{g}$ are indices in the algebra $\mathfrak{g}$ and γ^{ab} is a fixed invariant symmetric bilinear form on $\mathfrak{g}$. Throughout this paper we adopt the Einstein summation convention. The energy momentum tensor has the Sugawara form[2]

$$T^{\mathfrak{g}}(z) = \frac{\gamma_{ab}}{2k + c_{\mathfrak{g}}} {:}\, J^a(z)J^b(z){:} \ , \tag{2}$$

where γ_{ab} is the inverse of γ^{ab}. This energy momentum tensor obeys the usual

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[1] We only consider the holomorphic sector of the CFT. The treatment of the antiholomorphic sector is completely analogous.

[2] By the normal ordered product $:A(z)B(z):$ of two operators $A(z)$ and $B(z)$ we mean the following:

$$:A(z)B(z): \equiv \oint_{C_z} \frac{dw}{2\pi i} \frac{1}{w-z} A(w)B(z) \ ,$$

where C_z is a positively oriented contour in the complex w plane encircling z.

OPE

$$T^{\mathfrak{g}}(z)T^{\mathfrak{g}}(w) = \frac{c(\mathfrak{g},k)}{2(z-w)^4} + \frac{2}{(z-w)^2}T^{\mathfrak{g}}(w) + \frac{1}{z-w}\partial T^{\mathfrak{g}}(w) + \text{reg} \ , \tag{3}$$

where

$$c(\mathfrak{g},k) \equiv \frac{2kd_{\mathfrak{g}}}{2k+c_{\mathfrak{g}}} \ , \tag{4}$$

and $c_{\mathfrak{g}}$ is the eigenvalue of the quadratic Casimir operator $\gamma_{ab}J_0^aJ_0^b$ in the adjoint representation of $\mathfrak{g}$.

The next ingredient is a WZNW CFT with group $H \subset G$ and level $-(k+c_{\mathfrak{h}})$. This is defined by a set of currents $\widetilde{J}^i(z)$ obeying the OPE

$$\widetilde{J}^i(z)\widetilde{J}^j(w) = \frac{-(k+c_{\mathfrak{h}})\gamma^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k}{z-w}\widetilde{J}^k(w) + \text{reg} \ , \tag{5}$$

where γ^{ij} is the restriction of γ^{ab} to $\mathfrak{h}$. The energy momentum tensor again has the standard Sugawara form

$$\widetilde{T}^{\mathfrak{h}}(z) = \frac{-\gamma_{ij}}{2k+c_{\mathfrak{h}}}\gamma_{ij}\colon \widetilde{J}^i(z)\widetilde{J}^j(z)\colon \ , \tag{6}$$

where γ_{ij} is the inverse of γ^{ij}. This energy momentum tensor obeys the usual OPE

$$\widetilde{T}^{\mathfrak{h}}(z)\widetilde{T}^{\mathfrak{h}}(w) = \frac{c(\mathfrak{h},-k-c_{\mathfrak{h}})}{2(z-w)^4} + \frac{2}{(z-w)^2}\widetilde{T}^{\mathfrak{h}}(w) + \frac{1}{z-w}\partial\widetilde{T}^{\mathfrak{h}}(w) + \text{reg} \ . \tag{7}$$

The third ingredient is a set of $d_{\mathfrak{h}}$ anticommuting free (b,c) systems of spins (1,0) respectively with OPE given by

$$b^i(z)c_j(w) = \frac{\delta^i_j}{z-w} + \text{reg} = c_j(z)b^i(w) \ . \tag{8}$$

Their energy momentum tensor has the standard form

$$T^{\text{gh}}(z) = -\colon b^i\partial c_i\colon \tag{9}$$

and obeys the standard OPE

$$T^{\mathrm{gh}}(z)T^{\mathrm{gh}}(w) = \frac{-d_{\mathfrak{h}}}{(z-w)^4} + \frac{2}{(z-w)^2}T^{\mathrm{gh}}(w) + \frac{1}{z-w}\partial T^{\mathrm{gh}}(w) + \mathrm{reg}\ . \tag{10}$$

The final ingredient of the theory is the constraint which couples these three independent CFTs:

$$J_{\mathrm{tot}}{}^i(z) \equiv J^i(z) + \widetilde{J}^i(z) + J^i_{\mathrm{gh}}(z)\ , \tag{11}$$

with

$$J^i_{\mathrm{gh}}(z) \equiv f^{ij}{}_k {:}\, b^k(z)c_j(z) {:}\ . \tag{12}$$

The OPEs obeyed by these currents can be easily read from the ones given above and are given by (5) together with

$$J^i(z)J^j(w) = \frac{k\gamma^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k}{z-w}J^k(w) + \mathrm{reg}\ , \tag{13}$$

$$J^i_{\mathrm{gh}}(z)J^j_{\mathrm{gh}}(w) = \frac{c_{\mathfrak{h}}\gamma^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k}{z-w}J^k_{\mathrm{gh}}(w) + \mathrm{reg}\ . \tag{14}$$

Adding the central charges we see that in fact they cancel so that the constraint is first class. This just reiterates the fact that we gauged an anomaly free subgroup. Because of this fact we can build a BRST charge and we are guaranteed that it is square-zero.

BRST Quantization and Proof of Equivalence

We define the BRST operator as the contour integral of the BRST current. That is,

$$Q = \oint_{C_0} \frac{dz}{2\pi i}\, j_{\mathrm{BRST}}(z)\ , \tag{15}$$

where

$$j_{\mathrm{BRST}}(z) = {:}\, c_i(z)[J^i(z) + \widetilde{J}^i(z) + \frac{1}{2}J^i_{\mathrm{gh}}(z)] {:}\ . \tag{16}$$

Therefore to quantize the holomorphic part of this CFT we merely look for suitable representations of the relevant operator algebras. In this case this involves representations of two affine Lie algebras: $\widehat{\mathfrak{g}}$ at level k and $\widehat{\mathfrak{h}}$ at level $-(k + c_{\mathfrak{h}})$; whereas the ghosts—being free fields— are quantized trivially. The physical subspace is then defined as the cohomology of the BRST operator at zero ghost number.

We proceed now to the discussion of the Virasoro algebras appearing in this construction. The total energy momentum tensor $T(z)$ is given by the sum of three commuting terms:

$$T(z) = T^{\mathfrak{g}}(z) + \widetilde{T}^{\mathfrak{h}}(z) + T^{\mathrm{gh}}(z) \ . \tag{17}$$

Adding up the central charges we notice that the total central charge is

$$\begin{aligned} c &= \frac{2kd_{\mathfrak{g}}}{2k + c_{\mathfrak{g}}} + \frac{2(-k - c_{\mathfrak{h}})d_{\mathfrak{h}}}{2(-k - c_{\mathfrak{h}}) + c_{\mathfrak{h}}} - 2d_{\mathfrak{h}} \\ &= \frac{2kd_{\mathfrak{g}}}{2k + c_{\mathfrak{g}}} - \frac{2kd_{\mathfrak{h}}}{2k + c_{\mathfrak{h}}} \ , \end{aligned} \tag{18}$$

which coincides with the central charge of the G/H coset CFT. However the energy momentum tensor of the coset CFT is not the same as $T(z)$. In fact, it is given by

$$T^{\mathrm{GKO}}(z) = T^{\mathfrak{g}}(z) - T^{\mathfrak{h}}(z) \ , \tag{19}$$

where

$$T^{\mathfrak{h}}(z) = \frac{\gamma_{ij}}{2k + c_{\mathfrak{h}}} : J^i(z) J^j(z) : \ . \tag{20}$$

We can therefore split $T(z)$ as a sum of two commuting terms $T^{\mathrm{GKO}}(z) + T'(z)$ where

$$T'(z) = T^{\mathfrak{h}}(z) + \widetilde{T}^{\mathfrak{h}}(z) + T^{\mathrm{gh}}(z) \ . \tag{21}$$

Notice that $T'(z)$ has zero central charge.

Now, the BRST charge can be checked to commute with both $T^{\mathrm{GKO}}(z)$ and $T'(z)$. Hence they are physical operators; that is, they induce operators in the physical space. The conjecture of **[5]** is that $T'(z)$ induces the zero operator on physical states. This is equivalent to showing that there exists an operator $\Theta(z)$ such that

$$\{Q\,,\,\Theta(z)\} = T'(z)\ . \tag{22}$$

Before constructing $\Theta(z)$ we will need to know the OPEs between the BRST current and the other relevant operators or, equivalently, the (anti)commutation relations of the relevant operators with the BRST operator. One easily finds the following:

$$\{Q\,,\,b^i(z)\} = {J_{\rm tot}}^i(z)\ ; \tag{23}$$

$$[Q\,,\,J^i(z)] = k\gamma^{ij}\partial c_j(z) - f^{ij}{}_k c_j(z) J^k(z)\ ; \tag{24}$$

$$[Q\,,\,\widetilde{J}^i(z)] = -(k+c_{\mathfrak{h}})\gamma^{ij}\partial c_j(z) - f^{ij}{}_k c_j(z)\widetilde{J}^k(z)\ . \tag{25}$$

This suffices to prove (22) for

$$\Theta(z) = \frac{\gamma_{ij}}{2k+c_{\mathfrak{h}}} b^i(z)(J^j(z) - \widetilde{J}^j(z))\ , \tag{26}$$

which explicitly shows that $T'(z)$ is chain homotopic to zero.

<u>Conclusion</u>

In this paper we have proven the equivalence between the CFT which arises after gauging an anomaly free subgroup $H \subset G$ in a WZNW model and the CFT which arises out of the coset G/H via the GKO construction. The equivalence consists in showing that the GKO and gauged WZNW energy momentum tensors coincide on the physical states; when these are defined, as is usual in BRST quantization, as (a subspace of) the cohomology of the BRST operator.

The actual proof consisted in explicitly constructing a chain homotopy connecting the zero operator and the difference of the energy momentum tensors. As such the proof is extremely simple conceptually but it lacks the interesting side-effects in the method of proof proposed in [**5**] , which first proves the positive definiteness of the physical scalar product. This was carried out successfully for the special case of H abelian in [**5**] —using the quartet mechanism of [**7**] —and in [**8**] —using the homological methods of [**10**] and [**11**] . It is an interesting open problem to determine the positive definiteness of the physical scalar product for general H. It is also interesting to relate this work to that of Felder[14] on minimal models, since these can be obtained via coset constructions[1]. This is currently under study.

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