BRST Cohomology and its Applications to Two Dimensional Conformal Field Theory

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José Miguel Figueroa-O'Farrill

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José Miguel Figueroa-O'Farrill

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend the acceptance of the dissertation.

Martin Roček, Associate Professor of Physics
Institute for Theoretical Physics, Advisor

Peter van Nieuwenhuizen, Leading Professor of Physics
Institute for Theoretical Physics, Chairman

Roderich Engelmann, Professor of Physics
Department of Physics

H. Blaine Lawson, Professor of Mathematics
Department of Mathematics

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Abstract of the Dissertation

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We discuss BRST cohomology from a symplectic setting and we show it corresponds to the homological reduction of the Poisson algebra $C^\infty(M)$ of a symplectic manifold M to the Poisson algebra $C^\infty(\widetilde{M})$ of its symplectic reduction by a coisotropic submanifold defined as the zero locus of first class constraints. We characterize the BRST cohomology topologically and prove a duality theorem. We extend the BRST construction to vector bundles on M. We introduce the notion of a Poisson module and show that this extended BRST construction is precisely homological reduction of Poisson modules; hence all Poisson structures are preserved. This allows us to reduce prequantum data and define BRST quantization in the framework of geometric quantization. We discuss invariant polarizations and show that in some cases a choice of polarization for M forces a choice of polarization on the ghosts. In particular, we can always choose a polarization for the ghosts such that the BRST operator is at most trilinear in the ghosts. We prove a duality theorem for quantum BRST cohomology. We apply BRST cohomology to two dimensional conformal field theories. We discuss some general properties of quantum BRST complexes; proving a Hodge decomposition theorem which implies a duality theorem. We compute the operator cohomology in terms of the cohomology on states. We investigate the consequences of a vanishing theorem for BRST cohomology. We reformulate the no-ghost theorem and interpret the partition function as the character-valued index of an operator associated to the BRST operator. We then investigate the BRST cohomology of the NSR string. After identifying it with a particular semi-infinite cohomology we prove a vanishing theorem for a relative subcomplex which allows us to compute the cohomology of the full complex from the relative one. We give a simple proof of the no-ghost theorem. Finally we investigate the BRST cohomology of the gauged WZNW model, after identifying it with a particular semi-infinite cohomology. When the gauged subgroup is abelian we can prove a vanishing theorem and a no-ghost theorem. From this we compute the chiral partition of the theory for all subgroups but for a special kind of representations.

Esta tesis está dedicada
a Edmée O'Farrill, mi madre y amiga;
y a la memoria de mi abuelo Juan O'Farrill (1903-1986).

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Chapter One:

Introduction

Gauge theories play a pivotal rôle in the current paradigm of fundamental physics. With ample empirical evidence supporting them, quantum electrodynamics (QED), the Weinberg-Salam model, and quantum chromodynamics (QCD)—all of them gauge theories—form the basis of our mathematical modeling of subatomic phenomena. Furthermore, our classical (and, thus far, only) understanding of gravitational interactions is based on general relativity: which is a gauge theory. So secure, in fact, we feel within the framework of gauge theories that virtually all candidates for the theory of everything (TOE) proposed during the last three decades have been based on gauge theories. Perhaps the great advantage of gauge theories is that, in the absence of hard empirical data, symmetry principles have proved in the past to be invaluable tools in the description of interactions: the heart of field-theoretic modeling. Therefore the study of the gauge theories is interesting.

Our description of gauge theories is synonymous with redundancy. Although there are perfectly valid descriptions of gauge theories which only involve the physically relevant degrees of freedom, these descriptions are not satisfactory from the point of view of local field theory, since they violate desirable properties such as locality and/or covariance. For instance, in Yang-Mills theory the truly physical degrees of freedom are the non-integrable phase factors of Wu & Yang (now called Wilson loops) but these objects are non-local. In the context of lagrangian field theory, this redundancy manifests itself in the fact that the lagrangian is degenerate; that is, the velocities are not all independent and, as a consequence, the defining equations for the canonical momenta cannot be inverted to solve for the velocities in terms of the canonical momenta. Since, from the point of view of phase space formulation, the sole purpose of the lagrangian is to give special coordinates to the phase space: namely the coordinates and their associated canonical momenta, a degenerate lagrangian does not coordinatize all of phase space but only a given subvariety, called a constraint "surface". Hence gauge theories give rise to constrained dynamical systems.

For example, in the phase space treatment of QED, the momentum π_0 canonically conjugate to the time component of the 4-potential A_μ is automatically zero. Moreover, the hamiltonian time evolution does not preserve the constrained surface $\pi_0 = 0$ unless Gauss's law $\operatorname{div} E = 0$ is also imposed. However we already see that the definition of the constrained surface depends on the Lorentz frame and pursuing this description would yield a theory which is not manifestly Lorentz covariant. Since Lorentz covariance is a consistency condition of any sensible quantum theory, we would have to check this by hand at the end of the day. Moreover, it is not trivial to solve for Gauss's law as a constraint.

Therefore it is desirable to have a method by which we can work with constrained dynamical systems without having to explicitly solve for the constraints and hence lose covariance and/or locality. BRST quantization accomplishes precisely this and, in doing so, introduces unphysical degrees of freedom, in the form of ghosts which, if the quantization is successful, "eat up" the unphysical degrees of freedom present already in the local covariant formulation. Of course, this is not how things developed historically so, having told the punch line, let me now tell the joke.

1. Ghosts and BRS Transformations

Ghost fields (so called because they appeared only in internal lines in Feynman diagrams) were introduced by Feynman[1] in pure Yang-Mills in order to preserve 1-loop unitarity. Let us briefly recall his argument. Suppose we are interested in checking the transversality of the two-point function (suppressing the group indices on the vector fields for simplicity):

$$\Pi^{\mu\nu}(p) \equiv \int dx\, e^{-ip\cdot x} \langle 0|T\, A^\mu(x) A^\nu(0)|0\rangle \;; \tag{I.1.1}$$

that is, that $p_\mu p_\nu \Pi^{\mu\nu}(p) = 0$. This is an identity which can be checked in perturbation theory. In fact, introducing a complete set of states $\{|n\rangle\}$ we can turn this into

$$\begin{aligned} p_\mu p_\nu \Pi^{\mu\nu}(p) &= -\int dx\, e^{-ip\cdot x} \langle 0|T\, \partial\cdot A(x) \partial\cdot A(0)|0\rangle \\ &= -\int dx\, e^{-ip\cdot x} \sum_n \big(\theta(x_0)\langle 0|\partial\cdot A(x)|n\rangle\langle n|\partial\cdot A(0)|0\rangle \\ &\quad + \theta(-x_0)\langle 0|\partial\cdot A(0)|n\rangle\langle n|\partial\cdot A(x)|0\rangle\big) \;, \end{aligned} \tag{I.1.2}$$

which is an identity that can be checked loopwise. At tree level the two-point function is transverse if we choose the Lorentz gauge. At one loop there are two ways we can check this identity. First we can compute the relevant one loop diagram in a given gauge. This is equivalent to computing the RHS of (I.1.2) with a particular choice of $\{|n\rangle\}$: namely, those solutions to the classical equations of motion in that gauge. Alternatively we can choose for our complete set $\{|n\rangle\}$ only physical states. For example, this would correspond to using only transversely polarized gluons. It is a basic axiom in quantum mechanics that the physical states are complete, so the equivalence between these two computations is equivalent to the the completeness of the set $\{|n\rangle\}$ defined by the intermediate states in the one-loop diagram, *i.e.*, unitarity. Performing the first calculation we see that indeed the one-loop propagator is transverse; whereas performing the second calculation we find that $p_\mu p_\nu \Pi^{\mu\nu}(p) \neq 0$. In order to obtain zero, Feynman, in his usual clever way, added by hand more intermediate states which could contribute the same as the extra unphysical states of the gluons but with the opposite sign. Following the usual Feynman rules these would correspond to a pair of conjugate Lorentz scalar fermionic fields $(\bar{c}, c)$ in the adjoint representation of the gauge algebra (in order to cancel the group factors) coupled to the gluon via

$$\partial\bar{c} \cdot \big[A\,,\,c\big]\ , \tag{I.1.3}$$

where $\cdot$ stands for both Lorentz and group contractions and $\big[\,,\,\big]$ stands for the adjoint action of A (which is Lie algebra valued) on c. The fields $(\bar{c}, c)$ are known as ghosts, and their raison d'être is that of turning an overcomplete basis into a complete one; not by getting rid of the extra states but by adding some more. In other words, instead of getting rid of "excess probability" we add negative probability by the excess amount.

It wasn't until four years later, in 1967, that Feynman's heuristic rules could be derived from an action principle. This was done in a much celebrated paper by Faddeev & Popov[**2**]. To understand their solution let us briefly take a look at the problem. Let us consider the functional integral

$$Z = \int [dA]\ e^{iS_{\rm YM}[A]}\ , \tag{I.1.4}$$

where $S_{\rm YM}[A]$ is, for simplicity, the pure Yang-Mills action. Because $S_{\rm YM}$ is invariant under gauge transformations, the action is not damped along the gauge directions and therefore the path integral makes less sense than usual. What Faddeev & Popov did is to modify the region of integration in such a way that we take into account contributions from only one representative from each physically inequivalent field configuration. Their trick is the following.

Let F be an algebra valued local function depending on A and its derivatives and such that given any $A(x)$ there exists a unique element $g(x) \in G$ such that $F(A^g) = 0$, where A^g means the transformed of A by g. Define the following object

$$\triangle_F^{-1}[A] = \int [dg]\ \delta\left[F(A^g)\right]\ , \tag{I.1.5}$$

where $[dg]$ is the functional Haar measure. We define $\triangle_F[A]$ as the formal inverse of (I.1.5). As a consequence of the invariance of the Haar measure, $\triangle_F^{-1}[A]$—and hence $\triangle_F[A]$—is gauge invariant.

Cleverly inserting unity, Z becomes

$$\begin{aligned} Z &= \int [dA]\ \triangle_F\,[A]\ \triangle_F^{-1}\,[A]\, e^{iS_{\rm YM}[A]} \\ &= \int [dA][dg]\,\delta\,[F(A^g)]\ \triangle_F\,[A]\, e^{iS_{\rm YM}[A]}\ . \end{aligned} \tag{I.1.6}$$

Performing a gauge transformation by g^{-1} in Z and noticing that except for the term $\delta\,[F(A)]$ everything else is invariant we find

$$\begin{aligned} Z &= \int [dA][dg]\,\delta\,[F(A)]\ \triangle_F\,[A]\exp\ iS_{\rm YM}[A] \\ &= \mathcal{V}_G \int \delta\,[F(A)]\ \triangle_F\,[A]\exp\ iS_{\rm YM}[A]\ , \end{aligned} \tag{I.1.7}$$

where $\mathcal{V}_G$, the volume of the group of gauge transformations, is a constant that can (and will) be reabsorbed in the normalization of Z.

Now let's calculate $\triangle_F[A]$. Because $F(A) = 0$ for precisely one g, we can make a formal change of variables from $[dg]$ to $[dF]$. Thus,

$$\begin{aligned} \triangle_F^{-1}[A] &= \int [dg]\ \delta\,[F(A^g)] \\ &= \int [dF]\ \det\left(\frac{\delta g}{\delta F}\right)\ \delta\,[F(A^g)] \\ &= \det\left(\frac{\delta g}{\delta F}\right)\Bigg|_{F=0}\ . \\ \therefore \triangle_F[A] &= \det\left(\frac{\delta F}{\delta g}\right)\Bigg|_{F=0}\ . \end{aligned} \tag{I.1.8}$$

This allows us to rewrite Z as follows

$$Z = \int [dA]\ \det\left(\frac{\delta F}{\delta g}\right)\Bigg|_{F=0}\ \delta\,[F(A)]\ e^{iS_{\rm YM}[A]}\ . \tag{I.1.9}$$

Although Z written in this way seems to depend manifestly on the gauge choice F one can easily show that this dependence is indeed fictitious.

Since $\left(\frac{\delta F}{\delta g}\right)\Big|_{F=0}$ is the variation of F with respect to a group element which is close to the identity, we can parametrize it by the gauge algebra. Letting ϖ be parameters of the algebra such that for group elements sufficiently close to the identity $g(x) = \exp\,\varpi(x)$ we have that

$$\left(\frac{\delta F}{\delta g}\right)\Big|_{F=0} = \left(\frac{\delta F}{\delta \varpi}\right)\Big|_{F=0} \stackrel{\text{def}}{=} \mathcal{M}_F \ . \tag{I.1.10}$$

Now we can exponentiate the determinant of $\mathcal{M}_F$ in Z by the introduction of Grassmann-odd algebra valued scalar fields $(\bar{c}, c)$—the so-called Faddeev-Popov ghosts—via

$$\det \mathcal{M} = \int [dcd\bar{c}]\ e^{-i\langle \bar{c}\cdot\mathcal{M}\cdot c\rangle} \ . \tag{I.1.11}$$

Because Z is defined up to an overall normalization factor we can multiply by the following term:

$$\int [df]\ e^{-\frac{i}{2\alpha g^2}\langle f\cdot f\rangle} \ , \tag{I.1.12}$$

where f is algebra valued, α is a real constant, and $\langle\cdot\rangle$ means integration over spacetime and trace over the algebra. Because of the invariance of Z under changes in the gauge choice we can shift $F(A)$ to $F(A) - f$ and performing the integral over f using the $\delta\,[F(A) - f]$ we find that Z may be rewritten —dropping overall normalization factors— as:

$$Z = \int [dAdcd\bar{c}]\ e^{i\left(S_{\text{YM}}[A] - \frac{1}{2\alpha g^2}\langle F\cdot F\rangle - \langle \bar{c}\cdot\mathcal{M}_F\cdot c\rangle\right)} \ . \tag{I.1.13}$$

In the particular case of the covariant gauge $F(A) = \partial \cdot A$, the change in the gauge fixing function is the following:

$$\begin{aligned}
\delta F(A) &= (\partial \cdot A^g) \\
&= \partial \cdot D\varpi \\
\therefore\ \frac{\delta F}{\delta \varpi} &= \partial \cdot D = \mathcal{M}_F \ .
\end{aligned} \tag{I.1.14}$$

Therefore in this case the action becomes, after rescaling the gauge field by the coupling constant,

$$Z = \int [dAdcd\bar{c}]\ e^{i\left(S_{\text{YM}}[A] - \frac{1}{2\alpha}\langle F\cdot F\rangle + \langle \partial\bar{c}\cdot Dc\rangle\right)} \ , \tag{I.1.15}$$

which recovers the Feynman rules for the ghosts.

Two remarks are in order. The first is that since the ghosts are now fields in the action, we are not free to have them run only along internal lines. In fact, we must make sure that they decouple from physical amplitudes. The second remark, which is somewhat related to the first, is that the "quantum" action

$$S_{\mathrm{Q}}[A,c,\overline{c}] = S_{\mathrm{YM}}[A] + S_{\mathrm{FIX}}[A] + S_{\mathrm{FP}}[A,c,\overline{c}] \ , \qquad \text{(I.1.16)}$$

is no longer gauge-invariant. In fact it has a non-linear invariance: BRS; which will prove instrumental in proving unitarity, renormalizability, and gauge independence of the S-matrix.

In 1974, in the context of the abelian Higgs-Kibble model, Becchi, Rouet, & Stora[3] (BRS) discovered accidentally a non-linear invariance of the "quantum" action (with gauge fixing and Faddeev-Popov terms included). For the quantum action given by (I.1.16) the BRS transformations are given by

$$\begin{aligned} \delta A &= Dc \\ \delta c &= -\frac{1}{2}\big[c\,,\,c\big] \\ \delta\overline{c} &= \frac{1}{\alpha}\partial\cdot A \end{aligned}$$

On non-ghost fields (here only the gauge field) the BRST transformations are gauge transformations with the ghost for parameters. These transformations have the property that they leave invariant S_{Q} and that they are square-zero on the gauge field A and on the ghost field c. On the anti-ghost it is not square-zero; although it is proportional to the variation of the action with respect to the antighost, *i.e.*, to a field equation. To obtain a truly square-zero transformation we must introduce the Nakanishi-Lautrup auxiliary field b, which is a Lorentz scalar transforming in the coadjoint representation. These are introduced quadratically in the action in such a way that their (algebraic) field equations merely sets it equal to the gauge fixing function. Then one defines $\delta\overline{c} = b$ and $\delta b = 0$. Then $\delta^2 = 0$ manifestly and, moreover, if we substitute the gauge fixing lagrangian for $\frac{1}{2}\alpha b^2 + b\cdot\partial A$ we can then write the quantum lagrangian in the following manifestly invariant fashion:

$$L_{\mathrm{Q}} = L_{\mathrm{YM}} + \delta\left(\overline{c}\left(\frac{1}{2}\alpha b + \partial\cdot A\right)\right) \ . \qquad \text{(I.1.17)}$$

The BRS invariance of the quantum action gives rise to the Slavnov-Taylor identities which govern unitarity and the gauge-independence of the S-matrix. Moreover, BRS invariance is instrumental in the proof of renormalizability à la Lee[4] and Zinn-Justin[5].

BRS transformations were also implemented in other gauge theories such as gravity[6] [1]. However, in that same paper, the treatment of supergravity without auxiliary fields unearthed a new problem. Without auxiliary fields, the supergravity gauge algebra does not close off-shell[7] (hence the name open algebra) and the usual BRS procedure breaks down. It was not until the work of Nielsen[8] and Kallosh[9] that a satisfactory BRS treatment of supergravity was found. Later in works of de Wit & van Holten[10], Nielsen[11], Ore & van Nieuwenhuizen[12] [2], and, finally, Kugo & Uehara[13] the BRS treatment of lagrangian gauge theories was extended to general theories.

Another field about which we will have nothing to say is the understanding of anomalies within a BRS framework. This line of work has been pursued by many authors, notably Baulieu, Bonora & Cotta-Ramusino, Piguet & Sibold, Stora, Viallet, among others.

2. BRS becomes BRST: The Lebedev School

In 1975 and independently from the work of BRS, Tyutin[14] discovered the BRS transformations in the context of canonical quantization of gauge theories[3]. Thence forward BRS became BRST; although only recently has it started to catch on.

At about the same time there was a group at Lebedev composed of Batalin, Fradkin, and Vilkoviskii, working on phase space path integral quantization of gauge theories. Before the work of the Lebedev group, the phase space path integral quantization of gauge theories was based on the method developed by Faddeev[15], which was a path integral extension of the Dirac[16]-Bergmann[17] theory of contraints. Faddeev's construction is fairly straight forward. Suppose we have a dynamical system with first class constraints[4] $\{\phi_i\}$. Their zero locus defines the constrained submanifold. Because the constraints are first class, the restriction of the symplectic form (roughly the Poisson bracket) to this submanifold is degenerate. Essentially what happens is that the constraints generate gauge transformations which preserve this submanifold and in order to identify the physical configurations we must pick only one configuration from each gauge equivalence class. The way to do this is to choose subsidiary conditions $\{\chi_j\}$ obeying $\det\left\{\phi_i\,,\,\chi_j\right\} \neq 0$. Then the submanifold of

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[1] This was the first paper to feature "BRS" in its title.

[2] This was the first paper to feature "BRST" in its title.

[3] His preprint, written in Russian, was never published nor translated (as far as I know) although there seems to be a copy in the SLAC database and, moreover, Peter van Nieuwenhuizen assures me of its existence.

[4] We shall discuss constraints in more detail in Section II.2.

the constraint submanifold defined by the subsidiary conditions is symplectic. Faddeev's path integral consists of the induced measure on this symplectic manifold. Of course, this description is not manifestly gauge independent since it explicitly depends on the subsidiary conditions.

Using BRST ideas, namely extending the phase space by the inclusion of ghosts and the existence of the BRST transformations, Batalin & Vilkoviskii[18] constructed a manifestly gauge independent S-matrix for theories with irreducible first class constraints, *i.e.*, first class constraints which are independent. They defined the hamiltonian BRST operator Q which generated, via Poisson brackets in the extended phase space containing the ghosts, the BRST transformations. However their construction only worked for gauge algebras which closed.

For open gauge algebras the classical BRST operator was not square-zero. This was taken as a symptom that it was not complete and that it needed terms of higher order in the ghost fields (the Batalin-Vilkoviskii operator contained only terms linear and trilinear in the ghosts). This was first realized by Fradkin & Fradkina[19]; although the construction of the complete square-zero BRST operator for general irreducible first class constraints did not come until the seminal work of Henneaux[20], which is the English translation of his Thèse d'Aggregation.

For reducible constraints, on the other hand, the construction of the classical BRST charge has finally been achieved in the recent work of Fisch, Henneaux, Stasheff, & Teitelboim **[21]**; although quite a lot of work had been done by the Lebedev group (*cf. e.g.*, **[22]**, **[23]**, and **[24]**).

3. BRST Quantization: Principles and Applications

Although BRST methods were already quite established in the path integral quantization of gauge theories, no work had been done towards an axiomatic description of BRST quantization until, in 1979, Kugo & Ojima[25] wrote their monumental work on local covariant operator quantization of Yang-Mills theory. In this paper, Kugo & Ojima investigated the BRST quantization of Yang-Mills theory from the operator approach. They defined the physical space of the theory as the cohomology of the BRST operator and physical observables as the cohomology of the BRST operator acting on operators. They commented on the importance of the vanishing theorem for the BRST cohomology and they proved the unitarity of the physical S-matrix using their "quartet mechanism" which is essentially a Lie algebraic way to look at cohomology. Moreover they applied this to the problem of quark confinement which, from their point of view, relied on the fact that the chromomagnetic field is not a BRST invariant operator and hence it is not a physical observable. The depth of the work of Kugo & Ojima can hardly be overemphasized: the framework of BRST quantization that we have today is essentially unchanged from that in Kugo-Ojima's work.

The BRST quantization methods of Kugo & Ojima were immediately applied to gauge theories other than Yang-Mills, *e.g.*, supergravity[26] and p-forms[27],[28]. However, it is fair to say that the full weight of the Kugo-Ojima formalism was not felt by the community at large until the advent of string (field) theory when people were faced with the problem of actually constructing quantum theories; although already in the work of Nielsen[11] on supergravity one finds references to the ideas of Hata & Kugo[26].

The covariant quantization of the bosonic string à la BRST was done by Kato & Ogawa[29] in 1983 (although *cf.* also Hwang[30]) where they also proved the ghost-decoupling theorem using the methods of Kugo & Ojima. In particular the nilpotency of the BRST operator was only achieved in 26 dimensions and for the value of the intercept $\alpha_0 = 1$, which are precisely the values of these parameters for which the light-cone formulation becomes Lorentz covariant. Kato & Ogawa made some assumptions that were not necessary, as was further clarified by Henneaux[31] in 1986. Also in 1986 Freeman & Olive[32] proved that the BRST cohomology was given by the DDF states (a result also obtained by Kato & Ogawa) acting on the Fock vacuum and as a consequence that the physical space had a positive-definite norm, except at zero center of mass momentum. However, their results depended on the explicit computation of the BRST cohomology and, in particular, it relied heavily in early work on dual models. It seemed therefore that in order to get any information about the physical states (*e.g.*, positive-definiteness) one was forced to compute the cohomology beforehand: an unwieldy task in general.

Fortunately, in 1986, Frenkel, Garland, & Zuckerman[33] identified the quantum BRST operator in the open bosonic string with the operator computing the semi-infinite cohomology of the Virasoro algebra with coefficients in a particularly manageable representation. As a consequence of a very general vanishing theorem they could reproduce (without having to compute the cohomology) the aforementioned results of Kato & Ogawa, Henneaux, and Freeman & Olive. The details of their proof of the no-ghost theorem were also reproduced by Spiegelglas[34].

Some preliminary investigations of the BRST cohomology of the NSR string were done by Ohta[35] in 1986, where the critical dimension and the intercept were found; although the first conclusive results were given also by Ohta[36] using the methods of Freeman & Olive. However Ohta was sloppy in the treatment of the superconformal ghosts' zero modes in the Ramond sector and he obtained an infinite degeneracy. In 1987 this was shown to be incorrect by Henneaux[37] where he correctly computed the cohomology of the NSR string, although with an unnatural choice of representation for the superconformal zero modes. The extension of the Frenkel, Garland, & Zuckerman methods to the NSR string was done in 1988 by Figueroa-O'Farrill & Kimura[38], where a simple proof of the no-ghost theorem is given with the natural choice of representation for the superconformal ghosts' zero modes.

Perhaps the single most decisive factor in the acceptance of BRST quantization is the construction by Siegel[39] of a covariant second-quantized string theory using BRST methods. In the approach of Siegel, the ghost fields and the BRST operator play a deeper rôle than they had played so far in string theory: passing from being mere technical niceties to being an integral part of the logical thread of the theory. The papers of Siegel seem to have triggered an explosion[5] of interest in BRST quantization. We shall not have anything to say about this, except that the relevant papers are those of Siegel & Zwiebach[40], Banks & Peskin[41], Itoh, Kugo, Kunitomo, & Ooguri[42] for the free open bosonic string field theory; and Witten's celebrated paper[43] on the interacting theory. For the NSR string field theory there is a host of other papers culminating in the continuation to Witten's paper[44].

BRST quantization keeps playing a prominent rôle today: both in conformal field theory as in recent attempts at the covariant quantization of the superparticle and the Green-Schwarz superstring.

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[5] An interesting piece of statistical data on BRST publications is given the following table obtained from SLAC preprint database (SPIRES). The numbers represent the number of preprints received by SLAC with BRS or BRST in their title. Although it is not clear what can be inferred from such data, the only thing that appears unquestionable is the sudden rise of material after 1985.

Year	#	Year	#
1976	1	1983	12
1977	1	1984	13
1978	7	1985	15
1979	1	1986	74
1980	8	1987	87
1981	17	1988	104
1982	18	1989	≥ 36

4. Mathematical Formalism: BRST Cohomology

Although there had been some sporadic work on the cohomology of the BRST operator in the context of perturbative gauge theories: renormalization (Joglekar & Lee [**45**], Dixon [**46**]) and anomalies (Bonora & Cotta-Ramusino [**47**]); it was not until the seminal work of McMullan[48] in 1984 that the underlying cohomological story began to unfold. In 1986, McMullan recognized correctly the cohomological nature of the classical BRST operator of Batalin & Vilkoviskii and applied it to Yang-Mills theory[49]. Unfortunately the added complications present in his rigorous treatment of Yang-Mills obscured the very simple yet deep points he was putting forth. Later in 1987, in a paper with Browning[50], they treated the general case of open algebras and they gave a nice cohomological treatment of Henneaux's construction of the general BRST operator, in which the Koszul complex played a central rôle. In the simplest case of a closed algebra, their analysis showed that the BRST operator is the total differential of a double complex whose vertical operator (say) was the Koszul differential. In the general case there is no double complex but rather a graded filtered complex (see Section II.1 for vocabulary), to whose cohomology there converges a spectral sequence whose first term is the cohomology of a Koszul complex. However they did not pursue their analysis deeper.

At about the same time, Dubois-Violette[51] gave the first full, comprehensive treatment of BRST cohomology. He gave a geometric interpretation of BRST cohomology based on very deep relations between algebra and geometry. He showed that the BRST cohomology was isomorphic to a certain de Rham-type cohomology on the submanifold defined by the first class constraints. Unaware of this work, I rediscovered all of these results in 1988. Together with Kimura[52] we also investigated some of the consequences of this formalism and, in particular, of its geometric interpretation. Also in a 1988 paper, Henneaux & Teitelboim[53] reached the same conclusion with respect to the geometric interpretation. However they did not pursue the description any further. Finally (so far) in 1989, I[54] managed to put the geometric description of the classical BRST cohomology to good use and computed the classical BRST cohomology from initial data. The cleanest results happen in the case of a group action. It should be remarked that the results in [**54**] are also valid for the case of reducible gauge theories, for which a geometric description was found by Fisch, Henneaux, Stasheff, & Teitelboim[21].

Although the study of BRST cohomology had restricted itself to the symplectic setting, two things are worth remarking. On the geometric side, no use is made of the symplectic structure. The geometric results are all more general than the setting in which they appear. However the formalism takes a life of its own in the symplectic context. This, of course, was the key element in the work of Batalin, Fradkin, & Vilkoviskii and of Henneaux; although its true "symplecticity" was first noticed by Stasheff[55]. In that paper, he noticed that the BRST construction did not really pay much attention to the symplectic geometry itself, but rather to the Poisson structure of the algebra of functions. This led him to the study of constrained Poisson algebras in general and to the BRST construction in that setting[56].

This has opened up a whole new way to look at BRST cohomology from a Poisson algebraic point of view. In **[57]**, Figueroa-O'Farrill & Kimura have made use of this new way of thinking about BRST to define BRST quantization in a geometric quantization setting. Although the applications in that paper are geometric, the constructions are algebraic and in fact show promise of being generalizable to a new way of defining quantization of constrained systems extending the very interesting approach of Huebschmann[58] on algebraic quantization.

As for the cohomology of the quantum BRST operator, very little is known in general. The only results of a general nature are those of Frenkel, Garland, & Zuckerman[33] in the context of semi-infinite cohomology of graded Lie algebras; but there are not that many physical systems whose BRST cohomology corresponds to one of these semi-infinite cohomology theories; although, fortunately, strings are such systems.

5. Outline of Dissertation

This dissertation is divided into two more or less independent parts. The first part is dedicated to the study of BRST cohomology from the symplectic setting; whereas the second part is more concerned with the applications of BRST cohomology (as semi-infinite cohomology) to some two-dimensional conformal quantum field theories: the NSR string and the gauged WZNW model.

Chapter II consists of preliminary mathematical material which serves the purpose of setting up the vocabulary and the framework for the rest of the dissertation. An index of jargon has been provided at the end of the dissertation for the convenience of the reader. In Section II.1 we discuss the basic facts of homological algebra emphasizing the concrete aspects rather than the "abstract nonsense". There we quote all of the results on spectral sequences we shall use in the chapters that follow and prove the algebraic Künneth formula as an application we shall have ample opportunity to use. We also set the notation on Lie algebra cohomology. Section II.2 discusses symplectic reduction and the theory of constraints. The treatment is a geometrized version of Dirac's treatment; in particular, we derive the Dirac brackets from first principles. We make special mention, as familiar examples, of the moment map and of the symplectic reduction of a phase space.

Chapter III is concerned with the connections between classical BRST cohomology and the reduction of a symplectic manifold associated to a set of irreducible first class constraints. In Sections III.1 and III.2 we construct the BRST cohomology theory in an algebro-geometric way and give its geometric interpretation all in a way independent of the symplectic structure. In Section III.3 we make contact with the symplectic structure and show how natural the BRST construction is from this point of view. Finally in Section III.4 we discuss the topological characterization of classical BRST cohomology. This chapter describes work contained in the following papers: [**52**], [**54**], and [**57**].

Chapter IV discusses BRST quantization from a geometric quantization setting. The geometric quantization formalism is briefly reviewed in Section IV.1. In Sections IV.2 and IV.3 we describe the constructions. In Section IV.2 we describe prequantization which, although independent of the symplectic structure, once again becomes extremely natural when described in a symplectic setting. For this we need to introduce a new mathematical object: Poisson module. In Section IV.3 we discuss polarizations. We show that, contrary to what had been described in the existing literature, the polarizations of the matter and the ghosts are not independent. Finally in Section IV.4 we look at an immediate consequence of the formalism: a duality theorem. This chapter comes almost entirely out of [**57**].

Chapter V sits somewhat abridge the two parts of the dissertation. In it we describe broad and general properties of quantum BRST cohomology without tying ourselves down to any particular model; although we have in mind the applications to be found in the following chapters. In Section V.1 we define what we mean by a quantum BRST complex. Our definition is not too restrictive as can be judged by Chapters VI-VIII, which deal with special cases of such complexes. In Section V.2 we present a Hodge-style decomposition theorem which allows to include cohomology naturally as a subspace of the cocycles. This theorem has a lot of immediate consequences which we treat in the subsequent sections: "Poincaré" duality, the determination of the operator cohomology from a knowledge of the ordinary BRST cohomology in Section V.3 and the reformulation of the no-ghost theorem in Section V.4. The work described in this chapter is contained in [**59**].

Chapter VI describes the relation between BRST and the semi-infinite cohomology of graded Lie algebras. It is roughly a summary of the work of Frenkel, Garland, & Zuckerman, whose construction is described in Section VI.1. There we also prove their celebrated vanishing theorem. In Section VI.2 we apply this to the bosonic string and prove the no-ghost theorem.

Chapter VII contains the treatment of the BRST cohomology of the NSR string from the semi-infinite cohomology side. Section VII.1 proves the vanishing theorem for the relative subcomplex in the Neveu-Schwarz sector whereas in Section VII.3 we extend it to the full complex. The Ramond sector is trickier due to the presence of the superconformal ghosts' zero modes. We give a complete and rigorous treatment in Section VII.2 proving a vanishing theorem for a relative subcomplex, which in Section VII.3 will be extended to a vanishing theorem for the full complex. In Section VII.4 we give very simple proofs of the no-ghost theorems for both sectors. In the appendices we fill in details skipped in the main body. The work described here is more or less contained in [**38**]; although it also borrows from [**60**].

Chapter VIII is the last and discusses the BRST cohomology of the gauged WZNW model, which we describe from a conformal-field-theoretic point of view in Section VIII.1. In Section VIII.2 we identify its BRST cohomology with a certain relative semi-infinite cohomology and prove a vanishing theorem for a certain kind of representations. In particular, if the gauged subgroup is abelian the vanishing theorem holds for all representations. In the abelian case we can also prove a no-ghost theorem; and this we do in Section VIII.3, where we also compute the chiral partition function of the theory. We hope that this will lead to a proof of the CFT equivalence between the gauged WZNW model and the coset construction of GKO. This chapter is somewhat incomplete and contains some results which are yet unpublished, since they are work in progress; although parts of it can be found in [**61**].

Finally some remarks of a typographical nature. Chapters are numbered in uppercase roman numerals, sections in arabic numerals, and appendices in uppercase latin letters. A reference to a section or an appendix within the same chapter shall only contain its number; whereas a reference to a section or appendix on a different chapter shall also contain the chapter number. Subsections are not numbered. Equations and mathematica (*i.e.*, definitions, theorems, ...) are numbered sequentially within a section or appendix and also contain the number of the chapter (*e.g.*, (III.2.1) refers to the first numbered equation of the second section of the third chapter). Equations which appear to be in Section 0, appear at the beginning of the relevant chapter before the start of the first section (a minor TEXnichal glitch). We use Halmos' ■ notation to signify the end of a proof. Particularly interesting equations have been boxed for emphasis. Footnotes are numbered sequentally throughout the dissertation. **Boldfaced** words (except for that one) appear in the index. Finally, and in case the reader has not yet noticed, this dissertation has been typeset entirely in TEX[6]—the computer typesetting enviroment created by Donald Knuth—using a mix of macros defined by D. Knuth, V. Kaplunowsky, and myself.

[6] Straight TEX! None of these other wimpy versions that abound these days.

Chapter Two:

Mathematical Preliminaries

This dissertation borrows a lot of vocabulary, notation, concepts, and techniques from the surface of two major areas of mathematics: homological algebra and symplectic geometry. In an effort to make this work as self-contained as possible and not before some debate, I managed to convince myself that rather than succumbing to encyclopædistic tendencies and fill this dissertation with appendices which, on the one hand, will probably not be read; and, on the other hand, would upset the linear order of the discussion; I would devote a medium sized chapter to getting these prerequisites out of the way. Moreover, since if this chapter is to be read at all it should be done so at the beginning, I decided to make it the first real chapter in the dissertation. Needless to say, the reader is strongly urged to at least skim this chapter for notation.

This chapter is organized as follows. In Section 1 we review the basic facts of homological algebra. Although none of the concepts are too difficult (except perhaps spectral sequences), as usual with algebra, there are a lot of names. In this section we set our notation and vocabulary concerning differential complexes. In particular we discuss resolutions which will be very important conceptually throughout this dissertation. We then introduce the reader to spectral sequences. This is possibly the toughest concept in this chapter but it proves to be an invaluable tool when computing cohomology. As a special illustration we then take a look at the two canonical spectral sequences associated to a double complex and as an application of this we prove the algebraic Künneth formula. If the reader comes out with the compulsion that the first thing to try when faced with a double complex is to go ahead and compute the first two terms of the two spectral sequences, this chapter will have served its purpose. Finally, and because we will find ample use for these concepts, we briefly review the highlights of Lie algebra cohomology.

Section 2 is another introductory section which sets the language for the other important subject in this work: symplectic geometry. Everything is this section is familiar in one way or another to every working theoretical physicist; although the names I have used may not be so readily distinguishable. As a particularly nice application of the concepts and methods of symplectic geometry, we give a derivation from first principles of the Dirac bracket. We also cover symplectic reduction with respect to a coisotropic submanifold. This is not the most general case of symplectic reduction, but it is the one we shall be interested in. We then make contact with the theory of constraints. We prove that the constrained submanifold associated to a set of first (resp. second) class constraints is a coisotropic (resp. symplectic) submanifold. We then discuss a very special case of symplectic reduction: the one arising from the action of a Lie group. The first class constraints are nothing but the components of the moment map. Finally we discuss a special case of the moment map. This is the symplectic reduction of a phase space. In this case we show how any action of the configuration space automatically gives rise to an equivariant moment map in the configuration space which is linear in the momenta.

1. Basic Facts of Homological Algebra

In this section we assemble the basic definitions, notation, and facts of homological algebra that will be used in the sequel; as well as some less elementary material on spectral sequences which is nevertheless instrumental for this dissertation. We also give a brief introduction to the basic ideas of Lie algebra cohomology. These will come in handy when we discuss the semi-infinite cohomology of Feigin in Chapters VI-VIII. Homological algebra is a topic which lends itself easily to generalizations which would, however, only obscure the concepts of relevance to our discussion. Therefore we have attempted to suppress almost all evidence of "abstract nonsense" and keep the discussion as elementary as possible while still covering in detail the necessary background. Fuller treatments to which no justice could possibly be done in a few pages are to be found in the books by Lang **[62]**, Hilton & Stammbach **[63]**, and MacLane **[64]**. Lie algebra cohomology is treated in the books of Jacobson **[65]**, Hilton & Stammbach (*op. cit.*), and in the seminal paper of Chevalley & Eilenberg **[66]**. The cohomology of infinite dimensional Lie algebras is discussed with a wealth of examples in the book of Fuks **[67]**.

Basic Definitions

Homological algebra centers itself on the study of complexes and their cohomologies. Let C be a vector space and let $d : C \to C$ be a linear map which obeys $d^2 = 0$. Such a pair (C, d) is called a **differential complex**, and d is called the **differential**. Associated to the differential there are two subspaces of C:

$$Z \equiv \{v \in C \mid dv = 0\} = \ker d \tag{II.1.1}$$

$$B \equiv \{dv \mid v \in C\} = \operatorname{im} d \ , \tag{II.1.2}$$

the **kernel** and the **image** of d respectively. Because $d^2 = 0$, $B \subset Z$. The obstruction to the reverse inclusion is measured by the **cohomology** of d, written $H_d(C)$, and defined by

$$H_d(C) \equiv Z/B \; . \tag{II.1.3}$$

Whenever there is no risk of confusion we will omit all explicit mention of the differential and simply write $H(C)$ for the cohomology. The elements of C, Z, and B are called **cochains**, **cocycles**, and **coboundaries** respectively.

Therefore, $H(C)$ consists of equivalence classes of cocycles, where two cocycles v, w are said to be **cohomologous**—*i.e.*, in the same cohomology class— if their difference is a coboundary. In symbols,

$$[v] = [w] \iff v - w = du \quad (\exists u) \; . \tag{II.1.4}$$

In particular, a coboundary is cohomologous to zero. Although $H(C)$ is a vector space it is worth remarking that it is not a subspace of C. Rather it is a **subquotient**: the quotient of a subspace. Of course, we can always choose a set of cocycles $\{v_i\}$ whose cohomology classes $\{[v_i]\}$ form a basis for $H(C)$ and then complete this set to a basis $\{v_i, w_j\}$ for C. The subspace of C spanned by $\{v_i\}$ is isomorphic to $H(C)$ but this is not canonical. That is, there is no privileged representative cocycle for a given cohomology class. We will see later on, when we discuss BRST cohomology, that this is precisely the algebraic analog of picking a gauge. The situation may, of course, differ if C has some more structure, *e.g.*, an inner product. This will, in fact, be the main theme in Chapter V.

The life of a chain complex with so little structure is rather dull. To relieve this boredom let us add a grading. That is, suppose that C is a $\mathbb{Z}$–graded vector space

$$C = \bigoplus_{n\in\mathbb{Z}} C^n \tag{II.1.5}$$

and that d has degree one with respect to this grading

$$d : C^n \longrightarrow C^{n+1} \; . \tag{II.1.6}$$

We call (C, d) in this case a **graded complex**. A useful graphical depiction of a graded complex is a sequence of vector spaces with linear maps (arrows) between them:

$$\cdots \longrightarrow C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} C^1 \longrightarrow \cdots \; . \tag{II.1.7}$$

We can refine our notions of cocycle and coboundary as follows. Define the subspace Z^n of

n-cocycles and the subspace B^n of n-coboundaries as follows

$$Z^n \equiv Z \cap C^n = \{v \in C^n \mid dv = 0\} \tag{II.1.8}$$

$$B^n \equiv B \cap C^n = \left\{dv \mid v \in C^{n-1}\right\} \ . \tag{II.1.9}$$

Then the n^{th} cohomology group $H^n(C)$ is defined as the quotient

$$H^n(C) \equiv Z^n/B^n \ . \tag{II.1.10}$$

Clearly

$$H(C) = \bigoplus_{n\in\mathbb{Z}} H^n(C) \tag{II.1.11}$$

making the cohomology into a graded vector space. We will often call the degree n the **dimension**; and we refer to $H^n(C)$ as the cohomology of the complex (C, d) in n^{th} dimension.

Perhaps the prime example of a cohomology theory is that of de Rham. Let M be a differentiable manifold and let $\Omega(M)$ denote the graded ring of differential forms. The exterior derivative d is a differential of degree one. The cocycles are called **closed forms**, whereas the coboundaries are called **exact**. The de Rham cohomology is denoted $H_{dR}(M)$ and is one of the simplest topological invariants of M that one can compute.

Now let $\operatorname{End} C$ denote the vector space of **endomorphisms** of C; that is, the linear transformations of C. The $\mathbb{Z}$–grading of C induces a $\mathbb{Z}$–grading of $\operatorname{End} C$ in the obvious way. We say that a linear transformation $f \in \operatorname{End} C$ has degree n if

$$f : C^p \longrightarrow C^{p+n} \qquad \forall\, p \ ; \tag{II.1.12}$$

and we write $f \in \operatorname{End}_n C$. Clearly

$$\operatorname{End} C = \bigoplus_{n\in\mathbb{Z}} \operatorname{End}_n C \ . \tag{II.1.13}$$

We can turn $\operatorname{End} C$ into a Lie superalgebra by defining the bracket of homogeneous elements $f \in \operatorname{End}_i C$ and $g \in \operatorname{End}_j C$ as the graded commutator

$$\left[f \, , \, g\right] \equiv f \circ g - (-1)^{ij} g \circ f \ , \tag{II.1.14}$$

where $\circ$ stands for composition of linear transformations. In particular $d \in \operatorname{End}_1 C$ and hence the fact that $d^2 = 0$ is equivalent to the Lie algebraic statement that the subalgebra of $\operatorname{End} C$ it generates is abelian— a non-trivial statement since d is odd.

We can make a graded complex out of $\operatorname{End} C$ as follows. Define the linear map

$$\text{ad } d : \operatorname{End}_n C \to \operatorname{End}_{n+1} C \tag{II.1.15}$$

by

$$f \mapsto \big[d\,,\, f\big]\ . \tag{II.1.16}$$

Since $d^2 = 0$ and $(\text{ad } d)^2 = \text{ad } d^2$ the above map is a differential of degree one making $(\operatorname{End} C, \text{ad } d)$ into a graded complex. The cocycles are linear transformations of C which (anti)commute with d and are called **chain maps**; whereas the coboundaries are linear transformations which can be written as some (anti)commutator of d and are called **chain homotopic to zero**. If $f = \big[d\,,\, g\big]$ is chain homotopic to zero, g is called the **chain homotopy**. More generally, any two linear transformations (not necessarily chain maps) are said to be **chain homotopic** if their difference is a d (anti)commutator.

It turns out that we can understand the cohomology $H(\operatorname{End} C)$ in terms of $H(C)$ as follows. If $f \in \operatorname{End} C$ is a chain map, it induces a linear transformation f_* in $H(C)$ by

$$f_*[v] \equiv [fv]\ . \tag{II.1.17}$$

This linear transformation is clearly well-defined, *i.e.*, it does not depend on the choice of representative cocycle for the class $[v]$: for if $w = v + du$ then $fw = fv + fdu = fv \pm dfu$. Similarly if f and g are chain homotopic chain maps they induce the same map in $H(C)$. In fact, for any cocycle v, $fv - gv = \big[d\,,\, h\big]v = dhv$ and thus $[fv] = [gv]$, whence $f_* = g_*$. Therefore we have a natural linear map

$$H(\operatorname{End} C) \to \operatorname{End} H(C) \tag{II.1.18}$$

defined by

$$[f] \mapsto f_* \qquad \text{(II.1.19)}\ .$$

Two very natural questions pose themselves:

(i) Are all linear transformations of $H(C)$ induced by chain maps?

(ii) If a chain map induces the zero map in $H(C)$, is it necessarily chain homotopic to zero?

An affirmative answer to the first (resp. second) question is equivalent to the surjectivity (resp. injectivity) of the map $f \mapsto f_*$. Both answers are positive in the special case of C a finite dimensional vector space. We will give a proof in Chapter V in the context of the operator BRST cohomology.

Notice that $H(\mathrm{End}\, C)$ has a further algebraic structure. Namely it is a graded algebra with a multiplication

$$H^p(\mathrm{End}\, C) \otimes H^q(\mathrm{End}\, C) \longrightarrow H^{p+q}(\mathrm{End}\, C) \tag{II.1.20}$$

induced from composition of endomorphisms. To see this notice that

$$\mathrm{ad}\ d(\varphi \circ \psi) = (\mathrm{ad}\ d\,\varphi) \circ \psi + (-1)^g \varphi \circ (\mathrm{ad}\ d\psi) \qquad \text{for } \varphi \in \mathrm{End}_g C\ . \tag{II.1.21}$$

Therefore composition of endomorphisms maps

$$\begin{aligned} \ker\ \mathrm{ad}\ d \otimes \ker\ \mathrm{ad}\ d &\longrightarrow \ker\ \mathrm{ad}\ d \\ \ker\ \mathrm{ad}\ d \otimes \mathrm{im}\ \mathrm{ad}\ d &\longrightarrow \mathrm{im}\ \mathrm{ad}\ d\ , \end{aligned}$$

which makes the following operation well defined

$$[\varphi] \cdot [\psi] \longrightarrow [\varphi \circ \psi]\ . \tag{II.1.22}$$

Now we come to a very important concept which will underlie most of the work described in this dissertation: resolutions. In essence, a resolution of a given object consists of giving it a cohomological description in terms of simpler ones. The fundamental example of a resolution surfaces in Chapter III in our discussion of classical BRST cohomology; although its practical utility will become apparent in Chapters V-VIII. The main idea is very simple. Suppose for definiteness that we have a graded complex (C, d) with the property that all its cohomology resides in zeroth dimension. In other words,

$$H^n(C) = \begin{cases} 0 & \text{for } n \neq 0 \\ H & \text{for } n = 0 \end{cases}\ . \tag{II.1.23}$$

Then we say that the complex (C, d) provides a **resolution** of H. Of course, the utility of a resolution depends on the simplicity of the spaces C^n.

Let us see how one can use a resolution in order to simplify calculations. For this let us assume that C is a finite dimensional vector space so that $C^n = 0$ except for a finite number of n. Suppose further that f is a linear transformation of C which is also a chain map for d. We let f_* denote the linear transformation it induces on H. Then the following formula holds

$$\mathrm{Tr}_H f_* = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{Tr}_{C^n} f \ . \tag{II.1.24}$$

In particular if f is the identity we have

$$\dim H = \sum_{n\in\mathbb{Z}} (-1)^n \dim C^n \ , \tag{II.1.25}$$

which perhaps is more familiar if we realize that because of (II.1.23) $\dim H$ is really the **Euler characteristic** $\chi(C)$ of the complex (C, d):

$$\chi(C) \equiv \sum_{n\in\mathbb{Z}} (-1)^n \dim H^n(C) \ . \tag{II.1.26}$$

Formula (II.1.24) will be especially useful when we discuss no ghost theorems in Chapters VI-VIII.

A very special kind of resolution is one in which $C^n = 0$ for all $n > 0$. Then the complex can be pictured as follows

$$\cdots \longrightarrow C^{-2} \xrightarrow{d} C^{-1} \xrightarrow{d} C^0 \longrightarrow 0 \ . \tag{II.1.27}$$

The cohomology is given by

$$H^n(C) = \begin{cases} C^0/dC^{-1} \equiv H & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \ . \tag{II.1.28}$$

We call such resolutions **projective**. We can augment the complex as follows. We define d acting on C^0 to be the canonical surjection $C^0 \twoheadrightarrow C^0/dC^{-1}$ and we append this space as C^1 to the complex. This yields the following sequence

$$\cdots \longrightarrow C^{-2} \xrightarrow{d} C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} H \longrightarrow 0 \ , \tag{II.1.29}$$

which has the property that the kernel of any arrow is precisely the image of the preceding one. Hence this is an **exact sequence**. Therefore we see that a projective resolution of H consists in constructing an exact sequence with H sitting at the right.

Spectral Sequences

After this brief introduction to the most basic concepts of homological algebra it is upon us to introduce the reader to one of the most powerful gadgets at our disposal when trying to compute cohomologies: the spectral sequence. For the proofs of the theorems we quote in this section, the reader is referred to the books by Lang **[62]**, and Griffiths & Harris **[68]**. A more unified treatment of spectral sequences using Massey's concept of an exact couple can be found in the books by Bott & Tu **[69]**, and Hilton & Stammbach **[63]**. A complete treatment with applications can be found in the book by MacLane **[64]**.

Spectral sequences can be thought of as perturbation theory for cohomology, since it essentially allows us to approximate the cohomology of a complex by computing the cohomology of bigger and bigger chunks. By definition a **spectral sequence** is a sequence $\{(E_r, d_r)\}_{r=0,1,\ldots}$ of differential complexes where E_{r+1} is the cohomology of the preceding complex (E_r, d_r). In many cases of interest one has that for $r > R$, $E_r = E_{r+1} = \cdots = E_\infty$. In this case one says that the spectral sequence **spectral sequence** to E_∞ and one writes $(E_r) \Rightarrow E_\infty$.

The following is the typical use to which spectral sequences are put to in practice. Suppose we are interested in investigating the cohomology H of a certain complex. If we are lucky we may be able to show (if at all, usually by very general arguments) that there exists a spectral sequence converging to H, whose early (first and/or second) terms are easily computable. Thus one begins to approximate H. It may be that after the first or second term the differentials $\{d_r\}$ are identically zero. Then that term is already isomorphic to the limit term E_∞, in which case the spectral sequence is said to **spectral sequence** at the E_1 or E_2 terms. In that case we have reduced the computation of H to the computation of the cohomology of a much simpler complex. We will see plenty of examples of this phenomenon in the following chapters.

Sometimes however we are not so lucky and the spectral sequence does not degenerate early, yet it still provides us with a lot of useful information. In particular it can be used to obtain vanishing theorems. Let us elaborate on this. Throughout this work we will consider spectral sequences associated to graded complexes which will converge to the desired cohomology H in a way that will respect the grading. In other words, we will have convergence in each dimension: $(E_r^n) \Rightarrow H^n$ for all n. From the definition of the spectral sequence we notice that E_{r+1}^n is a subquotient of E_r^n and hence if for any r we have a vanishing of cohomology, say, $E_r^n = 0$ for some n, then the vanishing will persist and $H^n = 0$. This propagation of vanishing of cohomology is, in a nutshell, the essence of the vanishing theorems we will be concerned with in this work.

We now describe in some detail the spectral sequences with which we shall be concerned. Since they are all special cases of the spectral sequence which arises from a filtered complex, we start by considering these.

Let (C, d) be a differential complex. By a **filtration** of C we mean a sequence (not necessarily finite) of subspaces $FC = \{F^pC\}$ indexed by an integer p—called the **filtration degree**—such that, for all p, $F^pC \supseteq F^{p+1}C$ and such that $\cup_p F^pC = C$. We will deal exclusively with filtrations which are **filtration**: that is, there exist p_0 and p_1 such that

$$F^pC = \begin{cases} C & \text{for } p \leq p_0 \\ 0 & \text{for } p \geq p_1 \end{cases} . \tag{II.1.30}$$

If the differential respects the filtration, that is, $d\,F^pC \subseteq F^pC$, then (FC, d) is called a **filtered differential complex**.

Let FC be a bounded filtered complex. Then each F^pC is, in its own right, a complex under d and, therefore, its cohomology can be defined. The inclusion $F^pC \subseteq C$ induces a map in cohomology $H(F^pC) \to H(C)$ which, however, is generally not injective. To understand this notice that a cocycle in F^pC may be the differential of a cochain which does not belong to F^pC but to $F^{p-1}C$. Therefore the cohomology class it defines may not be trivial in $H(F^pC)$ but it may be in $H(C)$. Let us denote by $F^pH(C) \subseteq H(C)$ the image of $H(F^pC)$ under the aforementioned map. It is easy to verify that $FH(C)$ defines a filtration of $H(C)$ which is bounded if FC is.

To every filtered vector space FC we can associate a graded vector space $\mathrm{Gr}\, C = \bigoplus_p \mathrm{Gr}^p C$ where

$$\mathrm{Gr}^p C \equiv F^pC/F^{p+1}C\ . \tag{II.1.31}$$

It is easy to see that as vector spaces C and $\mathrm{Gr}\, C$ are isomorphic; although, since C is not necessarily graded, this isomorphism does not extend to an isomorphism of graded spaces.

If (FC, d) is a filtered differential complex then the associated graded space $\mathrm{Gr}\, C$ is also a complex whose differential is induced by d. Notice that if FC is bounded then $\mathrm{Gr}\, C$ is actually finite. Since d respects the filtration, upon passage to the quotient we obtain a map, also called d, which maps $d\colon \mathrm{Gr}^p C \to \mathrm{Gr}^p C$, whose cohomology is denoted by $H(\mathrm{Gr}\, C)$. Notice that although $\mathrm{Gr}\, C$ is graded, the differential has degree zero. This cohomology is usually easier to calculate than $H(C)$ or $H(FC)$; the reason being that the differential in the associated graded complex is usually a simpler operator: parts of d have positive filtration degree, mapping $F^pC \to F^{p+1}\,C$, in which case this is already zero in $\mathrm{Gr}^p C$.

The spectral sequence of a filtered complex relates the two spaces $\mathrm{Gr}\, H(C)$ and $H(\mathrm{Gr}\, C)$. In fact we have the following theorem:

Theorem II.1.32. *Let FC be a bounded filtered complex and $\operatorname{Gr} C$ its associated graded complex. Then there exists a spectral sequence $\{(E_r, d_r)\}$ of graded spaces*

$$E_r = \bigoplus_p E_r^p$$

with

$$d_r \colon E_r^p \to E_r^{p+r}$$

and such that

$$E_0^p \cong \operatorname{Gr}^p C \ ,$$
$$E_1^p \cong H(\operatorname{Gr}^p C) \ ,$$

and

$$E_\infty^p \cong \operatorname{Gr}^p H(C) \ .$$

Moreover the spectral sequence converges finitely to the limit term.

Now suppose that C is a graded complex and let FC be a filtration of C. In this case we can grade the filtration as follows: $F^pC = \bigoplus_n F^pC^n$ where $F^pC^n = F^pC \cap C^n$. The associated graded complex is now bigraded as follows $\operatorname{Gr} C = \bigoplus_{p,n} \operatorname{Gr}^p C^n$ with the obvious definition for $\operatorname{Gr}^p C^n$. Supposing that the filtration is bounded in each dimension we get a slightly modified version of the previous theorem:

Theorem II.1.33. *Let C be a graded complex, FC be a filtration which is bounded in each dimension and $\operatorname{Gr} C$ its associated graded complex. Then there exists a spectral sequence $\{(E_r, d_r)\}$ of bigraded spaces*

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

with

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and such that

$$E_0^{p,q} \cong \operatorname{Gr}^p C^{p+q} \ ,$$
$$E_1^{p,q} \cong H^{p+q}(\operatorname{Gr}^p C) \ ,$$

and

$$E_\infty^{p,q} \cong \operatorname{Gr}^p H^{p+q}(C) \ .$$

Moreover the spectral sequence converges finitely to the limit term.

There is a small *caveat* we must emphasize. The limit term of the spectral sequence is not the total cohomology but the graded object associated to the induced filtration. Of course, as vector spaces they are isomorphic but that is the end of the isomorphism. If the total cohomology has an extra algebraic structure (say it is an algebra, for instance) the theorem does not guarantee that the limit term E_∞ and the total cohomology as isomorphic as algebras.

The Spectral Sequences of a Double Complex

Two very important special cases of a filtered complex arise from a double complex. A **double complex** is a bigraded vector space $K = \bigoplus_{p,q} K^{p,q}$ (where, for definiteness, we take p,q integral; although this is not essential) and two differentials

$$D'\colon K^{p,q} \to K^{p+1,q} \tag{II.1.34}$$

$$D''\colon K^{p,q} \to K^{p,q+1} \tag{II.1.35}$$

which anticommute. It is often convenient to represent the double complex pictorially as follows

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & K^{p,q+1} & \xrightarrow{D'} & K^{p+1,q+1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle D''} & & \uparrow{\scriptstyle D''} & & \\
\cdots & \longrightarrow & K^{p,q} & \xrightarrow{D'} & K^{p+1,q} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array} \tag{II.1.36}$$

Hence we shall refer to D' and D'' as the horizontal and vertical differentials, respectively.

As far as the operator D' is concerned, the above double complex decomposes into a direct sum of graded complexes (the rows)

$$\cdots \longrightarrow K^{p,q} \xrightarrow{D'} K^{p+1,q} \longrightarrow \cdots \ ; \tag{II.1.37}$$

whose cohomology shall be denoted ${}'H^p(K^{\bullet,q})$ where the $\bullet$ just reminds us of which is the index running along with the cohomology we are taking. In other words,

$${}'H^p(K^{\bullet,q}) \equiv \frac{\ker\ D' : K^{p,q} \to K^{p+1,q}}{\mathrm{im}\ D' : K^{p-1,q} \to K^{p,q}} \ . \tag{II.1.38}$$

Since D'' anticommutes with D' (*i.e.*, it is a D'–chain map) it induces a map in ${}'H(K)$ which is also a differential since D'' is and which turns the columns of the double complex (after having taking D' cohomology) into graded complexes

$$\cdots \longrightarrow {}'H^p(K^{\bullet,q}) \xrightarrow{D''} {}'H^p(K^{\bullet,q+1}) \longrightarrow \cdots \ , \tag{II.1.39}$$

where, abusing a little the notation, we have called the differential also D''. We can therefore take D'' cohomology to obtain the spaces $''H^q('H^p(K))$ defined by

$$''H^q('H^p(K)) \equiv \frac{\text{ker } D'' : 'H^p(K^{\bullet,q}) \to 'H^p(K^{\bullet,q+1})}{\text{im } D'' : 'H^p(K^{\bullet,q-1}) \to 'H^p(K^{\bullet,q})} \ . \tag{II.1.40}$$

Reversing the roles of D' and D'' we obtain the cohomologies $'H^p(''H^q(K))$ by taking D' cohomology on the D'' cohomologies $''H^q(K^{p,\bullet})$.

What good are these cohomology groups? They will turn out to be first and second order approximations to the same "total" cohomology. Defining the total degree of vectors in $K^{p,q}$ as $p+q$ we may form a graded complex called the **total complex** and denoted by $\text{Tot}\, K = \bigoplus_n \text{Tot}^n K$ where

$$\text{Tot}^n K = \bigoplus_{p+q=n} K^{p,q} \ . \tag{II.1.41}$$

The differential in the total complex is $D = D' + D''$ and is called the **total differential**. Since the total differential has total degree 1

$$D \colon \text{Tot}^n K \to \text{Tot}^{n+1} K \ , \tag{II.1.42}$$

$(\text{Tot}\, K, D)$ becomes a graded complex. We shall deal exclusively with double complexes which satisfy the following finiteness condition: for each n there are only a finite number of non-zero $K^{p,q}$ with $p+q=n$.

There are two canonical filtrations associated to the graded complex $\text{Tot}\, K$. Define

$$'F^p \text{Tot}\, K = \bigoplus_q \bigoplus_{i \geq p} K^{i,q} \tag{II.1.43}$$

and

$$''F^q \text{Tot}\, K = \bigoplus_p \bigoplus_{j \geq q} K^{p,j} \ . \tag{II.1.44}$$

Fix n and define

$$'F^p \text{Tot}^n K = \bigoplus_{i \geq p} K^{i,n-i} \tag{II.1.45}$$

and

$$''F^q \text{Tot}^n K = \bigoplus_{j \geq q} K^{n-j,j} \ . \tag{II.1.46}$$

The finiteness condition for the double complex imply that the above filtrations are bounded for each n. Therefore, for each n, there exist p_0, p_1, q_0, and q_1—which depend on n—such that

$$'F^p\mathrm{Tot}^n K = \begin{cases} \mathrm{Tot}^n K & \text{for } p \leq p_0 \\ 0 & \text{for } p \geq p_1 \end{cases} , \tag{II.1.47}$$

and

$$''F^q\mathrm{Tot}^n K = \begin{cases} \mathrm{Tot}^n K & \text{for } q \leq q_0 \\ 0 & \text{for } q \geq q_1 \end{cases} . \tag{II.1.48}$$

By the previous theorem there is a spectral sequence associated to each of the filtrations defined above which converges finitely to the **total cohomology**, *i.e.*, the cohomology of the total complex $(\mathrm{Tot}\, K, D)$. What makes this example so important is that the earliest terms in the spectral sequence are easily described in terms of the original data (K, D', D''). In fact, one finds for the horizontal filtration:

Theorem II.1.49. *Associated to the filtration $'F\mathrm{Tot}\, K$ there exists a spectral sequence $\{('E_r, d_r)\}_{r=0,1,\dots}$ of bigraded vector spaces*

$$'E_r = \bigoplus_{p,q} {'E_r^{p,q}}$$

with

$$d_r\colon {'E_r^{p,q}} \to {'E_r^{p+r,q-r+1}}$$

such that

$$'E_0^{p,q} \cong K^{p,q} \ ,$$
$$'E_1^{p,q} \cong {''H^q}(K^{p,\bullet}) \ ,$$
$$'E_2^{p,q} \cong {'H^p}({''H^q}(K)) \ ,$$

and

$$'E_\infty^{p,q} \cong \mathrm{Gr}^p\, H^{p+q}(\mathrm{Tot}\, K) \ .$$

Similarly for the vertical filtration we have the following

Theorem II.1.50. *Associated to the filtration $''F\mathrm{Tot}\, K$ there exists a spectral sequence $\{(''E_r, d_r)\}_{r=0,1,\dots}$ of bigraded vector spaces*

$$''E_r = \bigoplus_{p,q} {''E_r^{q,p}}$$

with

$$d_r\colon {''E_r^{q,p}} \to {''E_r^{q+r,p-r+1}}$$

such that

$$''E_0^{q,p} \cong K^{p,q} \ ,$$
$$''E_1^{q,p} \cong {'H^p}(K^{\bullet,q}) \ ,$$
$$''E_2^{q,p} \cong {''H^q}({'H^p}(K)) \ ,$$

and

$$''E_\infty^{q,p} \cong \mathrm{Gr}^q\, H^{p+q}(\mathrm{Tot}\, K) \ .$$

As an application of the spectral theorems associated to a double complex let us prove a simple version of the algebraic Künneth formula. This formula relates the cohomology of a tensor product with the tensor product of the cohomologies. In general the relation between these two objects is governed by a universal coefficient theorem, but in the simple case we deal with, they will turn out to be isomorphic.

Suppose that (E, d) and (F, δ) are real **differential graded algebras**. That is, E (resp. F) is a real $\mathbb{Z}$-graded graded-commutative associative algebra $E = \bigoplus_{n\geq 0} E^n$ (resp. $F = \bigoplus_{n\geq 0} F^n$) such that each graded level is finite-dimensional and such that d (resp. δ) is a linear derivation on the algebra of degree 1 obeying $d^2 = 0$ (resp. $\delta^2 = 0$). Define a derivation D on $C \equiv E \otimes F$ as follows:

$$D(e \otimes f) = de \otimes f + (-1)^{\deg\, e} e \otimes \delta f \ . \tag{II.1.51}$$

It is easy to compute that $D^2 = 0$. C admits a bigrading $C^{p,q} \equiv E^p \otimes F^q$; although D does not have any definite properties with respect to it. Define $K^n \equiv \bigoplus_{p+q=n} C^{p,q}$. Then D has degree 1 with respect to this grading. In fact, C becomes a double complex under d and δ whose total complex is (K, D). Notice that for a fixed n, K^n consists of a finite number of $C^{p,q}$'s. Therefore the canonical filtrations associated to this double complex are bounded and we can use Theorem II.1.49 and Theorem II.1.50. One of the spectral sequences is enough to prove the Künneth formula so, for definiteness, we choose to use the horizontal filtration $'FK$. The $'E_1$ term in the spectral sequence is just the δ cohomology of the vertical complexes (indexed by p)

$$\cdots \longrightarrow C^{p,q-1} \xrightarrow{\delta} C^{p,q} \xrightarrow{\delta} C^{p,q+1} \longrightarrow \cdots \ . \tag{II.1.52}$$

But since $C^{p,q} = E^p \otimes F^q$, both E and F are vector spaces, and δ only acts on F^q, the cohomology of (II.1.52) is simply

$$'E_1^{p,q} = E^p \otimes H^q(F) \ . \tag{II.1.53}$$

The $'E_2$ term is the cohomology of the complexes (indexed by q)

$$\cdots \longrightarrow E^{p-1} \otimes H^q(F) \xrightarrow{d} E^p \otimes H^q(F) \xrightarrow{d} E^{p+1} \otimes H^q(F) \longrightarrow \cdots \ ; \tag{II.1.54}$$

which after similar reasoning allows us to conclude that its cohomology is simply

$$'E_2^{p,q} = H^p(E) \otimes H^q(F) \ . \tag{II.1.55}$$

Since the higher differentials d_r are essentially induced by the original differentials and these are already zero at the $'E_2$ level (since they are acting on their respective cohomologies) we see that the spectral sequence degenerates yielding the result

$$\boxed{H^n_D(E \otimes F) \cong \bigoplus_{p+q=n} H^p(E) \otimes H^q(F)} \tag{II.1.56}$$

which is the celebrated **Künneth formula**.

Lie Algebra Cohomology

A very interesting cohomology theory which is intimately linked to BRST cohomology is the cohomology theory of Chevalley & Eilenberg[66] for Lie algebras. For definiteness we shall only treat finite dimensional Lie algebras in this section.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathfrak{M}$ a $\mathfrak{g}$–module affording the representation

$$\begin{aligned} \mathfrak{g} \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (X, m) &\longrightarrow X \cdot m \ . \end{aligned} \tag{II.1.57}$$

Let $C^p(\mathfrak{g}, \mathfrak{M})$ denote the vector space of linear maps $\bigwedge^p \mathfrak{g} \to \mathfrak{M}$. That is, $C^p(\mathfrak{g}, \mathfrak{M}) \equiv \mathrm{Hom}(\bigwedge^p \mathfrak{g}, \mathfrak{M}) \cong \bigwedge^p \mathfrak{g}^* \otimes \mathfrak{M}$. The $C^p(\mathfrak{g}, \mathfrak{M})$ are called the p–**Lie algebra cochains** of $\mathfrak{g}$ with coefficients in $\mathfrak{M}$. Next we define a map $d : \mathfrak{M} \to C^1(\mathfrak{g}, \mathfrak{M})$ by $(dm)(X) = X \cdot m$ for all $X \in \mathfrak{g}$ and $m \in \mathfrak{M}$. Clearly, $\ker d = \mathfrak{M}^{\mathfrak{g}}$, *i.e.*, the $\mathfrak{g}$–invariant elements of $\mathfrak{M}$.

We now extend d to a map $d : C^1(\mathfrak{g}, \mathfrak{M}) \to C^2(\mathfrak{g}, \mathfrak{M})$ by defining it on monomials $\alpha \otimes m \in \mathfrak{g}^* \otimes \mathfrak{M} \cong C^1(\mathfrak{g}, \mathfrak{M})$ as

$$d(\alpha \otimes m) = d\alpha \otimes m - \alpha \wedge dm \ , \tag{II.1.58}$$

where $d\alpha \in \bigwedge^2 \mathfrak{g}^*$ is given by

$$(d\alpha)(X, Y) = -\alpha(\big[X \, , \, Y\big]) \ . \tag{II.1.59}$$

In other words, the map $d : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ is the negative transpose to the Lie bracket $\big[\, , \big] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. Next we extend d inductively to an odd derivation

$$\begin{aligned} &d : C^p(\mathfrak{g}, \mathfrak{M}) \to C^{p+1}(\mathfrak{g}, \mathfrak{M}) \\ &d(\omega \otimes m) = d\omega \otimes m + (-1)^p \omega \wedge dm \ . \end{aligned} \tag{II.1.60}$$

We claim that d so defined is actually a differential. Since d is an odd derivation, d^2 is an even derivation and one need only check it on generators: $\alpha \in \mathfrak{g}^*$ and $m \in \mathfrak{M}$. It is trivial to check that $d^2 m = 0$ due to the fact that $X \cdot (Y \cdot m) - Y \cdot (X \cdot m) = \big[X\,,\,Y\big] \cdot m$. Similarly, $d^2\alpha = 0$ due to the Jacobi identity. Therefore, $d^2 = 0$ and

$$C^0(\mathfrak{g}, \mathfrak{M}) \xrightarrow{d} C^1(\mathfrak{g}, \mathfrak{M}) \xrightarrow{d} C^2(\mathfrak{g}, \mathfrak{M}) \xrightarrow{d} \cdots \tag{II.1.61}$$

is a graded complex whose cohomology $H(\mathfrak{g}, \mathfrak{M})$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in $\mathfrak{M}$. In particular, $H^0(\mathfrak{g}, \mathfrak{M}) = \mathfrak{M}^{\mathfrak{g}}$.

In particular, if $\mathbb{R}$ denotes the trivial $\mathfrak{g}$ module, we have that $H(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}$. The first and second cohomology $H^1(\mathfrak{g}, \mathbb{R})$ and $H^2(\mathfrak{g}, \mathbb{R})$ have useful algebraic interpretations. Let $\alpha \in \mathfrak{g}^*$. Then $d\alpha = 0$ if and only if, for every $X, Y \in \mathfrak{g}$, $\alpha(\big[X\,,\,Y\big]) = 0$, *i.e.*, if the linear functional α is identically zero in the first derived ideal $\big[\mathfrak{g}\,,\,\mathfrak{g}\big]$. In other words, we have an isomorphism

$$H^1(\mathfrak{g}, \mathbb{R}) \cong \mathfrak{g}/\big[\mathfrak{g}\,,\,\mathfrak{g}\big] \ , \tag{II.1.62}$$

from which we deduce that $H^1(\mathfrak{g}, \mathbb{R}) = 0 \iff \big[\mathfrak{g}\,,\,\mathfrak{g}\big] = \mathfrak{g}$. Similarly, let $c \in \bigwedge^2 \mathfrak{g}^*$ obey $dc = 0$. This is equivalent to the cocycle condition

$$c(\big[X\,,\,Y\big], Z) + c(\big[Y\,,\,Z\big], X) + c(\big[Z\,,\,X\big], Y) = 0 \ , \tag{II.1.63}$$

for all $X, Y, Z \in \mathfrak{g}$. To interpret this algebraically, toss in an extra abstract generator k and consider the augmented space $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus k\mathbb{R}$ and define a new bracket by

$$\big[X\,,\,Y\big]_c = \big[X\,,\,Y\big] + c(X, Y)\,k \ , \tag{II.1.64}$$

and by the requirement that k be central. Then the cocycle condition (II.1.63) is equivalent to the Jacobi identities for the new bracket. Hence $\widehat{\mathfrak{g}}$ becomes a Lie algebra. In fact, it is a one-dimensional central extension of $\mathfrak{g}$. If $c = d\alpha$ for some linear functional $\alpha \in \mathfrak{g}^*$ then we can define $\widetilde{X} = X - \alpha(X)\,k \in \widehat{\mathfrak{g}}$ so that

$$\big[\widetilde{X}\,,\,\widetilde{Y}\big]_c = \widetilde{\big[X\,,\,Y\big]} \ ; \tag{II.1.65}$$

hence the central element drops out. Therefore $H^2(\mathfrak{g}, \mathbb{R})$ is in bijective correspondence with the equivalence classes of non-trivial central extensions of $\mathfrak{g}$.

There is a classic theorem in Lie algebra cohomology known as the **Weyl lemma**:

Theorem II.1.66. *If* $\mathfrak{g}$ *is a finite dimensional real semisimple Lie algebra then* $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$.

Cohomologywise semisimple Lie algebras are not very exciting. In fact, an equivalent characterization of semisimple finite dimensional Lie algebras is that their cohomology groups $H^p(\mathfrak{g}, \mathfrak{M})$ vanish for any non-trivial irreducible module $\mathfrak{M}$.

We shall have more to say about Lie algebra cohomology in Chapter VI when we relate BRST to the semi-infinite cohomology of Feigin.

2. Symplectic Reduction and Dirac's Theory of Constraints

In this section we establish the vocabulary and notation concerning symplectic geometry and phrase Dirac's theory of constraints in a slightly more geometric language. We also discuss symplectic reduction, as this will be a dominant theme in our treatment of classical BRST cohomology. This section is not meant to be expository but rather a brief reacquaintance with the classical mechanics of constrained systems from a slightly more geometric approach in the coordinate-free language of modern differential geometry. Any and all proofs missing from our treatment can be found in varying degrees of mathematical sophistication in the books by Arnold **[70]**, Abraham & Marsden **[71]**, Guillemin & Sternberg **[72]**, and in the excellent notes of Weinstein **[73]**. The classical treatment of constraints is to be found in Dirac's wonderful notes **[16]**.

We start by setting up the notation we will adhere to throughout the rest of our discussion. We then discuss symplectic reduction with respect to a coisotropic submanifold, which will be the geometric framework in which Dirac's theory of first class constraints will be treated. We end the section with a look at a very important special case of first class constraints: those arising from a moment map. Since we are eventually interested in classical BRST cohomology we are mostly concerned with first class constraints. However, second class constraints have an equally solid geometric underpinning, known as symplectic restriction, which, in an attempt to offer the reader unfamiliar with this language another reference point, we have decided to cover as well.

Elementary Symplectic Geometry

A **symplectic manifold** is a pair (M, Ω) consisting of a differentiable manifold M and a closed smooth non-degenerate 2–form Ω. The condition of non-degeneracy refers to the property that the induced map $\Omega^\flat$ taking vector fields to 1–forms and defined by $X \mapsto \Omega(X, \cdot)$ is an isomorphism. In other words, that if $\Omega(X, Y) = 0$ for all vector fields Y, then this implies that $X = 0$. Notice that this requires M to be even dimensional.

The prime example of a symplectic manifold is the cotangent bundle T^*N of a differentiable manifold. This corresponds to the phase space of the configuration space N.

Choose local coordinates q^i for N and let p^i denote coordinates for the covectors. Then the symplectic form for T^*N is given by $\Omega = -d\theta$, where θ is the canonical 1-form on T^*N given locally by $\sum_i p^i\, dq^i$.

The **symplectic form** Ω allows us to define a bracket in the ring $C^\infty(M)$ of smooth functions on M as follows. Given a function $f \in C^\infty(M)$ we define its associated **hamiltonian vector field** X_f as the unique vector field on M satisfying

$$\Omega^\flat(X_f) + df = 0 \ . \tag{II.2.1}$$

We then define the **Poisson bracket** of two functions $f, g \in C^\infty(M)$ as

$$\{f\,,\,g\} = \Omega(X_f, X_g) \ . \tag{II.2.2}$$

The Poisson bracket is clearly antisymmetric and, moreover, because Ω is closed, obeys the Jacobi indentity. Therefore it makes $C^\infty(M)$ into a Lie algebra. Since functions can be added and multiplied, $C^\infty(M)$ is also a commutative, associative algebra; and both of these structures are further linked by the following relation

$$\{f\,,\,gh\} = \{f\,,\,g\}h + g\{f\,,\,h\} \ , \tag{II.2.3}$$

valid for any $f, g, h \in C^\infty(M)$. A commutative, associative algebra possessing, in addition, a Lie bracket obeying (II.2.3) is called a **Poisson algebra**.

A classic theorem of Darboux says that locally on any symplectic manifold we can always find coordinates (p^i, q^i) such that the symplectic form takes the classic form

$$\Omega = \sum_i dq^i \wedge dp^i \ . \tag{II.2.4}$$

Therefore if f is a smooth function, its hamiltonian vector field is given by

$$X_f = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial}{\partial q^i} \right) \ , \tag{II.2.5}$$

and if f, g are smooth functions their Poisson bracket takes the familiar form

$$\{f\,,\,g\} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} \right) \ , \tag{II.2.6}$$

which is nothing but $X_f(g)$. Therefore Darboux's theorem just says that locally any symplectic manifold looks just like a phase space of a linear configuration space.

Now fix a point $p \in M$ and look at the vector space T_pM of tangent vectors to M at p; *i.e.*, the space of velocities at p. The symplectic form—being tensorial—restricts nicely to a non-degenerate antisymmetric form on T_pM, making it into a **symplectic vector space**. In a symplectic vector space V, there are four kinds of subspaces which merit our attention. If W is a subspace of V, we let $W^\perp$ denote its **symplectic complement** relative to the symplectic form Ω:

$$W^\perp = \{X \in V \mid \Omega(X,Y) = 0 \ \forall Y \in V\} \ . \tag{II.2.7}$$

Notice that if W is one dimensional, $W \subseteq W^\perp$ due to the antisymmetry of Ω. Subspaces W obeying $W \subseteq W^\perp$ are called **isotropic** and they necessarily obey $\dim W \leq \frac{1}{2}\dim V$. On the other hand, if $W \supseteq W^\perp$, W is called **coisotropic** and it must obey $\dim W \geq \frac{1}{2}\dim V$. If W is both isotropic and coisotropic, then it is its own symplectic complement, it obeys $\dim W = \frac{1}{2}\dim V$ and it is called a **lagrangian** subspace. Finally, if $W \cap W^\perp = 0$, W is called **symplectic**.

Notice that if W is isotropic and, in particular, lagrangian, the restriction of Ω to W is identically zero; whereas if W is symplectic, Ω restricts nicely to a symplectic form. In particular, symplectic subspaces are even dimensional. The most interesting case for us is when W is coisotropic. In this case Ω restricts to a non-zero antisymmetric bilinear form on W but which, nevertheless, is degenerate since any vector in $W^\perp \subseteq W$ is symplectically orthogonal to all of W. But it then follows that the quotient $W/W^\perp$ inherits a well defined symplectic form and hence becomes a symplectic vector space. The passage from V to $W/W^\perp$ (which is a subquotient) is known as the **symplectic reduction** of V relative to the coisotropic subspace W. The next subsection is devoted to the globalization of this procedure.

Symplectic Reduction

A submanifold M_o of a symplectic manifold M is called **isotropic**, **coisotropic**, **lagrangian**, or **symplectic** according to whether at all points $p \in M_o$, T_pM_o is an isotropic, coisotropic, lagrangian, or symplectic subspace of T_pM, respectively.

Suppose that M_o is a coisotropic submanifold of M and let $i : M_o \hookrightarrow M$ denote the inclusion. We let $\Omega_o \equiv i^*\Omega$ denote the pull back of the symplectic form of M onto M_o. It defines a distribution (in the sense of Frobenius), which we call $TM_o^\perp$, as follows. For $p \in M_o$ we let $(TM_o^\perp)_p = (T_pM_o)^\perp$. We will first show that this distribution is involutive. To this effect, let $X, Y \in TM_o^\perp$. Since Ω_o is closed, for all vector fields Z tangent to M_o, we have that

$$
\begin{aligned}
0 =& d\Omega_o(X,Y,Z) \\
=& X\Omega_o(Y,Z) - Y\Omega_o(X,Z) + Z\Omega_o(X,Y) \\
& - \Omega_o(\big[X\,,\,Y\big],Z) + \Omega_o(\big[X\,,\,Z\big],Y) - \Omega_o(\big[Y\,,\,Z\big],X)\ .
\end{aligned}
\tag{II.2.8}
$$

But all terms except the fourth are automatically zero since they involve Ω_o contractions between TM_o and $TM_o^\perp$. Therefore the fourth term is also zero, whence $\big[X\,,\,Y\big] \in TM_o^\perp$. Therefore, by Frobenius' theorem, $TM_o^\perp$ are the tangent spaces to a foliation of M_o which we denote $\mathcal{M}_o^\perp$. We define $\widetilde{M} \equiv M_o/\mathcal{M}_o^\perp$ to be the space of leaves of the foliation and we let $\pi : M_o \twoheadrightarrow \widetilde{M}$ be the natural surjection mapping a point in M_o to the unique leaf it belongs to. Then locally (and also globally, if the foliation is sufficiently well behaved) $\widetilde{M}$ is a smooth manifold, whose tangent space at a leaf is isomorphic to $T_pM_o/T_pM_o^\perp$ for any point p lying in that leaf. We can therefore give $\widetilde{M}$ a symplectic structure $\widetilde{\Omega}$ by demanding that $\pi^*\widetilde{\Omega} = \Omega_o$. In other words, let $\widetilde{X}$, $\widetilde{Y}$ be vectors tangent to $\widetilde{M}$ at a leaf. To compute $\widetilde{\Omega}(\widetilde{X},\widetilde{Y})$ we merely lift $\widetilde{X}$ and $\widetilde{Y}$ to vectors X_o and Y_o tangent to M_o at a point p in the leaf and then compute $\Omega_o(X_o,Y_o)$. The result is clearly independent of the particular lift, since the difference of any two lifts is in $TM_o^\perp$; and, moreover, it is also independent of the particular point p of the leaf since, if Z is a tangent vector to the leaf, the Lie derivative of Ω_o by Z:

$$
\mathcal{L}_Z\Omega_o = d\,\imath(Z)\Omega_o + \imath(Z)d\Omega_o \tag{II.2.9}
$$

vanishes since $d\Omega_o = 0$ and $\imath(Z)\Omega_o = 0$. Therefore $(\widetilde{M},\widetilde{\Omega})$ becomes a symplectic manifold (at least locally) and it is called the **symplectic reduction** of (M,Ω) relative to the coisotropic submanifold (M_o,Ω_o).

Suppose now that M_o is a symplectic submanifold of M and let $i : M_o \hookrightarrow M$ denote the inclusion. We can give M_o a symplectic structure merely by pulling back Ω to M_o. Hence, if $\Omega_o \equiv i^*\Omega$, (M_o,Ω_o) becomes a symplectic manifold, called the **symplectic restriction** of M onto M_o. In this case we can work out fairly explicitly the Poisson bracket of M_o in terms of the Poisson bracket of M: obtaining, as a special case, the celebrated Dirac bracket. We will impose, for convenience, the additional technical assumption that M_o is a closed imbedded submanifold of M. This is necessary and sufficient[**74**] to be able to extend any smooth function on M_o to a smooth function on M and to guarantee that all smooth functions on M_o can be obtained by restriction of smooth functions on M. Most cases that arise in practice satisfy this condition; although this could be precisely why these are the cases that arise in practice.

Let f and g be smooth functions on M_o and let us extend them to smooth functions on M which, allowing ourselves some notational abuse, will also be denoted by f and g, respectively. Let X_f and X_g be their respective hamiltonian vector fields on M, *i.e.*, computed with Ω. Since M_o is symplectic, the tangent space of M at every point $p \in M_o$ can written as the following direct sum

$$T_pM = T_pM_o \oplus (T_pM_o)^\perp ,$$

according to which a vector field X can be decomposed as the sum of two vectors: X_T, tangent to M_o; and $X^\perp$ symplectically perpendicular to M_o. Then the Poisson bracket of the two functions f and g on M_o is simply given by

$$\{f\,,\,g\}_o = \Omega(X_f - X_f^\perp, X_g - X_g^\perp)\ . \tag{II.2.10}$$

Now suppose that $\{Z_\alpha\}$ is a local basis for $TM_o^\perp$.[7] Then, given any vector X we can expand its normal part $X^\perp$ as linear combinations of the Z_α whose coefficients are easily determined as follows. Write

$$X^\perp = \sum_\alpha \lambda^\alpha X_\alpha\ . \tag{II.2.11}$$

Then notice that

$$\Omega(X, Z_\alpha) = \Omega(X^\perp, Z_\alpha) = \sum_\beta \lambda^\beta \Omega(Z_\beta, Z_\alpha)\ . \tag{II.2.12}$$

Because M_o is a symplectic submanifold, the square matrix $\mathbb{M}$ whose entries are given by $\mathbb{M}_{\alpha\beta} = \Omega(Z_\alpha, Z_\beta)$ is invertible. Let $\mathbb{M}^{\alpha\beta}$ be defined by

$$\sum_\beta \mathbb{M}_{\alpha\beta}\mathbb{M}^{\beta\gamma} = \delta_\alpha^\gamma\ . \tag{II.2.13}$$

Then the coefficients λ^α are given by

$$\lambda^\beta = \sum_\alpha \Omega(X, X_\alpha)\mathbb{M}^{\alpha\beta}\ . \tag{II.2.14}$$

---

[7] A sufficient and necessary condition[**75**] for the existence of a global basis is for M_o to be expressible as the zero locus of $(\dim M - \dim M_o)$ smooth functions $\{\chi_\alpha\}$. In that case, the global basis is just given by the hamiltonian vector fields associated to the $\{\chi_\alpha\}$. In general one can easily show that there exist functions $\{\chi_\alpha\}$ which locally describe M_o as their zero locus and whose hamiltonian vector fields provide a local basis for the normal vectors.

Plugging (II.2.14) into (II.2.11) and this into (II.2.10) we find that

$$\{f\,,\,g\}_o = \{f\,,\,g\} - \sum_{\alpha\beta} \Omega(X_f, Z_\alpha)\mathbb{M}^{\alpha\beta}\Omega(Z_\beta, X_g)\ . \tag{II.2.15}$$

If we further suppose that the $\{Z_\alpha\}$ are the hamiltonian vector fields associated (via Ω) to functions $\{\chi_\alpha\}$, then

$$\boxed{\{f\,,\,g\}_o = \{f\,,\,g\} - \sum_{\alpha\beta}\{f\,,\,\chi_\alpha\}\mathbb{M}^{\alpha\beta}\{\chi_\beta\,,\,g\}}\ , \tag{II.2.16}$$

where $\mathbb{M}^{\alpha\beta}$ is now the matrix inverse to the $\{\chi_\alpha\,,\,\chi_\beta\}$. Therefore, $\{\ ,\ \}_o$ in nothing but the **Dirac bracket** associated to the second class constraints $\{\chi_\alpha\}$.

First and Second Class Constraints

The purpose of this subsection is to show that the submanifold defined by a set of first class (resp. second class) constraints is coisotropic (resp. symplectic). But first we review Dirac's treatment of constraints. Throughout this subsection (M, Ω) shall be a fixed symplectic manifold on which we have singled out a privileged set of smooth functions $\{\psi_a\}$ which are called **constraints**. That is, the allowed "phase space" of the relevant dynamical system is the zero locus of the constraints

$$\{p \in M \mid \psi_a(p) = 0\ \forall\, a\}\ . \tag{II.2.17}$$

Of course the truly physically relevant information that the constraints convey is their zero locus. Any other set of functions with the same zero locus gives an equivalent description of the physics and this is why, in the modern literature (*cf.* **[71]** and references therein) on constrained dynamics, it is often the subvariety defined by (II.2.17) which is called the constraint. However in practice one needs an algebraic description of the constraints and there the $\{\psi_a\}$ play a crucial rôle; although we should (and will) at the end of the day make sure that none of our constructions depend on the particular choice of functions $\{\psi_a\}$.

Following Dirac let us denote by Ψ the linear subspace of $C^\infty(M)$ generated by the $\{\psi_a\}$; in other words, Ψ consists of linear combinations of the $\{\psi_a\}$ with constant coefficients. Let us also denote by J the ideal of $C^\infty(M)$ they generate. That is, linear combinations of the $\{\psi_a\}$ whose coefficients are arbitrary smooth functions. Then let F be a maximal subspace of Ψ with the property that

$$\{F\,,\,\Psi\} \subset J\ . \tag{II.2.18}$$

Let $\{\phi_i\}_{i=1}^l$ be a basis for F. The $\{\phi_i\}$ are linear combinations with constant coefficients of the $\{\psi_a\}$. Dirac calls the aforementioned basis for F **first class constraints**. Let the subspace S of Ψ complementary to F be spanned by $\{\chi_\alpha\}_{\alpha=1}^k$. Dirac calls these functions **second class constraints**. In terms of these functions, (II.2.18) just says that

$$\{\phi_i\,,\,\phi_j\} = f_{ij}{}^k\phi_k + f_{ij}{}^\alpha\chi_\alpha \tag{II.2.19}$$

$$\{\phi_i\,,\,\chi_\alpha\} = f_{i\alpha}{}^j\phi_j + f_{i\alpha}{}^\beta\chi_\beta \tag{II.2.20}$$

for arbitrary smooth functions $f_{ij}{}^k$, $f_{ij}{}^\alpha$, $f_{i\alpha}{}^j$, and $f_{i\alpha}{}^\beta$.

Dirac goes on to prove[16] that the matrix of functions $\{\chi_\alpha\,,\,\chi_\beta\}$ is nowhere degenerate. This, we will now show, is nothing but the statement that the submanifold defined by the second class constraints is symplectic. We will work under the additional technical assumption that zero is a regular value for the function $\Xi : M \to \mathbb{R}^k$ whose components are the second class constraints, *i.e.*, $\Xi(m) = (\chi_1(m), \ldots, \chi_k(m))$. This will guarantee[74] that the submanifold $N \equiv \Xi^{-1}(0)$ defined by the second class constraints is a closed imbedded submanifold of M. Then the vectors tangent to N are precisely those vectors which are perpendicular to the gradients of the constraints. That is, X is a tangent vector to N if, and only if, $d\chi_\alpha(X) = 0$ for all α. By the definition of the hamiltonian vector fields associated to the constraints, and denoting these by Z_α, the above condition translates into

$$X \in TN \iff \Omega(X, Z_\alpha) = 0\ \forall\,\alpha\ . \tag{II.2.21}$$

Let us denote by $\langle Z_\alpha\rangle$ the span of the vector fields Z_α. Then $TN = \langle Z_\alpha\rangle^\perp$. Since $\Omega(Z_\alpha, Z_\beta) = \{\chi_\alpha\,,\,\chi_\beta\}$ is non-degenerate, $\langle Z_\alpha\rangle \cap TN = 0$. Taking symplectic complements, $TN \cap TN^\perp = 0$, whence N is a symplectic submanifold of M. Therefore we can restrict ourselves to the symplectic manifold N with the Poisson bracket given by (II.2.16).

We now restrict the first class constraints $\{\phi_i\}$ to N. Allowing a little abuse of notation we still denote them $\{\phi_i\}$. Due to (II.2.19) and (II.2.16) they are still first class constraints. We again put them together in a function $\Phi : N \to \mathbb{R}^l$ and assume that 0 is a regular value of Φ, so that the submanifold $N_o \equiv \Phi^{-1}(0)$ defined by them is a closed imbedded submanifold. We now claim that N_o is a coisotropic submanifold of N. Again the tangent vectors to N_o are those vectors tangent to N such that they are annihilated by the gradients of the constraints

$$X \in TN_o \iff d\phi_i(X) = 0 \;\forall\, i \tag{II.2.22}$$

which, using the definition of the hamiltonian vector fields $\{X_i\}$ associated to the constraints $\{\phi_i\}$, translates into

$$TN_o = \langle X_i \rangle^\perp \;. \tag{II.2.23}$$

But—since the constraints are first class—

$$d\phi_i(X_j) = \big\{\phi_i \,,\, \phi_j\big\} = c_{ij}^k \phi_k \;, \tag{II.2.24}$$

which is zero on N_o. Therefore the X_i are tangent to N_o. This is equivalent, taking the symplectic complement of (II.2.23), to

$$TN_o^\perp \subset TN_o \;; \tag{II.2.25}$$

and, hence, to the coisotropy of N_o in N.

The Moment Map

A very special example of first class constraints arises in some cases when (M, Ω) admits a group action which preserves the symplectic structure. A diffeomorphism φ of M is called a **symplectomorphism** if $\varphi^*\Omega = \Omega$, *i.e.*, if it preserves the symplectic structure. Let $\mathrm{Symp}(M)$ denote the Lie subgroup of $\mathrm{Diff}(M)$ consisting of symplectomorphisms. Its Lie algebra $\mathfrak{symp}(M)$ is the Lie subalgebra of the Lie algebra of smooth vector fields on M consisting of those vector fields X obeying $\mathcal{L}_X\Omega = 0$. Such vector fields are called **symplectic**. Since Ω is closed this is equivalent to $\imath(X)\Omega$ being closed. Hence $\mathfrak{symp}(M)$ is the image of the closed 1–forms via the map $\Omega^\sharp$ inverse to $\Omega^\flat$. The image of the exact 1–forms is an ideal $\mathfrak{ham}(M) \subseteq \mathfrak{symp}(M)$ known as the **hamiltonian vector fields**. In fact, more is true:

$$\big[\mathfrak{symp}(M) \,,\, \mathfrak{symp}(M)\big] \subseteq \mathfrak{ham}(M) \;. \tag{II.2.26}$$

Now suppose that G is a Lie group acting on M via symplectomorphisms. Then this action defines a Lie algebra morphism $\mathfrak{g} \to \mathfrak{symp}(M)$ sending a vector $X \in \mathfrak{g}$ to a symplectic vector field $\widetilde{X}$. If for all $X \in \mathfrak{g}$, $\widetilde{X}$ is a hamiltonian vector field, then the G action is called **hamiltonian**. Notice that because of (II.2.26), if $\big[\mathfrak{g} \,,\, \mathfrak{g}\big] = \mathfrak{g}$—*i.e.*, if $H^1(\mathfrak{g}, \mathbb{R}) = 0$—then this is automatically satisfied. Also if all closed forms are exact, *i.e.*, $H^1_{dR}(M) = 0$, the action is also hamiltonian. Hence we see that the obstructions to a symplectic action being hamiltonian are cohomological in nature.

Suppose then that the G action is hamiltonian. That is, there exist functions ϕ_X for each $X \in \mathfrak{g}$ obeying

$$\imath(\widetilde{X})\Omega + d\phi_X = 0 \ . \tag{II.2.27}$$

The existence of these functions provides a linear map $\mathfrak{g} \to C^\infty(M)$, sending $X \to \phi_X$ which, nevertheless, may fail to be a Lie algebra morphism. To identify the obstruction in this case let us compute.

$$\begin{aligned} d\{\phi_X\,,\,\phi_Y\} =& d\Omega(\widetilde{X},\widetilde{Y}) \\ =& d\imath(\widetilde{Y})\imath(\widetilde{X})\Omega \\ =& \mathcal{L}_{\widetilde{Y}}\imath(\widetilde{X})\Omega && \text{since } \widetilde{X} \in \mathfrak{symp}(M) \\ =& [\mathcal{L}_{\widetilde{Y}}\,,\,\imath(\widetilde{X})]\Omega && \text{since } \widetilde{Y} \in \mathfrak{symp}(M) \\ =& \imath([\widetilde{Y}\,,\,\widetilde{X}])\Omega \\ =& d\phi_{[X\,,\,Y]} \ . \end{aligned}$$

Therefore,

$$c(X,Y) \equiv \{\phi_X\,,\,\phi_Y\} - \phi_{[X\,,\,Y]} \tag{II.2.28}$$

is locally constant. We shall assume for simplicity that M is connected and hence it is an honest constant. It is evident that c is antisymmetric and also that it obeys the cocycle conditions

$$c([X\,,\,Y],Z) + c([Y\,,\,Z],X) + c([Z\,,\,X],Y) = 0 \ . \tag{II.2.29}$$

Therefore it defines a projective representation of $\mathfrak{g}$. Notice that ϕ_X are defined up to a constant (*cf.* (II.2.27)) and hence $c(X,Y)$ is defined up to the addition of a term $b([X\,,\,Y])$ where b is an arbitrary linear functional on $\mathfrak{g}$. If by redefining the ϕ_X in this way we can shift c to zero, we have an honest representation and we say that the action is **Poisson**. If this is the case, the $\{\phi_i\}$, associated to a basis $\{X_i\}$ for $\mathfrak{g}$, are first class constraints. In particular, if $H^2(\mathfrak{g},\mathbb{R}) = 0$, $\mathfrak{g}$ admits non non-trivial central extension and the action is, again, Poisson. So we see again that the obstruction is cohomological in nature. A very nice derivation of these obstructions in terms of equivariant cohomology is given in the notes of Weinstein[73].

Let us suppose that we have a Poisson action of G on (M,Ω). We define the **moment map** $\Phi : M \to \mathfrak{g}^*$ dual to $\mathfrak{g} \to C^\infty(M)$ by

$$\langle \Phi(m), X\rangle = \phi_X(m) \ , \tag{II.2.30}$$

where $\langle,\rangle$ is the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The Poisson property of the action guarantees that this map is **moment map**: intertwining between the action of $\mathfrak{g}$ on M and the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Let $M_o \equiv \Phi^{-1}(0)$. If 0 is a regular value then M_o is a G–invariant coisotropic closed imbedded submanifold of M. In particular, the symplectic Killing vectors $\widetilde{X}$ are tangent to M_o and they define a foliation $\mathcal{G}$ of M_o whose leaves are the orbits of the G action, *i.e.*, the **gauge orbits**. The space of orbits $\widetilde{M} \equiv M_o/\mathcal{G}$ is (at least locally) a symplectic manifold and is a special case of the symplectic reduction of Marsden & Weinstein[**76**].

Symplectic Reduction of a Phase Space

In physics most symplectic manifolds are phase spaces, *i.e.*, cotangent bundles T^*N of a suitable configuration space N. Moreover many of the symmetries that arise in the study of dynamical systems are already symmetries of the configuration space. For example, in Yang-Mills the configuration space is the (convex) space $\mathfrak{A}$ of gauge fields (=connection 1-forms in a principal bundle over spacetime) and the gauge transformations $\mathfrak{G}$ have a well defined action on the connections. The physical configuration space is the space of gauge orbits $\mathfrak{A}/\mathfrak{G}$. Another example is given by bosonic string theory. The configuration space is the space of smooth maps $\mathrm{Map}(S^1, M)$ from the string to spacetime; whereas the physical configurations cannot distinguish between two smooth maps which are related by a reparametrization of the string. Hence the physical configurations are the space of orbits under $\mathrm{Diff}\, S^1$. Finally another example is general relativity in the hamiltonian description. Fixing a spacelike hypersurface Σ in spacetime, the configuration space is the "superspace" consisting of riemannian metrics on Σ. Just like in the string, to obtain the physical configurations we must identify configurations which are related by a diffeomorphism of Σ.

It turns out that whenever the configuration space N admits a smooth group action, the action automatically lifts to the phase space T^*N in such a way that it does not just preserves the symplectic form, but it also gives rise to an equivariant moment map which is linear in the momenta. That the action on N lifts to a symplectic action on T^*N follows from the fact that the canonical 1-form θ on T^*N is a diffeomorphism invariant of N. In other words, let $\varphi : N \to N$ be a diffeomorphism and let $T^*\varphi$ denote the induced diffeomorphism on T^*N. Then $(T^*\varphi)^*\theta = \theta$. Hence it also preserves the symplectic form $\Omega = -d\theta$.

So let G act on N via diffeomorphisms. Then if $X \in \mathfrak{g}$ is a vector in the Lie algebra, it gives rise to a Killing vector $\widetilde{X}$ on N and a Killing vector $\widehat{X}$ in T^*N. Since the canonical 1–form θ is G invariant, we have that

$$\begin{aligned} 0 =& \mathcal{L}_{\widehat{X}}\theta \\ =& d\imath(\widehat{X})\theta + \imath(\widehat{X})d\theta \end{aligned}$$

$$=d\imath(\widehat{X})\theta - \imath(\widehat{X})\Omega \ ,$$

Hence $\imath(\widehat{X})\Omega = d\imath(\widehat{X})\theta$, whence the hamiltonian function associated to X is $\phi_X = -\theta(\widehat{X})$. Therefore the G action is hamiltonian. But for $X, Y \in \mathfrak{g}$,

$$\begin{aligned}
\phi_{[X\,,Y]} &= -\,\imath([\widehat{X}\,,\,\widehat{Y}])\theta \\
&= -\,[\mathcal{L}_{\widehat{X}}\,,\,\imath(\widehat{Y})]\theta \\
&= -\,\mathcal{L}_{\widehat{X}}\imath(\widehat{Y})\theta && \text{since } \mathcal{L}_{\widehat{X}}\theta = 0 \\
&= -\,\imath(\widehat{X})d\imath(\widehat{Y})\theta \\
&= \imath(\widehat{X})\imath(\widehat{Y})d\theta && \text{since } \mathcal{L}_{\widehat{Y}}\theta = 0 \\
&= \Omega(\widehat{X}, \widehat{Y}) \\
&= \{\phi_X\,,\,\phi_Y\} \ .
\end{aligned} \tag{II.2.31}$$

Therefore the action is also Poisson.

The induced equivariant moment map is easy to write down explicitly. Let $\alpha \in T^*N$ be thought of as a 1-form on N at the point $\widetilde{\pi}(\alpha) \in N$, where $\widetilde{\pi} : T^*N \twoheadrightarrow N$ is the canonical projection sending a covector on N to the point on which it is defined. Then the moment map $\Phi : T^*N \to \mathfrak{g}^*$ is given by

$$\langle \Phi(\alpha), X\rangle = \langle \alpha, \widetilde{X}\rangle_{\widetilde{\pi}(\alpha)} \ , \tag{II.2.32}$$

where the right hand side of this equation refers to the dual pairing between tangent vectors and covectors on N at the point $\widetilde{\pi}(\alpha)$. Given local coordinates (p, q) on T^*N associated to local coordinates q for N, we have that the components of the moment map are

$$\phi_X(p, q) = p_i \widetilde{X}^i(q) \ , \tag{II.2.33}$$

whence linear in the momenta. Conversely, if a transformation on phase space induces a transformation on the configuration space, its associated hamiltonian function (which always exists locally) must be linear in the momenta, since its Poisson brackets with a function on configuration space $f(q)$ cannot depend on the momenta.

The symplectic reduction in this case, $\Phi^{-1}(0)/\mathcal{G}$, is nothing but the phase space of the reduced configuration space:

$$\Phi^{-1}(0)/\mathcal{G} \cong T^*(N/G) \ ; \tag{II.2.34}$$

hence the name **reduced phase space**.

Chapter Three:

Classical BRST Cohomology

In this chapter we discuss the BRST construction in a classical mechanics setting. Classical BRST is a cohomology theory which, in a sense to be made precise below, is dual to symplectic reduction. As explained in Section II.2, in symplectic reduction one starts with a symplectic manifold (M,Ω) and a given coisotropic submanifold $i : M_o \hookrightarrow M$ and constructs another symplectic manifold $\widetilde{M}$ defined as the space of leaves of the characteristic (null) foliation associated to the 2–form $i^*\Omega$ on M_o. What the BRST construction achieves is a cohomological description of this procedure. That such a cohomological description exists should not come as a complete surprise since after all both symplectic reduction and cohomology are subquotients. The goal of the BRST construction is to make this heuristic observation precise; and in order to do so we must learn how to describe geometric objects algebraically.

Dual to a manifold M we have the commutative algebra $C^\infty(M)$ of its smooth functions which characterize it completely. The correspondence goes roughly as follows. To every point $p \in M$ there corresponds an ideal $I(p)$ of $C^\infty(M)$ consisting of those functions vanishing at p. Since it is the kernel (via the evaluation map) of a homomorphism onto a field this ideal is maximal. Moreover with respect to any topology on $C^\infty(M)$ relative to which the evaluation map is continuous, $I(p)$ is closed. Hence we have an assignment of a maximal closed ideal of $C^\infty(M)$ to every point in M. It turns out that these are all the maximal closed ideals there are. So that as a set M is just the set $\mathcal{M}$ of maximal closed ideals of $C^\infty(M)$. In fact, one can topologize and give a differentiable structure to $\mathcal{M}$ in such a way that the set isomorphism $\mathcal{M} \cong M$ is really a diffeomorphism.

Similarly if $i : M_o \hookrightarrow M$ is a submanifold, it can be described by an ideal $I(M_o)$ consisting of the smooth functions vanishing on M_o. Clearly $I(M_o) = \cap_{p\in M_o} I(p)$. For a special type of submanifolds M_o, $I(M_o)$ is finitely generated. This corresponds to submanifolds which are described as the regular zero locus of a set of smooth functions. Then these functions generate $I(M_o)$ over $C^\infty(M)$. This will be the case of interest in this chapter. The rôle of the submanifold M_o will be played by the zero locus of a set of first class constraints on a symplectic manifold.

The BRST construction will follow three steps. The first step is to construct a cohomological description (a resolution) of the smooth functions on M_o from the smooth functions on M. The second step, which is independent from the first, is to describe cohomologically the functions on $\widetilde{M}$ starting from the functions on M_o. Finally the third step combines these two into a cohomology theory (BRST) which describes the smooth functions on $\widetilde{M}$ from the smooth functions (plus some extra ingredients) on M.

This chapter is organized as follows. In Section 1 we study the first step of the subquotient: the restriction to the subspace. Suppose $i : M_o \hookrightarrow M$ is a closed embedded submanifold of codimension k corresponding to the zero set (assumed regular) of a smooth function $\Phi : M \to \mathbb{R}^k$. We then define a Koszul-like complex associated to this embedding, which will play a central rôle in the constructions of the BRST cohomology theory. This complex yields a free acyclic resolution for $C^\infty(M_o)$ thought of as a $C^\infty(M)$-module. We give a novel proof of the acyclicity of this complex in which we introduce a double complex completely analogous to the Čech-de Rham complex introduced by Weil in order to prove the de Rham theorem. We call it the Čech-Koszul complex.

In Section 2 we tackle the second step of the subquotient: the quotient of the subspace. We define a cohomology theory associated to the foliation determined by the null distribution of $i^*\Omega$ on M_o. This is a de Rham-like cohomology theory of differential forms (co)tangent to the leaves of the foliation (vertical forms) relative to the exterior derivative along the leaves of the foliation (vertical derivative). If the foliation fibers onto a smooth manifold $\widetilde{M}$—the symplectic quotient of M by M_o—the zeroth cohomology is naturally isomorphic to $C^\infty(\widetilde{M})$. We then lift this cohomology theory via the Koszul resolution obtained in Section 1 to a cohomology theory (BRST) in a certain bigraded complex. The existence of this cohomology theory must be proven since the vertical derivative does not lift to a differential operator, *i.e.*, its square is not zero. However its square is chain homotopic to zero (relative to the Koszul differential) and the acyclicity of the Koszul resolution allows us to construct the desired differential.

In Section 3 we place the BRST construction in a truly symplectic setting. It should be emphasized that the BRST procedure *per se* is not really tied down to symplectic geometry. It should be amply evident from Sections 1 and 2, that we never make essential use of the symplectic structure of M. However when we take advantage of the symplectic structure, the BRST construction becomes so much more natural and manageable from a computational point of view. In this section we first review the basics of Poisson superalgebras and we then show that the BRST cohomology constructed in Section 2 is naturally expressed in this context. This allows us to prove that not only the ring and module structures are preserved under BRST cohomology but, more importantly, the Poisson structures also correspond. In fact, the BRST cohomology can be interpreted as the cohomology of an inner derivation on the ring of "smooth" functions of a De Witt supermanifold; although we will not follow this point of view here.

Finally in Section 4 we compute the classical BRST cohomology in terms of initial data. In particular we show that the BRST cohomology only depends on the constrained submanifold $i : M_o \hookrightarrow M$ eliminating in this way the fictitious dependence on the actual form of the constraints used to define it. The cleanest results arise from the case of a group action. We show that the classical BRST cohomology is given by the smooth functions on the reduced symplectic manifold taking values in the de Rham cohomology of the Lie group. We also prove a duality theorem for the BRST cohomology.

1. The Čech-Koszul Complex

We saw in our discussion on symplectic reduction that the reduction process was essentially a subquotient, consisting of two steps:

(i) restriction to the constrained submanifold; and

(ii) identifying points lying in the same leaf of the foliation; *i.e.*, taking a topological quotient.

In this section we describe algebraically the "restriction" part of the process. It is of a more general nature than the symplectic reduction, as should be amply evident to the reader. In particular, we never make use of the symplectic structure. So throughout this section M is an arbitrary smooth manifold and the "constraints" are arbitrary smooth functions. The key idea of this section is to construct a projective resolution for the smooth functions of the constrained submanifold M_o in terms of the smooth functions of M. This will allow us to, in effect, work with the functions on M_o without actually having to restrict ourselves to M_o.

For M_o a closed imbedded submanifold, any smooth function on M_o extends to a smooth function on M and the difference of any two such extensions vanishes on M_o. Hence if we let $I(M_o)$ denote the (multiplicative) ideal of $C^\infty(M)$ consisting of functions which vanish at M_o, we have the following isomorphism

$$C^\infty(M_o) \cong C^\infty(M)/I(M_o) \ . \qquad \text{(III.1.1)}$$

This is still not satisfactory since $I(M_o)$ is not a very manageable object. It will turn out that

$I(M_o)$ is precisely the ideal J generated by the constraints. Still this is not completely satisfactory because we would rather work with the constraints themselves than with the ideal they generate. The solution of this problem relies on a construction due to Koszul[77],[78]. We will see that there is a differential complex (the Koszul complex)

$$\cdots \longrightarrow K^2 \longrightarrow K^1 \longrightarrow C^\infty(M) \longrightarrow 0 \ , \tag{III.1.2}$$

whose cohomology in positive dimensions is zero and in zero dimension is precisely $C^\infty(M_o)$. We shall refer to this fact as the **quasi-acyclicity** of the Koszul complex. It will play a fundamental rôle in all our constructions.

The Local Koszul Complex

We will first discuss the construction on $\mathbb{R}^m$ and later we will globalize to M. We start with an elementary observation.

Lemma III.1.3. *Let $\mathbb{R}^m$ be given coordinates*

$$(y,x) = (y^1,\ldots,y^k,x^1,\ldots,x^{m-k}) \ .$$

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function such that $f(\mathbf{0},x) = 0$. Then there exist k smooth functions $h_i : \mathbb{R}^m \to \mathbb{R}$ such that $f = \sum_{i=1}^k \phi_i\, h_i$, where the ϕ_i are the functions defined by $\phi_i(y,x) = y^i$.

Proof: Notice that

$$\begin{aligned} f(y,x) &= \int_0^1 dt\, \frac{d}{dt} f(ty,x) \\ &= \int_0^1 dt\, \sum_{i=1}^k y^i\,(D_i\, f)(ty,x) \\ &= \sum_{i=1}^k y^i \int_0^1 dt\,(D_i\, f)(ty,x) \\ &= \sum_{i=1}^k \phi_i(y,x) \int_0^1 dt\,(D_i\, f)(ty,x) \ , \end{aligned}$$

where D_i is the i^{th} partial derivative. Defining

$$h_i(y,x) \stackrel{\text{def}}{=} \int_0^1 dt\,(D_i\, f)(ty,x) \tag{III.1.4}$$

the proof is complete. ■

Therefore, if we let $P \subset \mathbb{R}^m$ denote the subspace defined by $y^i = 0$ for all i, the ideal of $C^\infty(\mathbb{R}^m)$ consisting of functions which vanish on P is precisely the ideal generated by the functions ϕ_i.

Definition III.1.5. Let R be a commutative ring with unit. A sequence $(\phi_i)_{i=1}^k$ of elements of R is called **regular** if for all $j = 1, \ldots, k$, ϕ_j is not a zero divisor in R/I_{j-1}, where I_j is the ideal generated by $\phi_1, \ldots, \phi_j$ and $I_0 = 0$. In other words, if $f \in R$ and for any $j = 1, \ldots, k$, $\phi_j f \in I_{j-1}$ then $f \in I_{j-1}$ to start out with. In particular, ϕ_1 is not identically zero.

Proposition III.1.6. *Let $\mathbb{R}^m$ be given coordinates*

$$(y, x) = (y^1, \ldots, y^k, x^1, \ldots, x^{m-k}) \ .$$

Then the sequence (ϕ_i) in $C^\infty(\mathbb{R}^m)$ defined by $\phi_i(y, x) = y^i$ is regular.

Proof: First of all notice that ϕ_1 is not identically zero. Next suppose that $(\phi_1, \ldots, \phi_j)$ is regular. Let P_j denote the hyperplane defined by $\phi_1 = \cdots = \phi_j = 0$. Then by Lemma III.1.3, $C^\infty(P_j) = C^\infty(\mathbb{R}^m)/I_j$. Let $[f]_j$ denote the class of a $f \in C^\infty(\mathbb{R}^m)$ modulo I_j. Then ϕ_{j+1} gives rise to a function $[\phi_{j+1}]_j$ in $C^\infty(P_j)$ which, if we think of P_j as coordinatized by

$$(y^{j+1}, \ldots, y^k, x^1, \ldots, x^{m-k}) \ ,$$

turns out to be defined by

$$[\phi_{j+1}]_j(y^{j+1}, \ldots, y^k, x^1, \ldots, x^{m-k}) = y^{j+1} \ . \tag{III.1.7}$$

This is clearly not identically zero and, therefore, the sequence $(\phi_1 \ \ldots, \phi_{j+1})$ is regular. By induction we are done. ∎

We now come to the definition of the Koszul complex. Let R be a ring and let $\Phi = (\phi_1, \ldots, \phi_k)$ be a sequence of elements of R. We define a complex $K(\Phi)$ as follows: $K^0(\Phi) = R$ and for $p > 0$, $K^p(\Phi)$ is defined to be the free R module with basis $\{b_{i_1} \wedge \cdots \wedge b_{i_p} \mid 0 < i_1 < \cdots < i_p \leq k\}$.

Define a map $\delta_K : K^p(\Phi) \to K^{p-1}(\Phi)$ by $\delta_K b_i = \phi_i$ and extending to all of $K(\Phi)$ as an R-linear antiderivation. That is, δ_K is identically zero on $K^0(\Phi)$ and

$$\delta_K(b_{i_1} \wedge \cdots \wedge b_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} \phi_{i_j} \, b_{i_1} \wedge \cdots \wedge \widehat{b_{i_j}} \wedge \cdots \wedge b_{i_p} \ , \tag{III.1.8}$$

where a $\widehat{\ }$ adorning a symbol denotes its omission. It is trivial to verify that $\delta_K^2 = 0$, yielding a complex

$$0 \longrightarrow K^k(\Phi) \xrightarrow{\delta_K} K^{k-1}(\Phi) \longrightarrow \cdots \longrightarrow K^1(\Phi) \longrightarrow R \longrightarrow 0 \ , \tag{III.1.9}$$

called the **Koszul complex**.

The following theorem is a classical result in homological algebra whose proof is completely straight-forward and can be found, for example, in **[62]**.

Theorem III.1.10. *If $(\phi_1, \ldots, \phi_k)$ is a regular sequence in R then the cohomology of the Koszul complex is given by*

$$\boxed{H^p(K(\Phi)) \cong \begin{cases} \mathbf{0} & \text{for } p > 0 \\ R/J & \text{for } p = 0 \end{cases}} , \tag{III.1.11}$$

where J is the ideal generated by the ϕ_i.

Therefore the complex $K(\Phi)$ provides an acyclic resolution (known as the **Koszul resolution**) for the R-module R/J. Therefore if $R = C^\infty(\mathbb{R}^m)$ and Φ is the sequence $(\phi_1, \ldots, \phi_k)$ of Proposition III.1.6, the Koszul complex gives an acyclic resolution of $C^\infty(\mathbb{R}^m)/J$ which by Lemma III.1.3 is just $C^\infty(P_k)$, where P_k is the subspace defined by $\phi_1 = \cdots = \phi_k = 0$. The $\{b_i\}$ in the Koszul complex are the classical **antighosts**.

Globalization: The Čech-Koszul Complex

We now globalize this construction. Let M be our original symplectic manifold and $\Phi : M \to \mathbb{R}^k$ be the function whose components are the first class constraints constraints, *i.e.*, $\Phi(m) = (\phi_1(m), \ldots, \phi_k(m))$. We assume that 0 is a regular value of Φ so that $M_o \equiv \Phi^{-1}(0)$ is a closed embedded submanifold of M. Therefore for each point $m \in M_o$ here exists an open set $U \in M$ containing m and a chart $\Psi : U \to \mathbb{R}^m$ such that Φ has components $(\phi_1, \ldots, \phi_k, x^1, \ldots, x^{m-k})$ and such that the image under Φ of $U \cap M_o$ corresponds exactly to the points $(\underbrace{0, \ldots, 0}_{k}, x^1, \ldots, x^{m-k})$. Let $\mathcal{U}$ be an open cover for M consisting of sets like these. Of course, there will be some sets $V \in \mathcal{U}$ for which $V \cap M_o = \varnothing$.

To motivate the following construction let's understand what is involved in proving, for example, that the ideal J generated by the constraints coincides with the ideal $I(M_o)$ of smooth functions which vanish on M_o. It is clear that $J \subset I(M_o)$. We want to show the converse. That is, if f is a smooth function vanishing on M_o then there are smooth functions h^i such that $f = \sum_i h^i \phi_i$. This is always true locally. That is, restricted to any set $U \in \mathcal{U}$ such that $U \cap M_o \neq \varnothing$, Lemma III.1.3 implies that there will exist functions $h^i_U \in C^\infty(U)$ such that on U

$$f_U = \sum_i \phi_i \, h^i_U \ , \tag{III.1.12}$$

where f_U denotes the restriction of f to U. If, on the other hand, $V \in \mathcal{U}$ is such that $V \cap M_o = \varnothing$, then not all of the ϕ_i vanish and the statement is also true. There is a certain ambiguity in the choice of h^U_i. In fact, if δ_K denotes the Koszul differential we notice that (III.1.12) can be written as $f_U = \delta_K h_U$, where $h_U = \sum_i h^i_U b_i$ is a Koszul 1-cochain on U. Therefore, the ambiguity in h_U is precisely a Koszul 1-cocycle on U, but by Theorem III.1.10, the Koszul complex on U is quasi-acyclic and hence every 1-cocycle is a 1-coboundary. What we would like to show is that this ambiguity can be exploited to choose the h_U in such a way that $h_U = h_V$ on all non-empty overlaps $U \cap V$. This condition is precisely the condition for h_U to be a Čech 0-cocycle. In order to analyze this problem it is useful to make use of the machinery of Čech cohomology with coefficients in a sheaf. For a review of the necessary material we refer the reader to **[69]**; and, in particular, to their discussion of the Čech-de Rham complex. Our construction is very close in spirit to that one: in fact, it should properly be called the Čech-Koszul complex.

Let $\mathcal{E}_M$ denote the sheaf of germs of smooth functions on M and let $\mathcal{K} = \bigoplus_p \mathcal{K}^p$ denote the free sheaf of $\mathcal{E}_M$-modules which appears in the Koszul complex: $\mathcal{K}^p = \bigwedge^p \mathbb{V} \otimes \mathcal{E}_M$, where $\mathbb{V}$ is the vector space with basis $\{b_i\}$. Let $C^p(\mathcal{U};\mathcal{K}^q)$ denote the Čech p-cochains with coefficients in the Koszul subsheaf $\mathcal{K}^q$. This becomes a double complex under the two differentials

$$\check{\delta} : C^p(\mathcal{U};\mathcal{K}^q) \to C^{p+1}(\mathcal{U};\mathcal{K}^q) \qquad \text{“Čech”}$$

and

$$\delta_K : C^p(\mathcal{U};\mathcal{K}^q) \to C^p(\mathcal{U};\mathcal{K}^{q-1}) \qquad \text{“Koszul”}$$

which clearly commute, since they are independent. We can therefore define the complex $CK^n = \bigoplus_{p-q=n} C^p(\mathcal{U};\mathcal{K}^q)$ and the differential $D = \check{\delta} + (-1)^p \delta_K$ on $C^p(\mathcal{U};\mathcal{K}^q)$. The total differential has total degree one $D : CK^n \to CK^{n+1}$ and moreover obeys $D^2 = 0$. Since the double complex is bounded, *i.e.*, for each n, CK^n is the direct sum of a finite number of $C^p(\mathcal{U};\mathcal{K}^q)$'s, Theorem II.1.49 and Theorem II.1.50 guarantee the existence of two spectral sequences converging to the total cohomology. We now proceed to compute them. In doing so we will find it convenient to depict our computations graphically. The original double complex is depicted by the following diagram:

$C^0(\mathcal{U};\mathcal{K}^2)$	$C^1(\mathcal{U};\mathcal{K}^2)$	$C^2(\mathcal{U};\mathcal{K}^2)$	
$C^0(\mathcal{U};\mathcal{K}^1)$	$C^1(\mathcal{U};\mathcal{K}^1)$	$C^2(\mathcal{U};\mathcal{K}^1)$	
$C^0(\mathcal{U};\mathcal{K}^0)$	$C^1(\mathcal{U};\mathcal{K}^0)$	$C^2(\mathcal{U};\mathcal{K}^0)$	

Upon taking cohomology with respect to the horizontal differential (*i.e.*, Čech cohomology) and using the fact that the sheaves $\mathcal{K}^q$ are fine, being free modules over the structure sheaf

$\mathcal{E}_M$, we get

$K^2(\Phi)$	0	0	
$K^1(\Phi)$	0	0	
$K^0(\Phi)$	0	0	

where $K^p(\Phi) \cong \bigwedge^p \mathbb{V} \otimes C^\infty(M)$ are the spaces in the Koszul complex on M. Taking vertical cohomology yields the Koszul cohomology

$H^2(K(\Phi))$	0	0	
$H^1(K(\Phi))$	0	0	
$H^0(K(\Phi))$	0	0	

Since the next differential in the spectral sequence necessarily maps across columns it must be identically zero. The same holds for the other differentials and we see that the spectral sequence degenerates at the E_2 term. In particular the total cohomology is isomorphic to the Koszul cohomology:

$$H^n_D \cong H^n(K(\Phi)) \ . \tag{III.1.13}$$

To compute the other spectral sequence we first start by taking vertical cohomology, *i.e.*, Koszul cohomology. Because of the choice of cover $\mathcal{U}$ we can use Theorem III.1.10 and Lemma III.1.3 to deduce that the vertical cohomology is given by

0	0	0	
0	0	0	
$C^0(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	$C^1(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	$C^2(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	

where $\mathcal{E}_M/\mathcal{J}$ is defined by the exact sheaf sequence

$$0 \to \mathcal{J} \to \mathcal{E}_M \to \mathcal{E}_M/\mathcal{J} \to 0 \ , \tag{III.1.14}$$

where $\mathcal{J}$ is the subsheaf of $\mathcal{E}_M$ consisting of germs of smooth functions belonging to the ideal generated by the ϕ_i. Because of our choice of cover, Lemma III.1.3 implies that $\mathcal{J}(U)$

agrees, for all $U \in \mathcal{U}$, with those smooth functions vanishing on $U \cap M_o$, and hence we have an isomorphism of sheaves $\mathcal{E}_M/\mathcal{J} \cong \mathcal{E}_{M_o}$, where $\mathcal{E}_{M_o}$ is the sheaf of germs of smooth functions on M_o. Next we notice that $\mathcal{E}_{M_o}$ is a fine sheaf and hence all its Čech cohomology groups vanish except the zeroth one. Thus the E_2 term in this spectral sequence is just

0	0	0	
0	0	0	
$C^\infty(M_o)$	0	0	

Again we see that the higher differentials are automatically zero and the spectral sequence collapses. Since both spectral sequences compute the same cohomology we have the following corollary.

Corollary III.1.15. *If 0 is a regular value for $\Phi : M \to \mathbb{R}^k$ the Koszul complex $K(\Phi)$ gives an acyclic resolution for $C^\infty(M_o)$. In other words, the cohomology of the Koszul complex is given by*

$$\boxed{H^p(K(\Phi)) \cong \begin{cases} 0 & \text{for } p > 0 \\ C^\infty(M_o) & \text{for } p = 0 \end{cases}}, \tag{III.1.16}$$

where $M_o \equiv \Phi^{-1}(0)$.

Notice that, in particular, this means that the ideal J generated by the constraints is precisely the ideal consisting of functions vanishing on M_o. This is because $C^\infty(M_o) \cong C^\infty(M)/I(M_o)$ since M_o is a closed embedded submanifold. On the other hand, Corollary III.1.15 implies that $C^\infty(M_o) \cong C^\infty(M)/J$. Hence the equality between the two ideals.

It may appear overkill to use the spectral sequence method to arrive at Corollary III.1.15. In fact it is not necessary and the reader is urged to supply a proof using the "tic-tac-toe" methods in **[69]**. This way one gains some valuable intuition on this complex. In particular, one can show that way that the sequence Φ is regular in $C^\infty(M)$ and that $J = I(M_o)$ without having to first prove Corollary III.1.15. Lack of spacetime prevents us from exhibiting both computations and the spectral sequence computation is decidedly shorter.

We now introduce a generalization of the Koszul complex which will be of much use in the sections to come. Let R be a ring and E an R-module. We can then define a complex

$K(\Phi; E)$ associated to any sequence $(\phi_1, \ldots, \phi_k)$ by just tensoring the Koszul complex $K(\Phi)$ with E, that is, $K^p(\Phi; E) = K^p(\Phi) \otimes_R E$ and extending δ_K to $\delta_K \otimes \mathbf{1}$. Let $H(K(\Phi); E)$ denote the cohomology of this complex. It is naturally an R-module. It is easy to show that if E and F are R-modules, then there is an R-module isomorphism

$$H(K(\Phi); E \oplus F) \cong H(K(\Phi); E) \oplus H(K(\Phi); F) \ . \tag{III.1.17}$$

Hence, if $F \cong \bigoplus_\alpha R$ is a free R-module then

$$H(K(\Phi); F) \cong \bigoplus_\alpha H(K(\Phi)) \ . \tag{III.1.18}$$

In particular if Φ is a regular sequence then the generalized Koszul complex with coefficients in a free R-module is quasi-acyclic. Now let P be a projective module, *i.e.*, P is a summand of a free module. Then let N be an R-module such that $P \oplus N = F$, F a free R-module. Then

$$H(K(\Phi); F) \cong H(K(\Phi); P) \oplus H(K(\Phi); N) \ , \tag{III.1.19}$$

which, since $H(K(\Phi); F)$ is quasi-acyclic, implies the quasi-acyclicity of $H(K(\Phi); P)$. How about $H^0(K(\Phi); P)$? By definition

$$H^0(K(\Phi); P) \cong R/J \otimes_R P \cong P/JP \ . \tag{III.1.20}$$

Therefore we have the following algebraic result

Theorem III.1.21. *If $\Phi = (\phi_1, \ldots, \phi_k)$ is a regular sequence in R, and P is a projective R-module, then the homology of the Koszul complex with coefficients in P is given by*

$$\boxed{H^p(K(\Phi); P) \cong \begin{cases} 0 & \text{for } p > 0 \\ P/JP & \text{for } p = 0 \end{cases}} \ , \tag{III.1.22}$$

where J is the ideal generated by the ϕ_i.

The relevance of considering projective modules will come when we discuss geometric quantization. There we will not just have to work with the smooth fucntions on $\widetilde{M}$ but also with sections of vector bundles over $\widetilde{M}$ and these are precisely[79] the finitely generated projective modules over $C^\infty(\widetilde{M})$.

We conclude this section with two philosophical remarks. First, it should be emphasized that the Koszul resolution is independent on the nature of the constraints as long as their zero locus was a regular set. In particular, we never made use of the fact that the constraints were first class or, for that matter, that M had a symplectic structure. Hence also in the case of second class constraints there is a Koszul resolution giving a cohomological description of the smooth functions of the constrained submanifold. This, to my knowledge, has not been used in the physics literature. It would seem to be the natural starting place to extend the BRST quantization to the case of second class constraints and hence give a unified cohomological description of the full Dirac theory.

Second, it is worth pointing out that the restriction to the constraints being regular is not really necessary. With a bit more work a resolution (called the Tate resolution) can be constructed in order to handle this case as well. The method of Tate[**80**] consists of adding new cochains to kill whatever cohomology might exist in positive dimension. These new cochains are the antighosts for the ghosts for ghosts in the treatment of reducible gauge theories. A complete description of this work can be found in the recent paper by Fisch, Henneaux, Stasheff, & Teitelboim [**21**].

2. Classical BRST Cohomology

In this section we complete the construction of the algebraic equivalent of symplectic reduction by first defining a cohomology theory (vertical cohomology) that describes the passage of M_o to $\widetilde{M}$ and then, in keeping with our philosophy of not having to work on M_o, we lift it via the Koszul resolution to a cohomology theory (classical BRST cohomology) which allows us to work with $\widetilde{M}$ from objects defined on M. We shall assume for convenience that the foliation defining $\widetilde{M}$ is such that $\widetilde{M}$ is a smooth manifold and $\pi : M_o \twoheadrightarrow \widetilde{M}$ is a smooth surjection. In other words, the foliation is actually a fibration $M_o \xrightarrow{\pi} \widetilde{M}$ whose fibers are the leaves.

Vertical Cohomology

Since $\widetilde{M}$ is obtained from M_o by collapsing each leaf of the null foliation $\mathcal{M}_o^\perp$ to a point, a smooth function on $\widetilde{M}$ pulls back to a smooth function on M_o which is constant on each leaf. Conversely, any smooth function on M_o which is constant on each leaf defines a smooth function on $\widetilde{M}$. Since the leaves are connected (Frobenius' theorem) a function is constant on the leaves if and only if it is locally constant. Since the hamiltonian vector fields $\{X_i\}$ associated to the constraints $\{\phi_i\}$ form a global basis of the tangent space to the leaves, a function f on M_o is locally constant on the leaves if and only if $X_i\, f = 0$ for all i. In an effort to build a cohomology theory and in analogy to the de Rham theory, we pick a global basis $\{c^i\}$ for the cotangent space to the leaves such that they are dual to the $\{X_i\}$, *i.e.*, $c^i(X_j) = \delta^i_j$. We then define the **vertical derivative** d_V on functions as

$$d_V f = \sum_i (X_i f)\, c^i \qquad \forall\, f \in C^\infty(M_o)\ . \tag{III.2.1}$$

Let $\Omega_V(M_o)$ denote the exterior algebra generated by the $\{c^i\}$ over $C^\infty(M_o)$. We will refer to them as **vertical forms**. We can extend d_V to a derivation

$$d_V : \Omega^p_V(M_o) \to \Omega^{p+1}_V(M_o) \tag{III.2.2}$$

by defining

$$d_V c^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i\, c^j \wedge c^k\ , \tag{III.2.3}$$

where the $\{f_{ij}{}^k\}$ are the functions appearing in the Lie bracket of the hamiltonian vector fields associated to the constraints: $\left[X_i\,,\, X_j\right] = \sum_k f_{ij}{}^k\, X_k$; or, equivalently, in the Poisson bracket of the constraints themselves: $\left\{\phi_i\,,\,\phi_j\right\} = \sum_k f_{ij}{}^k\, \phi_k$.

Notice that the choice of $\{c^i\}$ corresponds to a choice of connection on the fiber bundle $M_o \xrightarrow{\pi} \widetilde{M}$. Let V denote the subbundle of TM_o spanned by the $\{X_i\}$. It can be characterized either as ker π_* or as $TM_o^\perp$. A connection is then a choice of complementary subspace H such that $TM_o = V \oplus H$. It is clear that a choice of $\{c^i\}$ implies a choice of H since we can define $X \in H$ if and only if $c^i(X) = 0$ for all i. If we let pr_V denote the projection $TM_o \to V$ it is then clear that acting on vertical forms, $d_V = \mathrm{pr}_V^* \circ d$, where d is the usual exterior derivative on M_o.

It follows therefore that $d_V^2 = 0$. We call its cohomology the **vertical cohomology** and we denote it as $H_V(M_o)$. As we will see in Section 4, it can be computed in terms of the de Rham cohomology of the typical fiber in the fibration $M_o \xrightarrow{\pi} \widetilde{M}$. In particular, from its definition, we already have that

$$H^0_V(M_o) \cong C^\infty(\widetilde{M})\ . \tag{III.2.4}$$

<u>The BRST Construction</u>

However this is not the end of the story since we don't want to have to work on M_o but on M. The results of the previous section suggest that we use the Koszul construction. Notice that $\Omega_V(M_o)$ is isomorphic to $\bigwedge \mathbb{R}^k \otimes C^\infty(M_o)$ where $\mathbb{R}^k$ has basis $\{c^i\}$. The Koszul complex gives a resolution for $C^\infty(M_o)$. Therefore extending the Koszul differential as the identity on $\bigwedge\mathbb{R}^k$ we get a resolution for $\Omega_V(M_o)$. We find it convenient to think of $\mathbb{R}^k$ as $\mathbb{V}^*$, whence the resolution of $\Omega_V(M_o)$ is given by

$$\cdots \longrightarrow \bigwedge\mathbb{V}^* \otimes \mathbb{V} \otimes C^\infty(M) \xrightarrow{\mathbf{1}\otimes\delta_K} \bigwedge\mathbb{V}^* \otimes C^\infty(M) \longrightarrow 0 \ . \tag{III.2.5}$$

This gives rise to a bigraded complex $K = \bigoplus_{c,b} K^{c,b}$, where

$$\boxed{K^{c,b} \equiv \bigwedge^c\mathbb{V}^* \otimes \bigwedge^b\mathbb{V} \otimes C^\infty(M)} \ , \tag{III.2.6}$$

under the Koszul differential $\delta_K : K^{c,b} \to K^{c,b-1}$. The Koszul cohomology of this bigraded complex is zero for $b > 0$ by (III.1.18), and for $b = 0$ it is isomorphic to the vertical forms, where the vertical derivative is defined. Elements of $\bigwedge\mathbb{V}^*$ are the classical **ghosts**. Therefore we see that although the ghosts and antighosts are dual to each other the rôles they play in the BRST construction are very different.

The purpose of the BRST construction is to lift the vertical derivative to K. That is, to define a differential δ_1 on K which anticommutes with the Koszul differential, which induces the vertical derivative upon taking Koszul cohomology, and which obeys $\delta_1^2 = 0$. This would mean that the total differential $D = \delta_K + \delta_1$ would obey $D^2 = 0$ acting on K and its cohomology would be isomorphic to the vertical cohomology. This is possible only in the case of a group action, *i.e.*, when the linear span of the constraints closes under Poisson bracket. In general this is not possible and we will be forced to add further δ_i's to D to ensure $D^2 = 0$. The need to include these extra terms was first pointed out by Fradkin and Fradkina in **[19]**, as was pointed out to me by Marc Henneaux.

We find it convenient to define $\delta_0 = (-1)^c\delta_K$ on $K^{b,c}$. We define δ_1 on functions and ghosts as the vertical derivative[8]

$$\begin{aligned} \delta_1 f &= \sum_i (X_i f)\, c^i \\ &= \sum_i \{\phi_i\, ,\, f\}\, c^i \end{aligned} \tag{III.2.7}$$

and

[8] Notice that the vertical derivative is defined on M_o and hence has no unique extension to M. The choice we make is the simplest and the one that, in the case of a group action, corresponds to the Lie algebra coboundary operator.

$$\delta_1\, c^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i\, c^j \wedge c^k \ . \tag{III.2.8}$$

We can then extend it as a derivation to all of $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$. Notice that it trivially anticommutes with δ_0 since it stabilizes $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$ where δ_0 acts trivially. We now define it on antighosts in such a way that it commutes with δ_0 everywhere. This does not define it uniquely but a convenient choice is

$$\delta_1 e_i = \sum_{j,k} f_{kj}{}^i\, \omega^j \wedge e_k \ . \tag{III.2.9}$$

Notice that $\delta_1^2 \neq 0$ in general, although it does in the case where the $f_{ij}{}^k$ are constant. However since it anticommutes with δ_0 it does induce a map in δ_0 (*i.e.*, Koszul) cohomology which precisely agrees with the vertical derivative d_V, which does obey $d_V^2 = 0$. Hence δ_1^2 induces the zero map in Koszul cohomology. This is enough (see algebraic lemma below) to deduce the existence of a derivation $\delta_2 : K^{c,b} \to K^{c+2,b+1}$ such that $\delta_1^2 + \{\delta_0\,,\,\delta_2\} = 0$, where $\{\ ,\ \}$ denotes the anticommutator. This suggests that we define $D_2 = \delta_0 + \delta_1 + \delta_2$. We see that

$$D_2^2 = \delta_0^2 \oplus \{\delta_0\,,\,\delta_1\} \oplus (\delta_1^2 + \{\delta_0\,,\,\delta_2\}) \oplus \{\delta_1\,,\,\delta_2\} \oplus \delta_2^2\ , \tag{III.2.10}$$

where we have separated it in terms of different bidegree and arranged them in increasing c-degree. The first three terms are zero but, in general, the other two will not vanish. The idea behind the BRST construction is to keep defining higher $\delta_i : K^{c,b} \to K^{c+i,b+i-1}$ such that their partial sums $D_i = \delta_0 + \cdots + \delta_i$ are nilpotent up to terms of higher and higher c-degree until eventually $D_k^2 = 0$. The proof of this statement will follow by induction from the quasi-acyclicity of the Koszul complex, but first we need to introduce some notation that will help us organize the information.

Let us define $F^pK = \bigoplus_{c\geq p} \bigoplus_b K^{c,b}$. Then $K = F^0K \supseteq F^1K \supseteq \cdots$ is a filtration of K. Let $\operatorname{Der} K$ denote the derivations (with respect to the $\wedge$ product) of K. We say that a derivation has bidegree (i,j) if it maps $K^{c,b} \to K^{c+i,b+j}$. $\operatorname{Der} K$ is naturally bigraded

$$\operatorname{Der} K = \bigoplus_{i,j} \operatorname{Der}^{i,j} K\ , \tag{III.2.11}$$

where $\operatorname{Der}^{i,j} K$ consists of derivations of bidegree (i,j). This decomposition makes $\operatorname{Der} K$ into a bigraded Lie superalgebra under the graded commutator:

$$[\ ,\ ] : \operatorname{Der}^{i,j} K \times \operatorname{Der}^{k,l} K \to \operatorname{Der}^{i+k,j+l} K\ . \tag{III.2.12}$$

We define $F^p\operatorname{Der} K = \bigoplus_{i\geq p} \bigoplus_j \operatorname{Der}^{i,j} K$. Then $F\operatorname{Der} K$ gives a filtration of $\operatorname{Der} K$ associated to the filtration $F\,K$ of K.

The remarks immediately following (III.2.10) imply that $D_2^2 \in F^3\mathrm{Der}\,K$. Moreover, it is trivial to check that $\left[\delta_0\,,\,D_2^2\right] \in F^4\mathrm{Der}\,K$. In fact,

$$\left[\delta_0\,,\,D_2^2\right] = \left[D_2\,,\,D_2^2\right] - \left[\delta_1\,,\,D_2^2\right] - \left[\delta_2\,,\,D_2^2\right] \tag{III.2.13}$$

where the first term vanishes because of the Jacobi identity and the last two terms are clearly in $F^4\mathrm{Der}\,K$. Therefore the part of D_2^2 in $F^3\mathrm{Der}\,K/F^4\mathrm{Der}\,K$ is a δ_0-chain map: that is, $\left[\delta_0\,,\,\left\{\delta_1\,,\,\delta_2\right\}\right] = 0$. Since it has non-zero b-degree, the quasi-acyclicity of the Koszul complex implies that it induces the zero map in Koszul cohomology. By the following algebraic lemma (see below), there exists a derivation δ_3 of bidegree $(3,2)$ such that $\left\{\delta_0\,,\,\delta_3\right\} + \left\{\delta_1\,,\,\delta_2\right\} = 0$. If we define $D_3 = \sum_{i=0}^{3} \delta_i$, this is equivalent to $D_3^2 \in F^4\mathrm{Der}\,K$. But by arguments identical to the ones above we deduce that $\left[\delta_0\,,\,D_3^2\right] \in F^5\,\mathrm{Der}\,K$, and so on. It is not difficult to formalize these arguments into an induction proof of the following theorem:

Theorem III.2.14. *We can define a derivation $D = \sum_{i=0}^{k} \delta_i$ on K, where δ_i are derivations of bidegree $(i, i-1)$, such that $D^2 = 0$.*

Finally we come to the proof of the algebraic lemma used above.

Lemma III.2.15. *Let*

$$\cdots \longrightarrow K_2 \xrightarrow{\delta_0} K_1 \xrightarrow{\delta_0} K_0 \to 0 \tag{III.2.16}$$

denote the Koszul complex where $K_b = \bigoplus_c K^{c,b}$. Let $d : K_b \to K_{b+i}$, $(i \geq 0)$ be a derivation which commutes with δ_0 and which induces the zero map on cohomology. Then there exists a derivation $K : K_b \to K_{b+i+1}$ such that $d = \left\{\delta_0\,,\,K\right\}$.

Proof: Since $C^\infty(M)$ is an $\mathbb{R}$-algebra it is, in particular, a vector space. Let $\{f_\alpha\}$ be a basis for it. Then, since $\delta_0\,f_\alpha = 0$, $\delta_0\,d\,f_\alpha = 0$. Since d induces the zero map in cohomology, there exists λ_α such that $d\,f_\alpha = \delta_0\,\lambda_\alpha$. Define $K\,f_\alpha = \lambda_\alpha$. Similarly, since $\delta_0\,d\,c^i = 0$, there exists μ^i such that $d\,c^i = \delta_0\,\mu^i$. Define $K\,c^i = \mu^i$. Since $C^\infty(M)$ and the $\{c^i\}$ generate K_0, we can extend K to all of K_0 as a derivation and, by construction, in such a way that on K_0, $d = \left\{\delta_0\,,\,K\right\}$. Now, $\delta_0\,d\,b_i = d\,\delta_0\,b_i$. But since $\delta_0\,b_i \in K_0$, $\delta_0\,d\,b_i = \delta_0\,K\,\delta_0\,b_i$. Therefore $\delta_0\,(d\,b_i - K\,\delta_0\,b_i) = 0$. Since $d\,b_i \in K^{i+1}$ for some $i \geq 0$, the quasi-acyclicity of the Koszul complex implies that there exists ξ_i such that $d\,b_i - K\,\delta_0\,b_i = \delta_0\,\xi_i$. Define $K\,b_i = \xi_i$. Therefore, $d\,b_i = \left\{\delta_0\,,\,K\right\}b_i$. We can now extend K as a derivation to all of K. Since d and $\left\{\delta_0\,,\,K\right\}$ are both derivations and they agree on generators, they are equal. ■

Defining the total complex $K = \bigoplus_n K^n$, where $K^n = \bigoplus_{c-b=n} K^{c,b}$, we see that $D : K^n \to K^{n+1}$. Its cohomology is therefore graded, that is, $H_D = \bigoplus_n H_D^n$. D is the **classical BRST operator** and its cohomology is the **classical BRST cohomology**. The total degree is known as the **ghost number**. We now investigate the classical BRST cohomology; although a full description in terms of initial data will have to wait until Section 4. Notice that since all terms in D have non-negative filtration degree with respect to $F\,K$, there exists (Theorem II.1.32) a spectral sequence associated to this filtration which converges to the cohomology of D. The E_1 term is the cohomology of the associated graded object $\mathrm{Gr}^p K \equiv F^p K/F^{p+1} K$, with respect to the induced differential. The induced differential is the part of D of c-degree 0, that is, δ_0. Therefore the E_1 term is given by

$$E_1^{c,b} \cong \bigwedge{}^c \mathbb{V}^* \otimes H^b(K(\Phi)) \; . \tag{III.2.17}$$

That is, $E_1^{c,0} \cong \Omega_V^c(M_o)$ and $E_1^{c,b>0} = 0$.

The E_2 term is the cohomology of E_1 with respect to the induced differential d_1. Tracking down the definitions we see that d_1 is induced by δ_1 and hence it is just the vertical derivative d_V. Therefore, $E_2^{c,0} \cong H_V^c(M_0)$ and $E_2^{c,b>0} = 0$. Notice, however, that the spectral sequence is degenerate at this term, since the higher differentials $d_2, d_3, \ldots$ all have b-degree different from zero. Therefore we have proven the following theorem.

Theorem III.2.18. *The classical BRST cohomology is given by*

$$\boxed{H_D^n \cong \begin{cases} 0 & \text{for } n < 0 \\ H_V^n(M_o) & \text{for } n \geq 0 \end{cases}} \; . \tag{III.2.19}$$

In particular, $H_D^0 \cong C^\infty(\widetilde{M})$.

We have not yet made sure, as we said we should, that the BRST cohomology is independent of the explicit form of the constraints and, thus, that it depends only on the actual constrained submanifold $i : M_o \hookrightarrow M$. Actually since, by Theorem III.2.18, the classic BRST cohomology merely recovers the vertical cohomology we must make sure that it is the vertical cohomology which is independent of the form of the constraints. From its definition the vertical cohomology explicitly depends on the choice of connection H. In other words, whereas the vertical tangent space V is uniquely defined, its complement H is not. We must show that any other choice of connection yields the same vertical cohomology; although, of course, the complexes used to calculate it are different. Instead of proving this directly we will wait until Section 4. There we compute the vertical cohomology and the answer is manifestly independent of the choice of connection.

3. Poisson Structure of Classical BRST

So far in the construction of the BRST complex no use has been made of the Poisson structure of the smooth functions on M. In this section we remedy the situation. It turns out that the complex K introduced in the last section is a Poisson superalgebra and the BRST operator D can be made into a Poisson derivation. It will then follow that in cohomology all constructions based on the Poisson structures will be preserved. This will be of special importance in the context of geometric quantization since all objects there can be defined purely in terms of the Poisson algebra structure of the smooth functions. In this section we review the concepts associated to Poisson algebras. We define the relevant Poisson structures in K and explore its consequences.

Poisson Superalgebras and Poisson Derivations

Recall that a **Poisson superalgebra** is a $\mathbb{Z}_2$-graded vector space $P = P_0 \oplus P_1$ together with two bilinear operations preserving the grading:

$$\begin{aligned} P \times P &\to P \qquad \text{(multiplication)} \\ (a,b) &\mapsto ab \end{aligned}$$

and

$$\begin{aligned} P \times P &\to P \qquad \text{(Poisson bracket)} \\ (a,b) &\mapsto [a\,,\,b] \end{aligned}$$

obeying the following properties

(P1) P is an associative supercommutative superalgebra under multiplication:

$$\begin{aligned} a(bc) &= (ab)c \\ ab &= (-1)^{|a||b|}\, ba \ ; \end{aligned}$$

(P2) P is a Lie superalgebra under Poisson bracket:

$$\begin{aligned} [a\,,\,b] &= (-1)^{|a||b|}\,[b\,,\,a] \\ [a\,,\,[b\,,\,c]] &= [[a\,,\,b]\,,\,c] + (-1)^{|a||b|}\,[b\,,\,[a\,,\,c]] \ ; \end{aligned}$$

(P3) Poisson bracket is a derivation over multiplication:

$$[a\,,\,bc] = [a\,,\,b]c + (-1)^{|a||b|}\,b[a\,,\,c] \ ;$$

for all $a, b, c \in P$ and where $|a|$ equals 0 or 1 according to whether a is even or odd, respectively.

The algebra $C^\infty(M)$ of smooth functions of a symplectic manifold (M,Ω) is clearly an example of a Poisson superalgebra where $C^\infty(M)_1 = 0$. On the other hand, if $\mathbb{V}$ is a finite dimensional vector space and $\mathbb{V}^*$ its dual, then the exterior algebra $\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ posseses a Poisson superalgebra structure. The associative multiplication is given by exterior multiplication ($\wedge$) and the Poisson bracket is defined for $u,v\in\mathbb{V}$ and $\alpha,\beta\in\mathbb{V}^*$ by

$$\left[\alpha\,,\,v\right] = \langle\alpha,v\rangle \qquad \left[v\,,\,w\right] = 0 = \left[\alpha\,,\,\beta\right]\ , \tag{III.3.1}$$

where $\langle,\rangle$ is the dual pairing between $\mathbb{V}$ and $\mathbb{V}^*$. We then extend it to all of $\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ as an odd derivation. Therefore the classical ghosts/antighosts in BRST possess a Poisson algebra structure. In **[81]** it is shown that this Poisson bracket is induced from the supercommutator in the Clifford algebra $\mathrm{Cl}(\mathbb{V}\oplus\mathbb{V}^*)$ with respect to the non-degenerate inner product on $\mathbb{V}\oplus\mathbb{V}^*$ induced by the dual pairing.

To show that K is a Poisson superalgebra we need to discuss tensor products. Given two Poisson superalgebras P and Q, their tensor product $P\otimes Q$ can be given the structure of a Poisson superalgebra as follows. For $a,b\in P$ and $u,v\in Q$ we define

$$(a\otimes u)(b\otimes v) = (-1)^{|u||b|}\,ab\otimes uv \tag{III.3.2}$$

$$\left[a\otimes u\,,\,b\otimes v\right] = (-1)^{|u||b|}\,\left(\left[a\,,\,b\right]\otimes uv + ab\otimes\left[u\,,\,v\right]\right)\ . \tag{III.3.3}$$

The reader is invited to verify that with these definitions (P1)-(P3) are satisfied. From this it follows that $K = C^\infty(M)\otimes\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ becomes a Poisson superalgebra.

Now let P be a Poisson superalgebra which, in addition, is $\mathbb{Z}$-graded, that is, $P = \bigoplus_n P^n$ and $P^n\,P^m \subseteq P^{m+n}$ and $\left[P^n\,,\,P^m\right]\subseteq P^{m+n}$; and such that the $\mathbb{Z}_2$-grading is the reduction modulo 2 of the $\mathbb{Z}$-grading, that is, $P_0 = \bigoplus_n P^{2n}$ and $P_1 = \bigoplus_n P^{2n+1}$. We call such an algebra a **graded Poisson superalgebra**. Notice that P^0 is an even Poisson subalgebra of P.

For example, letting $K = C^\infty(M)\otimes\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ we can define $K^n = \bigoplus_{c-b=n} K^{c,b}$. This way K becomes a $\mathbb{Z}$-graded Poisson superalgebra. Although the bigrading is preserved by the exterior product, the Poisson bracket does not preserve it. In fact, the Poisson bracket obeys

$$\left[\,,\,\right] : K^{i,j}\times K^{k,l} \to K^{i+k,j+l}\oplus K^{i+k-1,j+l-1}\ . \tag{III.3.4}$$

By a **Poisson derivation** of degree k we will mean a linear map $D : P^n \to P^{n+k}$ such that

$$D(ab) = (Da)b + (-1)^{k|a|}\, a(Db) \tag{III.3.5}$$

$$D\big[a\,,\,b\big] = \big[Da\,,\,b\big] + (-1)^{k|a|}\,\big[a\,,\,Db\big]\ . \tag{III.3.6}$$

The map $a \mapsto \big[Q\,,\,a\big]$ for some $Q \in P^k$ automatically obeys (III.3.5) and (III.3.6). Such Poisson derivations are called **inner**. Whenever the degree derivation is inner, any Poisson derivation of non-zero degree is inner[**51**] as we now show. The degree derivation N is defined uniquely by $Na = na$ if and only if $a \in P^n$. In the case $P = K$, N is the ghost number operator which is an inner derivation $\big[G\,,\,\cdot\big]$, where $G = \sum_i c^i \wedge b_i$, where $\{b_i\}$ is a basis for $\mathbb{V}$ and $\{c^i\}$ denotes its canonical dual basis. Now if $a \in P^n$, and the degree of D is $k \neq 0$, it follows from (III.3.6) that

$$Da = \frac{-1}{k}\big[DG\,,\,a\big]\ , \tag{III.3.7}$$

and so D is an inner derivation. If, furthermore, D should obey $D^2 = 0$, and be of degree 1, $Q = -DG$ would obey $\big[Q\,,\,Q\big] = 0$. To see this notice that for all $a \in P^n$

$$D^2 a = \big[Q\,,\,\big[Q\,,\,a\big]\big] = \frac{1}{2}\big[\big[Q\,,\,Q\big]\,,\,a\big] = 0\ .$$

But for $a = G$ we get that $\big[Q\,,\,Q\big] = 0$.

<u>The BRST Operator as a Poisson Derivation</u>

The BRST operator D constructed in the previous section is a derivation over the exterior product. Nothing in the way it was defined guarantees that it is a Poisson derivation and, in fact, it need not be so. However one can show that the δ_i's — which were, by far, not unique — can be defined in such a way that the resulting D is a Poisson derivation, from which it would immediately follow that it is inner. It is easier, however, to show the existence of the element $Q \in K^1$ such that $D = \big[Q\,,\,\cdot\big]$. We will show that there exists $Q = \sum_{i\geq 0} Q_i$, where $Q_i \in K^{i+1,i}$, such that $\big[Q\,,\,Q\big] = 0$ and that the cohomology of the operator $\big[Q\,,\,\cdot\big]$ is isomorphic to that of D. This was first proven by Henneaux in [**20**] and later in a completely algebraic way by Stasheff in [**55**]. Our proof is a simplified version of this latter proof.

From the discussion previous to Theorem III.2.18 we know that the only parts of D which affect its cohomology are δ_0, which is the Koszul differential, and δ_1 acting on the Koszul cohomology. Hence we need only make sure that the Q_i we construct realize these differentials. Notice that if $Q_i \in K^{i+1,i}$, $\big[Q_i\,,\,\cdot\big]$ has terms of two different bidegrees $(i+1, i)$

and $(i, i-1)$. Hence the only term which can contribute to the Koszul differential is Q_0. There is a unique element $Q_0 \in K^{1,0}$ such that $\big[Q_0\,,\, b_i\big] = \delta_0\, b_i = \phi_i$. This is given by

$$Q_0 = \sum_i c^i\, \phi_i\ . \tag{III.3.8}$$

Notice that

$$\big[Q_0\,,\, b_i\big] = \delta_0\, b_i = \phi_i \tag{III.3.9}$$

$$\big[Q_0\,,\, c^i\big] = \delta_0\, c^i = 0 \tag{III.3.10}$$

$$\big[Q_0\,,\, f\big] = (\delta_0 + \delta_1)\, f = \sum_i \big[\phi_i\,,\, f\big]\, c^i\ . \tag{III.3.11}$$

There is now a unique $Q_1 \in K^{2,1}$ such that $\big[Q_1\,,\, c^i\big] = \delta_1\, c^i$, namely,

$$Q_1 = -\frac{1}{2} \sum_{i,j,k} f_{ij}{}^k\, c^i \wedge c^j \wedge b_k\ . \tag{III.3.12}$$

If we define $R_1 = Q_0 + Q_1$ we then have that

$$\big[R_1\,,\, b_i\big] = (\delta_0 + \delta_1)\, b_i \tag{III.3.13}$$

$$\big[R_1\,,\, c^i\big] = (\delta_0 + \delta_1)\, c^i \tag{III.3.14}$$

$$\big[R_1\,,\, f\big] = (\delta_0 + \delta_1 + \delta_2)\, f\ . \tag{III.3.15}$$

In particular, two things are imposed upon us: $\delta_2\, f$ and $\delta_1\, b_i$; the latter imposition agrees with the choice made in (III.2.9).

Letting FK denote the filtration of K defined in the previous section, and using the notation in which, if $O \in K$ is an odd element, O^2 stands for $\frac{1}{2}\big[O\,,\, O\big]$, the following are satisfied:

$$R_1^2 \in F^3 K \qquad \text{and} \qquad \big[Q_0\,,\, R_1^2\big] \in F^4 K\ . \tag{III.3.16}$$

That means that the part of R_1^2 which lives in F^3K/F^4K is a δ_0-cocycle, since the $(0,-1)$ part of Q_0 is precisely δ_0. By the quasi-acyclicity of the Koszul complex it is a coboundary, say, $-\delta_0\, Q_2$ for some $Q_2 \in K^{3,2}$. In other words, there exists $Q_2 \in K^{3,2}$ such that if $R_2 = Q_0 + Q_1 + Q_2$, then $R_2^2 \in F^4 K$. If this is the case then

$$\big[Q_0\,,\, R_2^2\big] = \big[R_2\,,\, R_2^2\big] - \big[Q_1\,,\, R_2^2\big] - \big[Q_2\,,\, R_2^2\big]\ . \tag{III.3.17}$$

But the first term is zero because of the Jacobi identity and the last two terms are clearly in F^5K due to the fact that, from (III.3.4),

$$\left[F^pK\,,\,F^qK\right] \subseteq F^{p+q-1}K\ . \tag{III.3.18}$$

Hence, $\left[Q_0\,,\,R_2^2\right] \in F^5K$, from where we can deduce the existence of $Q_3 \in K^{4,3}$ such that $R_3 = Q_0 + Q_1 + Q_2 + Q_3$ obeys $R_3^2 \in F^5K$, and so on. It is easy to formalize this into an induction proof of the following theorem.

Theorem III.3.19. *There exists $Q = \sum_i Q_i$, where $Q_i \in K^{i+1,i}$ such that $\left[Q\,,\,Q\right] = 0$.*

Now let $D = \left[Q\,,\,\cdot\right]$. Then $D^2 = 0$ and repeating the proof of Theorem III.2.18 we obtain the following.

Theorem III.3.20. *The cohomology of D is given by*

$$\boxed{H_D^n \cong \begin{cases} 0 & \text{for } n < 0 \\ H_V^n(M_o) & \text{for } n \geq 0 \end{cases}}\ . \tag{III.3.21}$$

In particular, $H_D^0 \cong C^\infty(\widetilde{M})$.

From now on we will take $D = \left[Q\,,\,\cdot\right]$ to be the classical BRST operator; although it is common in the physics literature to call Q the classical BRST operator.

We now come to an important consequence of the fact that the classical BRST operator is a (inner) Poisson derivation. It is easy to verify that this implies that ker D becomes a Poisson subalgebra of K and im D is a Poisson ideal of ker D. Therefore the cohomology space $H_D = \ker D/\operatorname{im} D$ naturally inherits the structure of a Poisson superalgebra. Moreover since K is a graded Poisson superalgebra and D is homogeneous with respect to this grading, the cohomology naturally becomes a graded Poisson superalgebra. In particular, H_D^0 is a Poisson subalgebra and H_D is naturally a graded Poisson module of H_D^0. In particular, since H_D^0 is isomorphic to $C^\infty(\widetilde{M})$ we see that the Poisson brackets get induced. Therefore if we wished to compute the Poisson brackets of two smooth functions on $\widetilde{M}$ we merely need to find suitable BRST cocycles representing them and compute the Poisson bracket in K. It is noteworthy to remark that it is not always possible to choose BRST cocycles which are ghost independent, *i.e.*, in $K^{0,0}$ so that the ghosts and antighosts are an integral ingredient in the formulation.

The Case of a Group Action

Since the case when the constraints arise from a moment map is of special interest, it is worth looking at its classical BRST operator in some detail. We will be able to relate the BRST cohomology with a Lie algebra cohomology group with coefficients in an infinite dimensional (differential) representation.

So let G be a Lie group and $\mathfrak{g}$ its Lie algebra and let there be a Poisson action of G M giving rise to an equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Let $\{b_i\}$ be a basis for $\mathfrak{g}$ and $\{c^i\}$ be the canonical dual basis for $\mathfrak{g}^*$. Notice that the dual of the moment map gives rise to a map $\mathfrak{g} \to C^\infty(M)$ sending $b_i \mapsto \phi_i$, where ϕ_i are the coefficients of the moment map relative to the $\{c^i\}$:

$$\langle \Phi(m), b_i \rangle = \phi_i(m) \ , \tag{III.3.22}$$

which is precisely the map δ_K in the Koszul complex. In particular, we can identify $\mathbb{V}$ with $\mathfrak{g}$. Since the action is Poisson, the functions $\{\phi_i\}$ represent the algebra under the Poisson bracket: $\left\{\phi_i \, , \, \phi_j\right\} = \sum_k f_{ij}{}^k \phi_k$, where the $f_{ij}{}^k$ are the structure constants of $\mathfrak{g}$ in the chosen basis. Let $Q = Q_0 + Q_1$ where Q_0 and Q_1 are given by (III.3.8) and (III.3.12), respectively. Since the $f_{ij}{}^k$ are constant and satisfy the Jacobi identity, $\left\{Q \, , \, Q\right\} = 0$, and hence the extra $Q_{i>1}$ are not necessary. Hence the classical BRST "operator" is

$$\boxed{Q = \sum_i c^i \, \phi_i - \frac{1}{2} \sum_{i,j,k} f_{ij}{}^k \, c^i \wedge c^j \wedge b_k} \ . \tag{III.3.23}$$

Notice that this is precisely the operator found by Batalin & Vilkoviskii [**18**].

We can now make contact with Lie algebra cohomology. The cohomology of the classical BRST operator is exactly the cohomology of the vertical derivative which is computed by the complex C defined by

$$C^\infty(M_o) \xrightarrow{D} \mathfrak{g}^* \otimes C^\infty(M_o) \xrightarrow{D} \textstyle\bigwedge^2 \mathfrak{g}^* \otimes C^\infty(M_o) \xrightarrow{D} \cdots \ , \tag{III.3.24}$$

where D is defined on the generators by

$$\begin{aligned} Df &= \sum_i c^i \otimes \left\{\phi_i \, , \, f\right\} \\ Dc^i &= -\frac{1}{2} \sum_{j,k} f_{ij}{}^k \, c^j \wedge c^k \ . \end{aligned}$$

Comparing with (II.1.61) we deduce that C is nothing but the space of Lie algebra cochains $C(\mathfrak{g}; C^\infty(M_o))$; and comparing with (II.1.59) we deduce that D is nothing but the Lie algebra coboundary operator. Hence, for the case of a Poisson group action, the classical Lie algebra cohomology is just the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the module $C^\infty(M_o)$: $H(\mathfrak{g}; C^\infty(M_o))$.

4. Topological Characterization

In Section 2 we saw that that there is a geometric interpretation for the classical BRST cohomology as the vertical cohomology acting on differential forms along the leaves of the foliation $\mathcal{M}_o^\perp$ defined by the first class constraints on the coisotropic submanifold M_o traced by their zero locus. In this section we use this geometric interpretation to compute the classical BRST cohomology.

The tangent bundle of M_o breaks up as $TM_o = T\mathcal{M}_o^\perp \oplus N\mathcal{M}_o^\perp$, where $T\mathcal{M}_o^\perp = TM_o^\perp$ is the tangent space to the foliation and $N\mathcal{M}_o^\perp$ is the normal bundle to the foliation. Let $T^*\mathcal{M}_o^\perp$ and $N^*\mathcal{M}_o^\perp$ denote the cotangent and conormal bundles to the foliation, respectively. Under this split, the differential forms, $\Omega(M_o)$, on M_o decompose as

$$\Omega(M_o) = \bigoplus_{p,q} \Omega^{p,q}(M_o) \ , \tag{III.4.1}$$

where $\Omega^{p,q}(M_o)$ is the space of smooth sections through the bundle

$$\bigwedge^p T^*\mathcal{M}_o^\perp \otimes \bigwedge^q N^*\mathcal{M}_o^\perp \ . \tag{III.4.2}$$

The exterior derivative on M_o has a piece

$$d_V : \Omega^{p,q}(M_o) \to \Omega^{p+1,q}(M_o) \ , \tag{III.4.3}$$

which is just the vertical derivative and whose cohomology, acting on the vertical forms $\Omega_V^p(M_o) \equiv \Omega^{p,0}(M_o)$, is precisely the classical BRST cohomology.

In **[82]** the Poincaré lemma for this complex is proven. That is, if ω is a d_V-closed vertical p-form (for $p \geq 1$), then around each point in M_o there exists a neighborhood U and a vertical $(p-1)$-form θ_U defined on U such that $\omega = d_V\theta_U$ on U. A vertical 0–form is just a function on M_o and it is d_V–closed if and only if it is constant on each leaf. Therefore a d_V–closed vertical 0–form is the pull back via π of a function on $\widetilde{M}$. Let $\mathcal{E}_{\widetilde{M}}$ be the sheaf of germs of smooth functions on $\widetilde{M}$ and let Ω_V denote the sheaf of germs of vertical forms on M_o. By the above remarks there is an acyclic resolution

$$0 \longrightarrow \pi^*\mathcal{E}_{\widetilde{M}} \longrightarrow \Omega^0_V \xrightarrow{d_V} \Omega^1_V \longrightarrow \cdots \tag{III.4.4}$$

where the first map is the inclusion. This identifies the vertical cohomology with the sheaf cohomology $H(M_o; \pi^*\mathcal{E}_{\widetilde{M}})$ and thus makes contact with the work of Buchdahl[**83**] on the relative de Rham sequence, of which the vertical cohomology is an important special case.

Buchdahl treats the case of an arbitrary smooth surjective map $f : Y \to X$ between two arbitrary (smooth, paracompact) manifolds. He then obtains a resolution for the pull-back sheaf $f^*\mathcal{E}_X$ in terms of **relative forms** Ω_f. Relative forms are differential forms along the fibers of f and the derivative is the exterior derivative along the fibers; where by a fiber we mean the preimage via f of a point in X. Hence vertical cohomology is a particular case of this construction for a very special f, Y and X. Buchdahl does not characterize the relative cohomology completely, but he proves two results that relate it to the cohomology of the fibers. In the case of vertical cohomology, his results (Propositions 1 and 2 in [**83**]) imply the following two theorems, where F is the typical fiber in the fibration $M_o \overset{\pi}{\longrightarrow} \widetilde{M}$ and $H(F)$ stands for the real de Rham cohomology of the typical fiber.

Theorem III.4.5. *$H^1(F) = \mathbf{0}$ implies $H^1_V(M_o) = \mathbf{0}$. If $H^{p-1}(F) = H^p(F) = \mathbf{0}$ for some $p > 1$, then $H^p_V(M_o) = \mathbf{0}$.*

Theorem III.4.6. *If for some $p \geq 1$, $H^p_V(M_o) = H^{p+1}_V(M_o) = \mathbf{0}$, then $H^p(F) = \mathbf{0}$.*

An easy corollary of these two theorems gives a characterization of the vanishing of the BRST cohomology for positive ghost number.

Corollary III.4.7. *A necessary and sufficient condition for the classical BRST cohomology to vanish for positive ghost number is that the gauge orbits have vanishing positive de Rham cohomology.*

In particular in the case of a compact orientable gauge orbit, Poincaré duality already forbids the vanishing of the BRST cohomology of top ghost number.

These results, although already providing a lot of information, are far from fully characterizing the BRST cohomology in terms of the topology of the gauge orbits and the gauge invariant observables. Since the case of interest to us is so special we can obtain stronger results. In fact, we can characterize the vertical cohomology from initial data.

The Main Theorem

To fix the notation, let $F \longrightarrow M_o \xrightarrow{\pi} \widetilde{M}$ be a smooth fiber bundle where the typical fiber, F, is connected. Let d_V denote the vertical derivative, $\Omega_V(M_o)$ the vertical forms, and $H_V(M_o)$ the vertical cohomology. By definition, the zeroth vertical cohomology, $H^0_V(M_o)$, consists of those smooth functions on M_o which are locally constant on the fibers; and since the fibers are connected, these functions are constant. The projection π induces an isomorphism, $\pi^* : C^\infty(\widetilde{M}) \to C^\infty(M_o)$, defined by $\pi^* f = f \circ \pi$, onto the smooth functions on M_o which are constant on the fibers. Therefore, there is an isomorphism

$$H^0_V(M_o) \cong C^\infty(\widetilde{M}) \ . \tag{III.4.8}$$

By its definition the vertical derivative d_V obeys

$$d_V(\omega \wedge \theta) = (d_V\omega) \wedge \theta + (-1)^p \omega \wedge (d_V\theta) \ , \tag{III.4.9}$$

for $\omega \in \Omega^p_V(M_o)$ and $\theta \in \Omega_V(M_o)$. Therefore $\wedge$ induces an operation in cohomology

$$\cup : H^p_V(M_o) \times H^q_V(M_o) \longrightarrow H^{p+q}_V(M_o) \ , \tag{III.4.10}$$

defined by $[\omega] \cup [\theta] = [\omega \wedge \theta]$. This operation is well defined because of (III.4.9) and makes the vertical cohomology into a graded ring. In particular,

$$\cup : H^0_V(M_o) \times H^q_V(M_o) \longrightarrow H^q_V(M_o) \tag{III.4.11}$$

makes $H_V(M_o)$ into a graded $H^0_V(M_o) \cong C^\infty(\widetilde{M})$ module.

Let $\mathcal{H}_V$ denote the sheaf of $C^\infty(\widetilde{M})$-modules on $\widetilde{M}$ defined by $\mathcal{H}_V(U) = H_V(\pi^{-1}U)$ for all open $U \subset \widetilde{M}$. By local triviality there exists an open cover $\mathcal{U}$ for $\widetilde{M}$ such that for all $U \in \mathcal{U}$, $\pi^{-1}U \cong U \times F$. Therefore $\mathcal{H}_V(U) \cong H_V(U \times F)$. By a theorem of Kacimi-Alaoui (III (1) in **[84]**) the vertical cohomology of a product is given simply by

$$\boxed{H_V(U \times F) \cong C^\infty(U) \otimes H(F)} \ , \tag{III.4.12}$$

where $H(F)$ is the real de Rham cohomology of F. This implies that $\mathcal{H}_V$ is a locally free sheaf and thus[85] the sheaf of germs of smooth sections of a vector bundle over $\widetilde{M}$ with fiber $H(F)$.

The task ahead is to determine the transition functions of this bundle. Let $\{\psi_U\}$ be the family of diffeomorphisms

$$\psi_U : \pi^{-1}U \longrightarrow U \times F \tag{III.4.13}$$

given by the local triviality of the original bundle $M_o \xrightarrow{\pi} \widetilde{M}$. The transition functions of this bundle are then given, for all $U \cap V \neq \varnothing$, by $g_{UV} = \psi_U \circ \psi_V^{-1}$, thought of as a map $g_{UV} : U \cap V \to \mathrm{Diff}\, F$.

Recall that there is a natural representation of Diff F as automorphisms of degree zero of the (graded) de Rham cohomology ring $H(F)$. If $\varphi \in \mathrm{Diff}\, F$ then the automorphism is defined by $[\omega] \mapsto [(\varphi^{-1})^* \omega]$. By the homotopy invariance of de Rham cohomology, two diffeomorphisms which are homotopic are represented by the same automorphism in $H(F)$. So any diffeomorphism which is homotopic to the identity will automatically induce the identity automorphism on cohomology.

Composing the transition functions $\{g_{UV}\}$ with this representation provides maps

$$(g_{UV}^{-1})^* : U \cap V \to \mathrm{Aut}\;\, H(F) \;, \tag{III.4.14}$$

which, as we will now see, are the transition functions of the bundle whose sheaf of sections is given by $\mathcal{H}_V$.

To see this notice that for all open sets $U \in \mathcal{U}$

$$(\psi_U^{-1})^* : H_V(\pi^{-1}U) \to H_V(U \times F) \cong C^\infty(\widetilde{M}) \otimes H(F) \;, \tag{III.4.15}$$

allows us to identify vertical cohomology classes on $\pi^{-1}U$ with $H(F)$-valued functions on U. Let ω be a d_V-closed vertical form and $[\omega]$ its class in vertical cohomology. Restricted to $U \cap V$ there are two ways in which one can identify $[\omega]$ with an $H(F)$-valued function on $U \cap V$: either by using the trivialization on U or the one on V. Let $f_U = [(\psi_U^{-1})^* \omega]$ and $f_V = [(\psi_V^{-1})^* \omega]$. The transition functions h_{UV} are precisely the automorphisms of the fiber $H(F)$ relating these two descriptions of the same object. That is, the transition functions obey $f_U = h_{UV} f_V$. But because

$$\begin{aligned} f_U &= [(\psi_U^{-1})^* \omega] \\ &= [(\psi_U^{-1})^* \circ \psi_V^* \circ (\psi_V^{-1})^* \omega] \\ &= [(\psi_U^{-1})^* \circ \psi_V^*\, f_V] \\ &= [(\psi_V \circ \psi_U^{-1})^*\, f_V] \\ &= [(g_{UV}^{-1})^*\, f_V] \;, \end{aligned} \tag{III.4.16}$$

the transition functions are in fact the ones in (III.4.14). Therefore we have proven the following theorem.

Theorem III.4.17. *As a module over $C^\infty(\widetilde{M})$ the BRST cohomology is isomorphic to the smooth sections of the associated bundle $M_o \times_\rho H(F) \longrightarrow \widetilde{M}$ associated to the representation $\rho : \mathrm{Diff}\, F \to \mathrm{Aut}\ H(F)$.*

Notice that this associated bundle decomposes naturally as a Whitney sum of vector bundles

$$M_o \times_\rho H(F) = \bigoplus_p M_o \times_\rho H^p(F) \qquad \text{(III.4.18)}$$

since diffeomorphisms do not alter the degree of a form.

As a corollary of this theorem we have that the vertical cohomology (and hence the classical BRST cohomology) does not depend on the explicit form of the constraints used to describe M_o. In fact, the inclusion $i : M_o \hookrightarrow M$ is all that the cohomology depends on. With this information alone we can determine the pullback 2-form $i^*\Omega$ and hence its null foliation $\mathcal{M}_o^\perp$ and this defines a fibration $F \longrightarrow M_o \xrightarrow{\pi} \widetilde{M}$. By Theorem III.4.17, this is all the classical BRST cohomology depends on.

<u>The Case of a Group Action</u>

When the constraints arise from the hamiltonian action of a connected Lie group G—*i.e.*the constraints are the coefficients of the moment map relative to a fixed basis for the Lie algebra of G—the bundle

$$\begin{array}{ccc} G & \longrightarrow & M_o \\ & & \big\downarrow \pi \\ & & \widetilde{M} \end{array} \qquad \text{(III.4.19)}$$

is in fact a principal G-bundle and the diffeomorphisms of G defined by the transition functions correspond to right multiplication by an element of the group. Since G is connected, right multiplication by any element $g \in G$ is homotopic to the identity. (*Proof:* Let $t \mapsto g(t)$ be a curve in G such that $g(0) = \mathbf{1}$ and $g(1) = g$. Right multiplication by $g(t)$ gives the desired homotopy.) By the homotopy invariance of de Rham cohomology, the transition functions of the associated bundle $M_o \times_\rho H(G) \longrightarrow \widetilde{M}$ are the identity maps and thus the bundle is trivial. This proves the following corollary.

Corollary III.4.20. *When the constraints arise from the hamiltonian action of a connected Lie group G, the BRST cohomology is isomorphic to the $H(G)$-valued functions on $\widetilde{M}$.*

The Case of Compact Fibers

Finally suppose that the fibers are compact. Since they are also orientable[9], Poincaré duality induces an isomorphism

$$\star : H^p(F) \to H^{n-p}(F) \ , \tag{III.4.21}$$

where n is the dimension of the fiber. This induces a duality in the BRST cohomology as follows. Let σ be a section through $M_o \times_\rho H^p(F)$. Define a section $\widetilde{\star}\sigma$ through $M_o \times_\rho H^{n-p}(F)$ by

$$(\widetilde{\star}\sigma)(m) = \star\,\sigma(m) \qquad \forall\ m \in \widetilde{M}\ . \tag{III.4.22}$$

This is an isomorphism and hence we have the following result.

Corollary III.4.23. *Let the typical fiber F be n-dimensional and compact. Then there is an isomorphism*

$$\boxed{H^p_V(M_o) \cong H^{n-p}_V(M_o)}\ . \tag{III.4.24}$$

It is worth remarking that for the case of reducible constraints the BRST operator also has the same geometric interpretation[**21**] and hence almost all the results of this section go through unchanged. The only exception is the last subsection where we needed orientability of the fibers. In the reducible case the fibers are no longer parallelizable. I ignore if they are generally orientable and hence, for reducible constraints, the hypothesis in Corollary III.4.23 must be amended to assume that the fibers are orientable.

[9] In fact, they are parallelizable since the $\{X_i\}$ provide a global basis for the tangent bundle.

Chapter Four: Geometric BRST Quantization

Perhaps the single most salient feature of the analysis of classical BRST presented in the previous chapter is the naturality of the BRST construction within the symplectic framework. The point of this chapter is to exploit this feature in an effort to define BRST quantization. There are two quantization prescriptions which can be defined intrinsically in terms of symplectic data: geometric quantization (*cf. e.g.*, **[86]**) and deformation theory (*cf. e.g.*, **[87]**). We will not have anything to say about the latter quantization scheme, except that I have been thus far unable to extend BRST theory in this direction. However the situation is quite different for geometric quantization; although the conclusions are not very optimistic. Let us elaborate on this point.

BRST quantization offers a possible solution to the quantization of systems whose hamiltonian (classical) description involves redundant degrees of freedom which give rise to first class constraints. The quantization of such systems consists in the successful completion of the following diagram:

$$\boxed{\begin{array}{ccc} (M,\Omega) & \xrightarrow[\text{quantization}]{\text{canonical}} & \mathcal{H} \\ \big\downarrow & & \big\downarrow \\ (\widetilde{M},\widetilde{\Omega}) & \xrightarrow[\text{quantization}]{\text{canonical}} & \mathcal{H}_{\text{phys}} \end{array}} \tag{IV.0.1}$$

where by successful we mean that the rightmost arrow can be constructed in such a way that the diagram commutes. That is, to go from the initial data (M,Ω) to the physically meaningful quantum theory on $\mathcal{H}_{\text{phys}}$ it does not matter which route we take: we could either first symplectically reduce (M,Ω) to $(\widetilde{M},\widetilde{\Omega})$ and then canonically quantize this latter symplectic manifold, or we could quantize (M,Ω) directly and then recover algebraically the physical states[10]. In practice the second route is preferred because either the symplectic reduction is hard to do explicitly or, even when it can be done one may lose desirable properties of the redundant formalism: locality, covariance,... However even when in practice there is only one way to effectively construct the quantum theory it is important to verify that the diagram is commutative. The only results of a general nature that hint at the commutativity of (IV.0.1) are the results of Guillemin & Sternberg **[88]** which essentially state its commutativity for the special case of M a (simply connected) compact Kähler manifold and $\widetilde{M}$ its reduction via the Poisson action of a (simply connected) compact Lie group. However, the conjecture of commutativity is further supported by our experience with certain exactly solvable systems, for instance, free string theory where we can do BRST quantization or go to the light-cone and both theories have been shown to have the same spectrum.

Two main facts make the BRST construction in the geometric quantization framework so uniquely suited for this problem: first of all is the fact that the construction is algebraic in nature and hence it is easier to compute with; and second, that geometric quantization is a fairly "continuous" process: both the operator algebra and the representation can be constructed from symplectic data, as opposed to the usual canonical quantization, which only provides the operator algebra.

In the previous chapter we discussed the leftmost arrow of (IV.0.1) in the BRST language and in this chapter we attempt to fill in the rest of the diagram. Unfortunately we are still far from achieving the desired goal. There is one major obstacle having to do with the inner products. We will comment on this in the appropriate section.

We should mention that the problem of defining a BRST quantization procedure has been recently analyzed in the literature[**81**]. In **[81]** the authors discuss the BRST quantization in the case of constraints arising from the action of an algebra and they focus only on the ghost part assuming that the quantization of the ghost and matter parts are independent. In this chapter we show that this is not always possible in geometric quantization, since the polarization of the matter forces, in some cases, a particular polarization of the ghosts.

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[10] Matters are further complicated by the fact that the horizontal arrows in (IV.0.1) do not exist in general. This is nothing but the statement that given a classical dynamical system there is no unique way to quantize it, or sometimes it cannot be quantized at all ("first quantization is not a functor", "there is nothing canonical about canonical quantization"). In fact, in all honesty, we really only understand the leftmost arrow. But we should not let this thwart our plans.

This chapter is organized as follows. Section 1 contains a brief review of the philosophy and methods of geometric quantization which barely skims the surface of this very active field. A lot more details than the ones given here can be found in the books by Woodhouse **[86]** and by Hurt **[89]**. However, quantization is a hard problem and geometric quantization is by far not a closed chapter in mathematical research. Therefore it is lacking the comprehensive treatises that flood other more successful branches of mathematics.

Section 2 deals with BRST prequantization. It essentially constructs the BRST cohomology theory to reduce the sections of the prequantum line bundle. It is of slightly more generality and, in fact, develops an algebraic machinery to provide a cohomological description to the reduction of (sections) of vector bundles and, hence, of most of the interesting geometric objects. If $E \longrightarrow M$ is a vector bundle and $i : M_o \hookrightarrow M$ is the constrained submanifold, the restriction of E to M_o gives rise to a bundle $i^{-1}E \longrightarrow M_o$ known as the pull-back bundle. We then construct a Koszul complex which provides an acyclic resolution of the smooth sections of this bundle. The acyclicity of this complex follows from the acyclicity of the Koszul complex for functions described in Section III.1 and from the fact that the smooth sections of any vector bundle is (finitely generated) projective over the ring of smooth functions.

We then go on to define a cohomology theory associated with the foliation determined by the null distribution of $i^*\Omega$ on M_o. This is a generalization of the vertical cohomology constructed in Section III.2, except that the vertical differential forms take values in a vector bundle over M_o which admits a representation of the vectors tangent to the foliation; *e.g.*, any bundle on which one can define the notion of a Lie derivative or which admits a connection relative to which the directions spanned by the vectors tangent to the foliation are flat. The zeroth cohomology is then a finitely generated projective module over $C^\infty(\widetilde{M})$ which corresponds to the module of smooth sections of some vector bundle over $\widetilde{M}$. We then lift this cohomology theory via the Koszul resolution to a BRST cohomology theory. The proof of existence of the BRST differential follows identical steps to the ones for the BRST cohomology on functions.

In an effort to make contact with symplectic concepts we then go on to show that the BRST cohomology just constructed has a very nice interpretation using the concept of a Poisson module. Therefore we first review this notion. I have not seen Poisson modules defined anywhere[11] but we feel our definition is the natural one. We then show that the BRST

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[11] Although in a very interesting recent paper by Johannes Huebschmann[**58**] they are seen to be special cases of a modules over a much more general kind of algebra than Poisson algebras: Rinehart's (A, R)–algebras.

cohomology just constructed is naturally a Poisson module over the BRST cohomology constructed in Section III.2. Since prequantization is essentially the fact that the sections of the prequantum line bundle form a Poisson module over the Poisson algebra of smooth functions, we are ready to construct BRST prequantization. If (M, Ω) is prequantizable then there is a hermitian line bundle with compatible connection such that its curvature is given by $-2\pi i\Omega$. Its pull back onto M_o is also a hermitian line bundle with compatible connection whose curvature is $-2\pi i(i^*\Omega)$. Since the flat directions of this connection coincide with the directions tangent to the null foliation, the smooth sections of this line bundle admit a representation of the vector fields tangent to the foliations and thus we can build a BRST cohomology theory. We prove that all the prequantum data gets induced à la BRST except for the inner product, which involves integration. We discuss the reasons why.

In Section 3 we discuss polarizations. We introduce the notion of an invariant polarization and show that in many cases the existence of invariant polarizations imposes stringent conditions on the constraints. We find that in many cases—pseudo-Kähler polarizations, G-action with G semisimple—a choice of polarization for M forces a polarization of the ghost part. This is to be compared with **[81]** where the quantization of the ghost and the matter parts are done separately. In particular one can always choose a polarization for the ghost part in which the quantum BRST operator simplifies enormously since it only contains linear and trilinear ghost terms.

Finally in Section 4 and modulo some minor technicalities which we discuss there, we prove a duality theorem for the quantum BRST cohomology. Some of the technicalities are connected to the potential infinite dimensionality of the spaces involved.

1. Geometric Quantization

Geometric quantization is an attempt to develop a mathematically consistent and invariant quantization scheme. It tries to overcome the problems of the more traditional "canonical" quantization. The canonical quantization of finite dimensional systems consists in finding a unitary irreducible representation of the Heisenberg algebra

$$\left[q^a\, ,\, p_b\right] = i\hbar\, \delta^a_b\ , \tag{IV.1.1}$$

where (q,p) are local coordinates for the phase space of the system we are quantizing and $i = \sqrt{-1}$. The Stone–Von Neumann theorem guarantees that there is essentially a unique such representation[12]. In this representation—taken, without loss of generality, to be $L^2(\mathbb{R}^n)$—q^a is represented by the multiplication operator $\psi(q) \mapsto q^a\,\psi(q)$; and p_b is represented by $\psi(q) \mapsto -i\hbar\psi'(q)$. A classical observable $f(p, q)$ is then represented by $f(-i\hbar\frac{\partial}{\partial q}, q)$.

This has two obvious problems. First, the operator $f(-i\hbar\frac{\partial}{\partial q}, q)$ requires for its definition that we give an ordering prescription, since q and $\frac{\partial}{\partial q}$ do not commute. And second, the Heisenberg algebra is not general coordinate invariant, so the above prescription depends on the choice of coordinates.

It was Dirac who first noticed the similarity between the algebraic structures in both quantum and classical mechanics. He observed that the Poisson bracket seemed to be the classical analogue of the quantum commutator. The fact that the Poisson bracket has an invariant meaning in the phase space allowed Dirac to reformulate canonical quantization in an invariant fashion. The Poisson bracket (*i.e.*, the symplectic structure) thus plays a fundamental rôle in the Dirac quantization approach. The Dirac quantization problem consists therefore in finding an irreducible representation of the Lie algebra (under Poisson bracket) of real smooth functions as (essentially) self-adjoint operators in a Hilbert space with the properties that the constant function with value 1 shall be represented by the identity operator and that, if (q, p) is local chart forming a canonically conjugate pair (*i.e.*, they obey the Heisenberg algebra), then they shall act irreducibly or at least, in case one wants to include internal degrees of freedom, with finite reducibility.

A celebrated theorem of Van Hove[**90**],[**71**] forbids the existence of such a representation; although he showed that one could find an irreducible representation of some subalgebra if one dropped the last condition on canonically conjugate pairs.

The geometric quantization program of Kostant[**91**] and Souriau[**92**] provides an invariant method of constructing such representations. The first part of the method, called prequantization, consists of dropping the irreducibility condition and constructing a representation of the Lie algebra of smooth functions as self-adjoint operators in a Hilbert space, purely in terms of symplectic data. The second part of the construction, called polarization, will take care of making this representation irreducible and in the process restricting the class of functions which can be quantized.

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12 Strictly speaking, the theorem guarantees the uniqueness up to unitary equivalence of the irreducible representations of the exponentiated (Weyl) form of the commutation relations.

Prequantization

Let (M, Ω) be a symplectic manifold. Since $d\Omega = 0$, the symplectic form defines a class in the real de Rham cohomology group $H^2_{dR}(M;\mathbb{R})$. We say Ω is integral if this class lies in the image of the map

$$H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R}) \cong H^2_{dR}(M;\mathbb{R}) \ . \tag{IV.1.2}$$

If this is the case we speak of an **integral symplectic manifold**.

If (M, Ω) is an integral symplectic manifold then there exists[91] at least one complex line bundle $E \longrightarrow M$ with a hermitian structure, *i.e.*, a sesquilinear map

$$\langle , \rangle : \Gamma(E) \times \Gamma(E) \to C^\infty_{\mathbb{C}}(M) \ , \tag{IV.1.3}$$

which is antilinear in the first factor and linear in the second; and with a connection

$$\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E) \ , \tag{IV.1.4}$$

such that

(PQ1) $\langle , \rangle$ is parallel with respect to ∇; that is, for all $\sigma, \tau \in \Gamma(E)$,

$$d\langle \sigma, \tau \rangle = \langle \sigma, \nabla\tau \rangle + \langle \nabla\sigma, \tau \rangle \ ;$$

(PQ2) the symplectic form and the curvature 2-form of the connection are related by

$$\mathrm{curv}(\nabla) = -2\pi i \Omega \ .$$

The triple $(E, \nabla, \langle , \rangle)$ satisfying the above properties will be called **prequantum data** for the symplectic manifold (M, Ω). Hence integral symplectic manifolds are also known as prequantizable symplectic manifolds.

Let $d\mu_L$ denote the Liouville measure on M. This is the measure induced by the volume form proportional to $\underbrace{\Omega \wedge \cdots \wedge \Omega}_{n}$ for M a $2n$-dimensional manifold. This allows us to define an inner product on $\Gamma(E)$ by integrating the pointwise inner product with respect to this measure:

$$(\sigma, \tau) \equiv \int_M \langle \sigma, \tau \rangle \, d\mu_L \ . \tag{IV.1.5}$$

Let $\Gamma_{L^2}(E)$ denote the Hilbert space completion of the subspace of $\Gamma(E)$ consisting of sections σ such that $\|\sigma\|^2 \equiv (\sigma, \sigma) < \infty$. This will become the prequantum Hilbert space.

The prequantization map assigning to a smooth function f an operator $O(f)$ in $\Gamma_{L^2}(E)$ is the following

$$f \mapsto O(f) \equiv \nabla_{X_f} + 2\pi i\, f \ , \tag{IV.1.6}$$

where X_f is the Hamiltonian vector field associated to f, that is, $\imath(X_f)\,\Omega + df = 0$. The prequantization map obeys the following

$$O(f)O(g)\sigma - O(g)O(f)\sigma = O(\{f\,,\,g\})\sigma \ , \tag{IV.1.7}$$

$$O(f)(g\sigma) = \{f\,,\,g\}\sigma + g\,O(f)\sigma \ , \tag{IV.1.8}$$

for all $\sigma \in \Gamma(E)$ and $f, g \in C^\infty(M)$. Moreover each $O(f)$ is a skew-symmetric operator. That is, if $\sigma, \tau \in \Gamma_{L^2}(E)$ are in the domain of $O(f)$ then

$$(O(f)\sigma, \tau) + (\sigma, O(f)\tau) = 0 \ . \tag{IV.1.9}$$

If, in addition, X_f is a complete vector field, $O(f)$ has a skew-self-adjoint extension[**86**] and generates, by Stone's theorem, a one parameter family of unitary operators in $\Gamma_{L^2}(E)$.

The prequantization map has the property that the only operator of the form $O(f)$ for some $f \in C^\infty(M)$ which commutes with all the other $O(g)$'s are the scalars, corresponding to $O(c)$ for c a constant function on M. Still this representation is highly reducible: roughly speaking it consists of integrable functions of both the momenta and the coordinates. Thus we need to cut down the size of $\Gamma_{L^2}(E)$. The way to do this is via polarizations.

Polarizations

To help fix the ideas, let us discuss the familiar example of M a cotangent bundle, say T^*N. In this case the symplectic form is exact and hence the prequantum line bundle is trivial. We can therefore choose a global non-vanishing section and hence identify the space of sections with the complex valued functions themselves. However this space is far too big: it contains functions of both position and momentum. We would like to end up with functions of just position. It is then clear what we must do. We must pick the subspace of the functions which are independent of the momentum, *i.e.*, they are constant on the fibers of the bundle $T^*N \xrightarrow{\pi} N$. In other words, they are the functions which are annihilated by the vector fields tangent to the fibers. These vector fields span an integrable distribution of TT^*N, being exactly the kernel of the derivative of the projection $\pi_* : TT^*N \to TN$. Moreover this distribution is lagrangian, since locally it is spanned by $\{\frac{\partial}{\partial p_a}\}$ in a basis where the symplectic form is $\sum_a dq^a \wedge dp_a$. This distribution, ker π_*, is the canonical real polarization (see later) of T^*N. A general polarization will consist of a suitable generalization of this object. Of course, in general, a symplectic manifold need not have a canonical polarization. In this sense, cotangent bundles are special.

We begin then by defining polarizations. By a **polarization** of a symplectic manifold (M, Ω) we will mean an involutive lagrangian complex subbundle of the complexified tangent bundle of M. In other words, let $F = \{m \mapsto F_m \subset T^{\mathbb{C}}_m M\}$ be a smooth involutive distribution such that F_m is a complex lagrangian subspace of $T^{\mathbb{C}}_m M \cong \mathbb{C} \otimes_{\mathbb{R}} T_m M$, made into a complex symplectic vector space by extending the symplectic form $\mathbb{C}$-linearly to a new symplectic form $\Omega_{\mathbb{C}}$. Then F is a polarization of the symplectic manifold (M, Ω) and (M, Ω, F) is called a **polarized symplectic manifold**.

Notice that if F is a polarization, so is $\overline{F}$. A polarization is **real** if $F = \overline{F}$. This is the case if and only if[**73**] $F = \mathbb{C} \otimes V$ for some involutive lagrangian subbundle V of TM. The canonical polarization of a cotangent bundle gives rise, upon complexification, to a real polarization. On the other extreme, a polarization is **totally complex** if $F \cap \overline{F} = 0$. In this case, $T^{\mathbb{C}} M \cong F \oplus \overline{F}$. Therefore $F \cap TM = F \cap iTM = 0$ and hence F is the graph of a bundle isomorphism $TM \to iTM$. That is, $F_m = \{X + iJ_m X \mid X \in T_m M\}$ for some isomorphism $J_m : T_m M \to T_m M$. Since F_m is a complex linear subspace we deduce that $J^2 = -\mathbf{1}$ and so it is a complex structure. Moreover since F is lagrangian, for all $X, Y \in TM$, we have

$$\begin{aligned} 0 &= \Omega_{\mathbb{C}}(X + iJX, Y + iJY) \\ &= \Omega(X, Y) - \Omega(JX, JY) + i\left(\Omega(X, JY) + \Omega(JX, Y)\right) \ . \end{aligned} \tag{IV.1.10}$$

From the real part of this equation we deduce that J is a symplectomorphism and from the imaginary part that $g(X, Y) \equiv \Omega(X, JY)$ is symmetric. Moreover it follows that J is orthogonal with respect to g. In fact, for all $X, Y \in TM$

$$\begin{aligned} g(JX, JY) &= \Omega(JX, J^2 Y) \\ &= \Omega(Y, JX) \\ &= g(Y, X) \\ &= g(X, Y) \ . \end{aligned}$$

Also, for all $X, Y \in TM$,

$$\left[X + iJX \,,\, Y + iJY\right] = \left[X \,,\, Y\right] - \left[JX \,,\, JY\right] + i\left[X \,,\, JY\right] + i\left[JX \,,\, Y\right] \ , \tag{IV.1.11}$$

which, since F is involutive, implies that

$$J\big[X\,,\,Y\big] - J\big[JX\,,\,JY\big] = \big[X\,,\,JY\big] + \big[JX\,,\,Y\big]\ . \qquad \text{(IV.1.12)}$$

In other words, the Nijenhuis tensor vanishes and, by the Newlander-Nirenberg theorem, (M, J) is a complex manifold whose holomorphic vector fields correspond to the sections $\Gamma(F)$ of F. Therefore, (M, Ω, F) becomes a pseudo-Kähler manifold, becoming Kähler only when g is positive definite. In this latter case we say F is a **positive definite polarization**.

Let (M, Ω, F) be a polarized symplectic manifold. Let us define A_F to be the following class of functions

$$A_F = \{f \in C^\infty_{\mathbb{C}}(M) \mid \overline{X}f = 0\ ,\ \forall\, X \in \Gamma(F)\}\ . \qquad \text{(IV.1.13)}$$

Alternatively we can characterize these functions as follows.

Proposition IV.1.14. *A_F consists precisely of those functions in $C^\infty_{\mathbb{C}}(M)$ whose associated hamiltonian vector fields are in $\Gamma(\overline{F})$.*

Proof: By definition, a function $f \in C^\infty_{\mathbb{C}}(M)$ belongs to A_F if and only if $\overline{X}f = 0$ for all $X \in \Gamma(F)$. But $\overline{X}f = df(\overline{X}) = \Omega_{\mathbb{C}}(\overline{X}, X_f)$. Hence $f \in A_F$ if and only if $X_f \in \Gamma(\overline{F}^{\perp})$. But $\overline{F}$ is lagrangian, so that $\overline{F}^{\perp} = \overline{F}$. Thus $f \in A_F$ if and only if $X_f \in \Gamma(\overline{F})$. ■

Corollary IV.1.15. *A_F is an abelian Poisson subalgebra of $C^\infty_{\mathbb{C}}(M)$.*

Proof: If $f, g \in A_F$ then for all $X \in \Gamma(F)$, $\overline{X}f = \overline{X}g = 0$. Therefore $\overline{X}(fg) = (\overline{X}f)g + f(\overline{X}g) = 0$, so $fg \in A_F$. Moreover, $\big\{f\,,\,g\big\} = \Omega_{\mathbb{C}}(X_f, X_g)$ which is zero by the previous Proposition and the fact that $\overline{F}$ is lagrangian. ■

In some cases, *e.g.*, F a totally complex polarization, A_F is a maximal abelian subalgebra of $C^\infty_{\mathbb{C}}(M)$. This, in fact, is sometimes taken to be the algebraic definition of a polarization of a Poisson algebra[93]. Let us now define another class of functions

$$N_F = \{f \in C^\infty_{\mathbb{C}}(M) \mid \big[X_f\,,\,\overline{Y}\big] \in \Gamma(\overline{F})\ ,\ \forall\, Y \in \Gamma(F)\}\ . \qquad \text{(IV.1.16)}$$

That is, N_F consist of those functions whose hamiltonian vector fields lie in the normalizer of $\Gamma(\overline{F})$. Hence by general properties of normalizers N_F is a Lie subalgebra of $C^\infty_{\mathbb{C}}(M)$ and that $A_F \subset N_F$ is an abelian ideal. However, in general, N_F is not a Poisson subalgebra of $C^\infty_{\mathbb{C}}(M)$.

Now let $\Gamma(E)$ denote the smooth sections of the prequantum line bundle on M. We now define the polarized sections

$$\Gamma_F(E) = \{\sigma \in \Gamma(E) \mid \nabla_{\overline{X}}\sigma = 0 \ , \ \forall\, X \in \Gamma(F)\} \ . \tag{IV.1.17}$$

If F is a totally complex polarization then E is a holomorphic line bundle and we can choose ∇ to be a holomorphic connection. In this case, $\Gamma_F(E)$ correspond to the holomorphic sections. It is easy to verify that N_F is the maximal Lie subalgebra of $C^\infty_{\mathbb{C}}(M)$ stabilizing $\Gamma_F(E)$.

The quantum Hilbert space in the geometric quantization program is precisely the Hilbert space completion of the polarized sections of finite norm. For general polarizations there may not be any polarized sections of finite norm. Indeed, suppose that P is not totally complex. Then the norm of a polarized section is invariant on the leaves of the foliation defined by the integrable distribution $D = P \cap \overline{P}$. If these leaves are non-compact then there may not be any sections of finite norm. This obstacle can be overcome via "half-form" quantization, which naturally yields objects which can be integrated in the space of leaves M/D. Half-form quantization consists in tensoring the prequantum line bundle with the bundle of half-forms in such a way that the point-wise inner product does not just yield a function to be integrated with respect to the Liouville form, but actually yields a form which can be integrated in the appropriate space. We will not discuss half-form quantization in this paper, except to note that, as we will see, our algebraic constructions extend naturally to this case as well. Supposing that the polarization is totally complex but not positive definite we may still find that there are no polarized sections of finite norm. In fact, let $\{z^a\}$ be a local holomorphic chart which trivializes the prequantum line bundle and relative to which the symplectic form is

$$\Omega = \frac{i}{2} g_{ab} dz^a \wedge d\overline{z}^b \ . \tag{IV.1.18}$$

The inner product, after identifying holomorphic sections with holomorphic functions, is just the usual L^2 inner product of holomorphic functions on an open set of $\mathbb{C}^n$ with measure $e^{-\frac{1}{2}g_{ab}z^a\overline{z}^b} dz d\overline{z}$. If g_{ab} is not positive definite, *i.e.*, if the polarization is not positive definite, the exponential grows in the negative directions and, if the image of the chart is unbounded in those directions, there will be no holomorphic integrable functions. Based on this considerations we will often restrict ourselves to positive definite (totally complex) polarizations.

Having defined the Hilbert space as the space of polarized integrable sections, we notice that the only readily quantizable observables are the (real) functions in N_F. In order to quantize other observables one must resort to ways of pairing Hilbert spaces obtained from different polarizations. We will not discuss this topic here but rather refer the interested reader to any of the standard references on geometric quantization[**86**],[**89**]. More recently, methods based on generalized Bergman kernels[**94**] have been used to be able to quantize observables others than the ones in N_F.

2. BRST Prequantization

In [**88**] Guillemin & Sternberg proved that in the case of a hamiltonian group action on a symplectic manifold (M, Ω) one could induce prequantum data on the reduced symplectic manifold. Their construction goes roughly as follows. Let M_o denote the constrained submanifold $\Phi^{-1}(0)$, Φ being the moment mapping in their case, and let $E_o \longrightarrow M_o$ denote the pull back bundle of E via the inclusion $i : M_o \hookrightarrow M$. They define a complex line bundle $\widetilde{E} \longrightarrow \widetilde{M}$, by defining its sheaf of sections to be the $\mathfrak{g}$-invariant sections[13] of E_o. On M_o, a section σ was $\mathfrak{g}$-invariant if and only if $\nabla_X \sigma = 0$ for all Killing vectors X. Hence if σ and τ are $\mathfrak{g}$-invariant sections, so is their pointwise inner product $\langle \sigma, \tau \rangle$, since

$$X\langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle \ . \tag{IV.2.1}$$

A connection $\widetilde{\nabla}$ is also constructed by constructing its connection 1-form patchwise. Finally, the inner product was defined integrating the pointwise inner product with the Liouville measure on $\widetilde{M}$.

In this section we discuss the BRST equivalent of that construction. We shall not have to restrict ourselves to the case of the group action. First we discuss the Koszul construction for sections of vector bundles on M. This is completely analogous to the case of functions. What we find is an acyclic resolution for the sections of the restriction of the bundle to the submanifold M_o. Next we discuss how to induce a bundle from M_o to its quotient $\widetilde{M}$. This is a natural extension to the vertical cohomology discussed in Section III.2 and consists in considering vertical cohomology with coefficients in a vector bundle. We then lift the vertical cohomology through the Koszul complex to define the prequantum BRST cohomology. Just as classical BRST found its most elegant setting in the framework of Poisson algebras, the prequantum BRST construction turns out to be expressed extremely

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[13] Strictly speaking they needed to assume that the $\mathfrak{g}$ action on the sections of E_o lifts to a G action, to guarantee that the invariant sections correspond to the sections of some vector bundle over $\widetilde{M}$. This assumption is unnecessary, as we will show in our construction.

naturally in the context of Poisson modules and hence we devote a short subsection to this concept. Finally we apply all this to the case of the prequantum line bundle.

The Koszul Complex for Vector Bundles

Let $E \xrightarrow{\pi} M$ be a complex vector bundle of rank r over M and let $\Gamma(E)$ denote the space of smooth sections. It is clear that $\Gamma(E)$ is a module over $C^\infty(M)$ where multiplication is defined pointwise using the linear structure on each fiber. It is straight-forward to prove that $\Gamma(E)$ is a free rank r $C^\infty(M)$-module if and only if E is a trivial bundle. It can be shown[**79**] that M has a finite cover trivializing E. Since on each set of the cover, E is trivial we see that $\Gamma(E)$ is finitely generated: just take as a set of generators the local sections on each cover multiplied by the appropriate elements of a partition of unity subordinate to the cover.

It can also be shown[**79**] that given a vector bundle $E \xrightarrow{\pi} M$ there exists another vector bundle $F \xrightarrow{\rho} M$ such that their Whitney sum $E \oplus F$ is trivial. Therefore $\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$ is a free $C^\infty(M)$ module and we see that $\Gamma(E)$ is a direct summand of a free module. In summary, $\Gamma(E)$ is a finitely generated projective $C^\infty(M)$-module.

After these remarks Theorem III.1.21 and Corollary III.1.15 provide an immediate corollary.

Corollary IV.2.2. *Let 0 be a regular value for $\Phi : M \to \mathbb{R}^k$ and let $E \xrightarrow{\pi} M$ be a smooth vector bundle over M. Then the Koszul complex $K(\Phi; \Gamma(E))$ gives an acyclic resolution of $\Gamma(E)/J\Gamma(E)$.*

We now turn to the geometric identification of $\Gamma(E)/J\Gamma(E)$. Let $i : M_o \hookrightarrow M$ denote the natural inclusion. Then if $E \xrightarrow{\pi} M$ is a vector bundle over M denote by $i^{-1}E \longrightarrow M_o$ the pull-back bundle via i. It will follow from the following theorem that $\Gamma(E)/J\Gamma(E)$ is isomorphic to $\Gamma(i^{-1}E)$. But first we need some remarks of a more general nature. Let $\psi : \widetilde{M} \to M$ be a smooth map between differentiable manifolds. It induces a ring homomorphism

$$\psi^* : C^\infty(M) \to C^\infty(\widetilde{M}) \tag{IV.2.3}$$

defined by $\psi^* f = f \circ \psi$ for $f \in C^\infty(M)$. This makes any $C^\infty(\widetilde{M})$-module (in particular $C^\infty(\widetilde{M})$ itself) into a $C^\infty(M)$-module, by **restriction of scalars**; that is, multiplication by $C^\infty(M)$ is effected by precomposing multiplication by $C^\infty(\widetilde{M})$ with ψ^*.

Now let $\widetilde{E} \xrightarrow{\widetilde{\pi}} \widetilde{M}$ and $E \xrightarrow{\pi} M$ be vector bundles of the same rank with the property that there is a bundle map given by the following commutative diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\varphi} & E \\ \Big\downarrow \widetilde{\pi} & & \Big\downarrow \pi \\ \widetilde{M} & \xrightarrow{\psi} & M \end{array} \tag{IV.2.4}$$

($i.e.$, φ is smooth fiber-preserving) with the property that φ restricts to a linear isomorphism on the fibers. Then we may form the following $C^\infty(M)$-module

$$C^\infty(\widetilde{M}) \otimes_{C^\infty(M)} \Gamma(E) \tag{IV.2.5}$$

which can be made into a $C^\infty(\widetilde{M})$-module by **extension of scalars**: left multiplication by $C^\infty(\widetilde{M})$. Define a map $\varphi^\sharp : \Gamma(E) \to \Gamma(\widetilde{E})$ by

$$(\varphi^\sharp \sigma)(\widetilde{m}) = (\varphi_{\widetilde{m}})^{-1} \left[\sigma(\psi(\widetilde{m}))\right] \ , \tag{IV.2.6}$$

for all $\widetilde{m} \in \widetilde{M}$ and $\sigma \in \Gamma(E)$. Then the following can be easily proven[79]

Theorem IV.2.7. *With the above notation, there exists an isomorphism of $C^\infty(\widetilde{M})$ modules*

$$C^\infty(\widetilde{M}) \otimes \Gamma(E) \xrightarrow{\cong} \Gamma(\widetilde{E}) \ , \tag{IV.2.8}$$

defined by $f \otimes \sigma \mapsto f \cdot \varphi^\sharp\, \sigma$ and where the tensor product is over $C^\infty(M)$. ∎

In the case we are interested in we have the following commutative bundle diagram

$$\begin{array}{ccc} i^{-1}E & \xrightarrow{j} & E \\ \Big\downarrow \pi_o & & \Big\downarrow \pi \\ M_o & \xrightarrow{i} & M \end{array} \tag{IV.2.9}$$

By Theorem IV.2.7, we have that

$$\Gamma(i^{-1}E) \cong C^\infty(M_o) \otimes_{C^\infty(M)} \Gamma(E) \ . \tag{IV.2.10}$$

But $C^\infty(M_o) \cong C^\infty(M)/J$, whence

$$\begin{aligned} \Gamma(i^{-1}E) &\cong C^\infty(M)/J \otimes_{C^\infty(M)} \Gamma(E) \\ &\cong \Gamma(E)/J\Gamma(E) \ , \end{aligned} \tag{IV.2.11}$$

where the last isomorphism is standard.

Therefore we conclude this subsection with the following important corollary:

Corollary IV.2.12. *If* $\mathbf{0}$ *is a regular value of* $\Phi : M \to \mathbb{R}^k$. *Then the Koszul complex* $K(\Phi; \Gamma(E))$ *gives an acyclic resolution of the module of smooth sections of the pullback of the bundle* $E \xrightarrow{\pi} M$ *via the natural inclusion* $i : \Phi^{-1}(\mathbf{0}) \to M$.

Proof: This is a direct consequence of Corollary IV.2.2 and the isomorphism of (IV.2.11). ■

Vertical Cohomology with Coefficients

Now suppose that $E \longrightarrow M$ is a vector bundle over M whose smooth sections $\Gamma(E)$ afford a representation of the Lie algebra structure of $C^\infty(M)$ given by the Poisson brackets. That is, we have an action

$$\begin{aligned} C^\infty(M) \times \Gamma(E) &\to \Gamma(E) \\ (f, \sigma) &\mapsto f \times \sigma \end{aligned} \tag{IV.2.13}$$

such that for all $f, g \in C^\infty(M)$ and $\sigma \in \Gamma(E)$

$$f \times (g \times \sigma) - g \times (f \times \sigma) = \big\{ f \, , \, g \big\} \times \sigma \ . \tag{IV.2.14}$$

Furthermore we demand that this action be a derivation with respect to the usual action of $C^\infty(M)$ on $\Gamma(E)$ given by pointwise multiplication. That is, for all $f, g \in C^\infty(M)$ and $\sigma \in \Gamma(E)$

$$f \times (g\,\sigma) = \big\{ f \, , \, g \big\}\, \sigma + g\,(f \times \sigma) \ \ . \tag{IV.2.15}$$

For example, if E admits a flat connection ∇ then we can define

$$f \times \sigma \equiv \nabla_{X_f} \sigma \ , \tag{IV.2.16}$$

where X_f is the hamiltonian vector field associated to f. The fact that ∇ is flat implies that

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{\big[X \, , \, Y\big]} \ , \tag{IV.2.17}$$

and hence it gives a representation. Similarly if we have a notion of Lie derivative on $\Gamma(E)$, we can define

$$f \times \sigma \equiv \mathcal{L}_{X_f} \sigma \ . \tag{IV.2.18}$$

Let $E_o \longrightarrow M_o$ denote the pullback of E via the inclusion $i : M_o \hookrightarrow M$. We can now define **vertical cohomology with coefficients** in E_o as follows. Define the E_o-valued vertical forms $\Omega_V(E_o)$ by

$$\Omega_V(E_o) \equiv \Omega_V(M_o) \otimes_{C^\infty(M_o)} \Gamma(E_o) \ . \qquad \text{(IV.2.19)}$$

We define ∇_V by

$$\nabla_V \, \sigma = \sum_i (\phi_i \times \sigma) \, c^i \qquad \text{(IV.2.20)}$$

and

$$\nabla_V \, c^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i \, c^j \wedge c^k \ , \qquad \text{(IV.2.21)}$$

for all $\sigma \in \Gamma(E_o)$. We then extend it to all of $\Omega_V(E_o)$ as a derivation. Just as for d_V, it is easy to verify that $\nabla_V^2 = 0$. We denote its cohomology by $H_V(E_o)$. Moreover notice that for all E_o-valued vertical forms θ

$$\nabla_V \, (f \, \theta) = d_V \, f \wedge \theta + f \, \nabla_V \, \theta \ . \qquad \text{(IV.2.22)}$$

Therefore, $H_V^0(E_o)$ becomes a $C^\infty(\widetilde{M})$-module (under pointwise multiplication) after the identification of $C^\infty(\widetilde{M})$ with $H_V^0(M_o)$. To see this notice that if $d_V \, f = 0$ and $\nabla_V \, \sigma = 0$, for some $\sigma \in \Gamma(E_o)$, $\nabla_V \, (f \, \sigma) = 0$. Moreover, it is easy to verify that this module is finitely generated and projective. Hence, by general arguments[79], it is the module of sections of some vector bundle over $\widetilde{M}$.

Now using the Koszul resolution for $\Gamma(E_o)$ given by Corollary IV.2.12 we would like to lift ∇_V to a differential D_∇ on the complex $K(E) \equiv \bigoplus_{c,b} K^{c,b}(E)$ where

$$K^{c,b}(E) \equiv \bigwedge^c \mathbb{V}^* \otimes \bigwedge^b \mathbb{V} \otimes \Gamma(E) \ . \qquad \text{(IV.2.23)}$$

This follows virtually identical steps as for the case of functions. We define

$$\nabla_0 \equiv (-1)^c \delta_K \otimes \mathbf{1} \qquad \text{on} \quad K^{c,b}(E) \ . \qquad \text{(IV.2.24)}$$

Then, just as before, we define ∇_1 by

$$\nabla_1 \, \sigma = \sum_i (\phi_i \times \sigma) \, c^i \ , \qquad \text{(IV.2.25)}$$

$$\nabla_1 \, c^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i \, c^j \wedge c^k \ , \qquad \text{(IV.2.26)}$$

and

$$\nabla_1 \, b_i = \sum_{j,k} f_{ji}{}^k \, c^j \wedge b_k \ . \tag{IV.2.27}$$

In this way $\nabla_0^2 = 0$ and $\{\nabla_0 \, , \, \nabla_1\} = 0$. Therefore filtering $K(E)$ in the same way as we filtered K and following isomorphic arguments to those leading to Theorem III.2.14 we prove the following theorem.

Theorem IV.2.28. *We can define a derivation $D_\nabla = \sum_{i=0}^k \nabla_i$ on $K(E)$, where ∇_i are derivations of bidegree $(i, i-1)$, such that $D_\nabla^2 = 0$.*

Another spectral sequence argument isomorphic to the one yielding Theorem III.2.18 allows us to compute the cohomology of D_∇.

Theorem IV.2.29. *The cohomology of D_∇ is given by*

$$\boxed{H^n_{D_\nabla} \cong \begin{cases} 0 & \text{for } n < 0 \\ H^n_V(E_o) & \text{for } n \geq 0 \end{cases}} \ . \tag{IV.2.30}$$

<u>Poisson Modules</u>

Let P be a Poisson superalgebra and $M = M_0 \oplus M_1$ a $\mathbb{Z}_2$-graded vector space. We will call M a P-module, or a **Poisson module** over P, if there exist two bilinear operations preserving the grading

$$P \times M \to M$$
$$(a, m) \mapsto a \cdot m$$

and

$$P \times M \to M$$
$$(a, m) \mapsto a \times m$$

obeying the following properties

(M1) $\cdot$ makes M a module over the associative structure of P:

$$a \cdot (b \cdot m) = (ab) \cdot m \ ;$$

(M2) $\times$ makes M into a module over the Lie superalgebra structure of P:

$$a \times (b \times m) - (-1)^{|a||b|} \, b \times (a \times m) = [a \, , \, b] \times m \ ;$$

(M3) $a \times (b \cdot m) = [a \, , \, b] \cdot m + (-1)^{|a||b|} \, b \cdot (a \times m)$; where $a, b \in P$ and $m \in M$.

In particular a Poisson algebra becomes a Poisson module over itself after identifying $a \cdot b$ with ab and $a \times b$ with $\left[a\,,\,b\right]$. Notice that equations (IV.2.14) and (IV.2.15) imply that $\Gamma(E)$ becomes a Poisson module over $C^\infty(M)$.

Just like the tensor product of two Poisson superalgebras can be made into a Poisson superalgebra, if M and N are Poisson modules over P and Q, respectively, their tensor product $M \otimes N$ becomes a $P \otimes Q$-module under the following operations:

$$(a \otimes u) \cdot (m \otimes n) = (-1)^{|u||m|}\,(a \cdot m) \otimes (u \cdot n) \tag{IV.2.31}$$

$$a \otimes u \times m \otimes n = (-1)^{|u||m|}\,\left(a \times m \otimes u \cdot n + a \cdot m \otimes u \times n\right)\ , \tag{IV.2.32}$$

for all $a \in P$, $u \in Q$, $m \in M$, and $n \in N$. Again the reader is invited to verify that with these definitions (M1)-(M3) are satisfied.

Since $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ is a Poisson module over itself and $\Gamma(E)$, for $E \longrightarrow M$ the kind of vector bundle discussed in the previous subsection, is a Poisson module over $C^\infty(M)$, their tensor product $K(E)$ becomes a Poisson module over K.

Just as we defined graded Poisson superalgebras, we can also define the analogous concept of a **graded Poisson module** over a graded Poisson superalgebra P in the obvious way: $M = \bigoplus_n M^n$, where $M_0 = \bigoplus_n M^{2n}$ and $M_1 = \bigoplus_n M^{2n+1}$, and both actions of P on M respect the grading. Therefore $K(E)$ becomes a graded Poisson module over K.

Now suppose that $D = \left[Q\,,\,\cdot\right]$ is an inner derivation of degree 1 on P, where $\left[Q\,,\,Q\right] = 0$. If let M be a P-module and define the endomorphism $\mathbb{D} : M \to M$ by

$$\mathbb{D}\,m = Q \times m \qquad \forall m \in M\ , \tag{IV.2.33}$$

it follows from (M2) and the fact that $\left[Q\,,\,Q\right] = 0$ that $\mathbb{D}^2 = 0$. Furthermore, using (M1)-(M3), it follows that its cohomology, $H_{\mathbb{D}}$, inherits the structure of a graded Poisson module over H_D. In particular, $H^0_{\mathbb{D}}$ is a Poisson module over H^0_D.

Poisson Structure of BRST Prequantization

Consider an arbitrary vector bundle $E \longrightarrow M$, whose smooth sections $\Gamma(E)$ form a Poisson module over $C^\infty(M)$. Then we can define $\mathbb{D}$ by (IV.2.33). Then $\mathbb{D}$ decomposes into $\mathbb{D} = \sum_i \nabla_i$, where $\nabla_i : K^{c,b} \to K^{c+i,b+i-1}$. We can recover the ∇_i from the Q_i by picking the contribution with the right bidegree. Using the explicit expressions for Q_0 and Q_1 it is easy to verify that ∇_0 and ∇_1 defined in this way agree precisely with the ones in (IV.2.24), (IV.2.25), (IV.2.26), and (IV.2.27).

This being enough to determine its cohomology, we deduce that the cohomology of $\mathbb{D}$ and that of the operator D_∇ of Theorem IV.2.28 are isomorphic. That is,

Theorem IV.2.34. *The cohomology of* $\mathbb{D}$ *is given by*

$$\boxed{H^n_{\mathbb{D}} \cong \begin{cases} 0 & \text{for } n < 0 \\ H^n_V(E_o) & \text{for } n \geq 0 \end{cases}} \ . \tag{IV.2.35}$$

We now proceed to see how the prequantum data gets induced à la BRST. First of all notice that the sections $\Gamma(E)$ of the prequantum line bundle form a Poisson module over $C^\infty(M)$. Therefore the discussion of the previous subsections goes through unaltered. That is, we construct the graded complex $K(E) = \bigoplus_n K^n(E)$ and the derivation

$$D_\nabla : K^n(E) \to K^{n+1}(E) \ , \tag{IV.2.36}$$

given by $\sigma \mapsto Q \times \sigma$, where $Q \in K^1$ is the element such that the classical BRST operator D is given by $\left[Q\, , \cdot\right]$. We then define $\widetilde{E} \longrightarrow \widetilde{M}$ by defining its space of smooth sections $\Gamma(\widetilde{E})$ as $H^0_{D_\nabla}$. It then becomes a $H^0_D \cong C^\infty(\widetilde{M})$ module as shown in the previous subsection. Since the prequantization map is precisely one of the maps incorporated in the Poisson module structure of $\Gamma(\widetilde{E})$ we have induced all the prequantum data except for the inner product. We will be able to induce a pointwise inner product but not an inner product. We will comment on the reasons why later on.

In order to induce a pointwise inner product on $H^0_{D_\nabla}$ it will be, first of all, necessary to define a pointwise inner product on $K(E)$. To motivate this construction let us first understand in Poisson terms the invariance of the pointwise inner product of two invariant sections. This invariance follows from the following fact. Since $\langle , \rangle$ is $\mathbb{R}$-linear in both slots it induces a map

$$\langle , \rangle : \Gamma(E) \otimes \Gamma(E) \to C^\infty_{\mathbb{C}}(M) \ ,$$

which is a $C^\infty(M)$-module homomorphism (a homomorphism of Poisson modules over $C^\infty(M)$). That is, if $\sigma, \tau \in \Gamma(E)$ and $f \in C^\infty(M)$ then

$$\left[f\, , \langle \sigma, \tau \rangle\right] = \langle f \times \sigma, \tau \rangle + \langle \sigma, f \times \tau \rangle \ . \tag{IV.2.37}$$

We would now like to extend $\langle , \rangle$ to a K-module homomorphism

$$\langle\!\langle , \rangle\!\rangle : K(E) \otimes K(E) \to K_{\mathbb{C}} \ . \tag{IV.2.38}$$

This boils down to essentially defining a linear map

$$\langle , \rangle : \bigwedge(\mathbb{V} \oplus \mathbb{V}^*) \otimes \bigwedge(\mathbb{V} \oplus \mathbb{V}^*) \to \bigwedge(\mathbb{V} \oplus \mathbb{V}^*) \ , \tag{IV.2.39}$$

satisfying, for all $\phi, \omega, \theta \in \bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$, the following relations

$$\langle \phi \wedge \omega, \theta \rangle = \phi \wedge \langle \omega, \theta \rangle = (-1)^{|\phi||\omega|} \langle \omega, \phi \wedge \theta \rangle \ , \tag{IV.2.40}$$

and

$$\big[\phi\,,\,\langle \omega, \theta \rangle\big] = \langle \big[\phi\,,\,\omega\big], \theta \rangle + (-1)^{|\phi||\omega|} \langle \omega, \big[\phi\,,\,\theta\big] \rangle \ . \tag{IV.2.41}$$

There is one obvious candidate:

$$\langle \omega, \theta \rangle = \omega \wedge \theta \ . \tag{IV.2.42}$$

However, although this pointwise inner product will turn out to play an important rôle when we discuss duality, it does not seem to be the natural pointwise inner product on $\Gamma(\widetilde{E})$. There are other variants of this inner product, differing from it in a degree dependent sign, which eliminate some of the $\pm$'s which will appear when we discuss duality. However these signs are not very relevant and for simplicity we will stick with this pointwise inner product.

At any rate, with this choice we have constructed a sesquilinear map

$$\langle\!\langle , \rangle\!\rangle : K(E) \times K(E) \to K_{\mathbb{C}} \ , \tag{IV.2.43}$$

which is invariant under the action of K. It is then clear that, if $Z(E)$ and $B(E)$ stand for the D_∇ cocycles and coboundaries respectively and Z and B stand for the D cocycles and coboundaries respectively, the mapping $\langle\!\langle , \rangle\!\rangle$ obeys

$$Z(E) \times Z(E) \to Z \ , \tag{IV.2.44}$$

$$Z(E) \times B(E) \to B \ , \tag{IV.2.45}$$

$$B(E) \times Z(E) \to Z \ ; \tag{IV.2.46}$$

from where it follows that it induces a well defined map in cohomology. In particular, since it is graded, it induces a map

$$\widetilde{\langle,\rangle} : H^0_{D_\nabla} \times H^0_{D_\nabla} \to H^0_D \otimes \mathbb{C} \ , \tag{IV.2.47}$$

which, under the relevant identifications, becomes a pointwise inner product

$$\widetilde{\langle,\rangle} : \Gamma(\widetilde{E}) \times \Gamma(\widetilde{E}) \to C^\infty_{\mathbb{C}}(\widetilde{M}) \ . \tag{IV.2.48}$$

This is the best that can be done about the inner product under the present circumstances. There is no inner product on $K(E)$ which induces, by evaluating it on D_∇ cocycles, the prequantum inner product on $\widetilde{M}$. The reason is the following. The inner product consists of integrating the pointwise inner product with respect to the Liouville measure. It is impossible that one can evaluate the inner product of sections of the prequantum bundle on $\widetilde{M}$ by merely picking representative sections on M and evaluating the inner product there. The reason is that functions on $\widetilde{M}$ are represented by functions on M whose restriction to M_o are constant on the leaves of the null foliation; but M_o has Liouville measure zero in M and hence two functions which agree on M_o but which disagree at will away from M_o have different integrals. Therefore the inner product would not be independent of the representatives. By tensoring the sections of the prequantum line bundle with half-forms (*cf.* [**86**]) the BRST cohomology of this new complex yields objects whose pointwise inner product can be integrated on $\widetilde{M}$ but, again, the integral does not lift to M.

3. Polarizations

Suppose that (M,Ω) admits a hamiltonian action of a connected Lie group G. A polarization F is **G-invariant** if the induced action of G on $T^{\mathbb{C}}M$ preserves F. Since G is connected, F is G-invariant if and only if for all Killing vectors X,

$$\big[X \, , \, Y\big] \in \Gamma(F) \qquad \forall\, Y \in \Gamma(F) \ . \tag{IV.3.1}$$

In particular, this implies that

$$\big\{\phi_X \, , \, f\big\} \in A_F \qquad \forall\, f \in A_F \ , \tag{IV.3.2}$$

from where it follows that A_F is a $\mathfrak{g}$-submodule of $C^\infty_{\mathbb{C}}(M)$.

In **[88]** Guillemin & Sternberg showed that a positive definite G-invariant polarization on (M, Ω) induces canonically a positive definite polarization $\widetilde{F}$ on the reduced manifold $(\widetilde{M}, \widetilde{\Omega})$. Let F_o be the subbundle of $T^{\mathbb{C}}M_o$ given by $F_o = F \cap T^{\mathbb{C}}M_o$. The by G-invariance, F_o induces an involutive lagrangian subbundle of $T^{\mathbb{C}}\widetilde{M}$. Since F is G-invariant, the $\{\phi_X\}$ belong to N_F and hence the polarized sections become a $\mathfrak{g}$-module. The polarized sections $\Gamma_{\widetilde{F}}(\widetilde{M})$ are precisely the $\mathfrak{g}$-invariant polarized sections on M_o, that is, $\Gamma_{F_o}(M_o)^{\mathfrak{g}}$.

In **[81]** Kostant & Sternberg discussed BRST quantization for the case of a group action in the spirit that quantizing the ghost part $(\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}))$ and the matter part $(C^\infty(M))$ could be done independently. The quantization of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ relied on the observation that the Clifford algebra $\mathrm{Cl}(\mathfrak{g} \oplus \mathfrak{g}^*)$ is naturally filtered and the associated graded object to this filtration is precisely $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. That is, we have a canonical isomorphism of vector spaces $c : \mathrm{Cl}(\mathfrak{g}^* \oplus \mathfrak{g}) \to \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$, which is to be thought of as classical limit. For finite dimensional $\mathfrak{g}$, $\mathrm{Cl}(\mathfrak{g}^* \oplus \mathfrak{g})$ has, up to isomorphism, a unique irreducible module S and thus it can be identified with $\mathrm{End}\, S$. A quantization of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ was defined in **[81]** to be a map $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \to \mathrm{End}\, S$ induced from a map $q : \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}^* \oplus \mathfrak{g})$ which is an inverse of c. No natural inverse exists and the choice of q is easily seen to correspond to an ordering prescription.

A model for S can be constructed by choosing a maximal isotropic subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$, that is, a subspace $N \subset \mathfrak{g} \oplus \mathfrak{g}^*$ such that $N^\perp = N$, for $\perp$ the orthogonal complement with respect to the inner product on $\mathfrak{g} \oplus \mathfrak{g}^*$ induced by the dual pairing. For example one could take N to be $\mathfrak{g}$ or $\mathfrak{g}^*$. Having chosen a maximal isotropic subspace N , let P be a complementary maximal isotropic subspace, *i.e.*, $\mathfrak{g} \oplus \mathfrak{g}^* = N \oplus P$. Then a model for S is given by

$$S \cong \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})/I(N) \cong \bigwedge P \ , \tag{IV.3.3}$$

where $I(N)$ is the ideal generated by N.

The crucial observation is that $\bigwedge P$ is a maximal abelian subalgebra (under Poisson bracket) of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ and, in analogy with the case of a symplectic manifold, can be considered to be the polarized functions and, in this case, the polarized sections. Hence a choice of maximal isotropic subspace N is equivalent to a choice of polarization.

We will see that in many cases, *e.g.*, the case of a totally complex polarization, a particular polarization is imposed on $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. In this context, therefore, the quantization of the matter and the ghosts part go hand in hand.

<u>Invariant Polarizations</u>

In analogy with the group case, we define a polarization F in the symplectic manifold (M,Ω) to be **invariant** with respect to a set of first class constraints $\{\phi_i\}$ if and only if $\phi_i \in N_F$ for all i. In other words, if $\left[X_i\,,\,Y\right] \in \Gamma(F)$ for all $Y \in \Gamma(F)$. Let F be an invariant polarization. We then consider the subspace of $K(E)$ consisting of polarized sections:

$$\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)\ . \tag{IV.3.4}$$

The existence of invariant polarizations may impose stringent conditions on the form of the constraints. For example, suppose that we consider the canonical real polarization ker π_* in the phase space ker T^*N. Locally, with respect to a **Darboux chart**—*i.e.*, one in which the symplectic form takes the form (II.2.4)—ker π_* is spanned by $\{\frac{\partial}{\partial p_a}\}$. Let X_f be a hamiltonian vector field which leaves this polarization invariant; that is,

$$\left[X_f\,,\,\frac{\partial}{\partial p_a}\right] \in \ker\,\pi_*\ . \tag{IV.3.5}$$

Using (II.2.5) this translates into

$$\frac{\partial^2 f}{\partial p_a \partial p_b} = 0\ , \tag{IV.3.6}$$

whence f is linear in the momenta. Therefore constraints leaving invariant the canonical real polarization of the phase space are linear in the momenta. In particular this means that their Poisson flows induce well defined transformations on the configuration space. That is, locally, the constraints take the form $\phi_i(p,q) = p_a X_i^a(q)$, where the $X_i^a(q)$ are (up to a sign) the coefficients of the horizontal (*i.e.*, tangent to N) components of the hamiltonian vector fields associated to the constraints. If these define a subbundle of TN—*i.e.*, if the subspace $\langle X_i\rangle$ spanned by them has constant dimension on N—then the constrained submanifold $M_o \subset T^*N$ is the annihilator $\langle X_i\rangle^o$ of the distribution defined by the horizontal components of the X_i. In particular, M_o is a subbundle of T^*N over N. Another remarkable peculiarity of the constraints is that the structure functions are independent of the momenta and hence induce well defined functions on the configuration space. This has a very interesting repercussion in the BRST construction since the classical BRST charge only contains linear and trilinear terms in the (anti)ghosts. Let Q_0 and Q_1 be defined by equations (III.3.8) and (III.3.12) respectively. Using the notation of Section III.3, let $R_1 = Q_0 + Q_1$. Then R_1^2 lies completely in $K^{3,1}$. Normally we would expect a term in $K^{4,2}$ corresponding to the Poisson brackets of the structure functions, but since these are independent of the momenta, their Poisson bracket vanishes. It is easy to show that the Jacobi identities of the Poisson brackets of the constraints makes R_1^2 into a Koszul cocycle and hence, by Corollary III.1.15, it is a Koszul coboundary. Therefore it is automatically zero when the constraints are imposed. In particular, since the constraints are linear in the momenta, R_1^2 is zero when the momenta are zero. But R_1^2 is independent of the momenta, hence it is zero automatically. Therefore $Q = Q_0 + Q_1$.

Similarly, suppose that a set of first class constraints $\{\phi_i\}$ leaves invariant a totally complex polarization. Since the manifold is pseudo-Kähler, around each point there is holomorphic chart such that the symplectic form has the form (IV.1.18) and such that the polarization is spanned by the holomorphic vector fields $\{\frac{\partial}{\partial z^a}\}$. If X_f is the hamiltonian vector field associated to a real function f, then the condition that it stabilizes the polarization translates into

$$\frac{\partial^2 f}{\partial z^a \partial z^b} = 0 \ , \tag{IV.3.7}$$

which, together with its complex conjugate, imply that f is at most linear in z^a and $\overline{z}^a$. That is, the constraints $\{\phi_i\}$ look like

$$\phi_i(z, \overline{z}) = a_i + b_{ia} z^a + \overline{b}_{ia} \overline{z}^a + c_{iab} z^a \overline{z}^b \ , \tag{IV.3.8}$$

where a_i, b_{ia}, $\overline{b}_{ia}$, and c_{iab} are constants which due to the reality of the constraints obey $a_i \in \mathbb{R}$, b_{ia} and $\overline{b}_{ia}$ are complex conjugates, and $c_{iab} = \overline{c_{iba}}$. In particular since they are locally given by polynomials, their zero locus is locally an algebraic subvariety of the image of the chart. Moreover, since we know that they close under Poisson brackets, this imposes strong conditions on the structure functions. Naïvely it may appear as though they would have to be constant but I am unable to prove it.

Polarized BRST Operator

We would like to consider the prequantum BRST operator D_∇ acting on this subspace and define the physical Hilbert space as its cohomology (at least for ghost number zero). However the BRST operator may not leave this subspace invariant. For example, let $b_k \otimes \sigma$ be an element in $\mathbb{V} \otimes \Gamma_F(E)$. Then

$$D_\nabla(b_k \otimes \sigma) = \sum_i Q_i \times (b_k \otimes \sigma) \ . \tag{IV.3.9}$$

If this is to belong to $\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)$ each piece with a different bidegree must belong to $\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)$. In particular the $(0,0)$ piece must be a polarized section. The $(0,0)$ piece is just $\phi_k \, \sigma$, which is a polarized section if and only if $\phi_k \in A_F$.

If the polarization is totally complex A_F corresponds to holomorphic functions. Since ϕ_k is a real non-constant function it cannot be holomorphic. Hence $\phi_k\,\sigma$ is not a polarized section and the BRST operator does not stabilize $\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)$. If we divide out by the ideal generated by $\mathbb{V}$ then this term is not present. So we are forced to polarize the ghosts in such a way that only $\bigwedge\mathbb{V}^*$ appears. We thus define the complex $K_F(E)$ by

$$K_F(E) = \bigwedge\mathbb{V}^* \otimes \Gamma_F(E)\ . \tag{IV.3.10}$$

It is then clear that the only part of D_∇ which survives is ∇_1 since it has b-degree zero. It clearly obeys $\nabla_1^2 = 0$, since it was only $\mathbb{V}$ which gave us a hard time at that.

The quantum BRST cohomology—for the case of a totally complex polarization—is therefore the cohomology of the complex

$$\cdots \longrightarrow \bigwedge^c\mathbb{V}^* \otimes \Gamma_F(E) \xrightarrow{\nabla_1} \bigwedge^{c+1}\mathbb{V}^* \otimes \Gamma_F(E) \longrightarrow \cdots\ . \tag{IV.3.11}$$

In the case of a group action ∇_1 agrees with the Lie algebra coboundary operator and hence the quantum BRST cohomology is precisely the Lie algebra cohomology with coefficients in the module of polarized sections: $H(\mathfrak{g};\Gamma_F(E))$.

There in one striking fact about this result. No part of the quantum BRST operator carries the information concerning the restriction to M_o; since, in fact, the only part with this information was the Koszul differential ∇_0. A heuristic explanation can be given as follows. In the case of a totally complex polarization, $\Gamma_F(E)$ are holomorphic sections. By the uniqueness of analytic continuation, a holomorphic section on M in completely determined once you know it on a real submanifold M_o; in other words, there is a unique extension to a holomorphic section on M.

Another instance when a polarization is imposed on the ghost part is in the case a semisimple group action and F is a polarization for which A_F is a maximal abelian subalgebra of $C^\infty_{\mathbb{C}}(M)$. Let $\mathfrak{h}$ denote the subalgebra of $\mathfrak{g}$ defined by

$$X \in \mathfrak{h} \iff \phi_X \in A_F\ . \tag{IV.3.12}$$

Notice that $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ since the map $X \mapsto \phi_X$ is a Lie algebra morphism and A_F is abelian. But moreover $\mathfrak{h}$ is an ideal. To see this notice that if $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$ then for all $f \in A_F$

$$\begin{aligned}\left\{\phi_{[X\,,Y]}\,,f\right\} &= \left\{\left\{\phi_X\,,\phi_Y\right\},f\right\}\\ &= \left\{\phi_X\,,\left\{\phi_Y\,,f\right\}\right\} + \left\{\left\{\phi_X\,,f\right\},\phi_Y\right\}\ .\end{aligned} \tag{IV.3.13}$$

Since F is invariant $\left\{\phi_Y\,,f\right\} \in A_F$ and both terms are Poisson brackets of elements in

A_F and because A_F is abelian they vanish. Since $\phi_{\left[X\,,\,Y\right]}$ commutes with A_F, it must, by maximality, be in A_F itself. Therefore $\left[X\,,\,Y\right] \in \mathfrak{h}$ and hence it is an ideal. If $\mathfrak{g}$ is semisimple it cannot have any abelian ideals and thus $\mathfrak{h}$ is zero. Therefore no ϕ_X belongs to A_F and, once again, D_∇ does not stabilize $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes \Gamma_F(E)$ unless we polarize the ghost part in such a way that only $\bigwedge\mathfrak{g}^* \otimes \Gamma_F(E)$ is left.

In the case of a general polarization it is not clear that a specific polarization of the ghost part is forced, but in any case we may always choose to polarize the ghosts in such a way that the resulting quantum BRST complex is (IV.3.11). With this choice of polarization the explicit form of the ∇_i for $i \neq 1$ is irrelevant. This makes the quantization much simpler than what one might have originally expected. Therefore we choose to work with this polarization unless otherwise stated.

We end this section with a brief comment about the inner product. We remarked in the previous section how it seems impossible to calculate the inner product on $\widetilde{M}$ by evaluating the inner product on M on suitable representatives, due to the geometric nature of the inner product. It may seem, at least with the chosen polarization, that this problem could be obviated since holomorphic sections are uniquely characterized by their behavior on M_o. Still, if the gauge orbits are not compact the integrable polarized sections on M and $\widetilde{M}$ may not correspond. Of course, one choose a quantization prescription in which one always compute everything on M, but it is then not clear how to show that this quantum theory agrees with the theory in $\widetilde{M}$.

There are other problems surrounding the inner product. Ideally, in the purest BRST philosophy, we would like to be able to compute everything pertaining to objects in $\widetilde{M}$ by choosing representative objects in M (cocycles of the relevant BRST cohomology) and computing with them in M. This requires certain compatibility conditions, namely that the cohomology inherits the relevant structure from the cochains. In the case of the polarized sections, we would like the inner product to be induced and we see that this does not occur in general, because of the problem with the integration. But even without the integral, we have the problem of defining a reasonable pointwise inner product. In the last section we gave one which descends to a pointwise inner product in cohomology, but it has a major drawback. The norm of the polarized sections (elements in the zeroth BRST cohomology) is always zero, since the inner product pairs cocycles of complementary ghost number. We have not been able to find a pointwise inner product which descends to cohomology and which is non-zero when restricted to just cocycles of zero ghost number.

4. Duality in Quantum BRST Cohomology

Duality theorems in BRST cohomology have recently attracted some attention. We saw in Section III.4 that Poincaré duality on the gauge orbits induced a duality in the classical BRST cohomology. We will see moreover in Chapter V that, provided that a mild finiteness condition holds, there is quite generally a duality for the quantum BRST cohomology in the context of a Fock space representation. The results do not depend on the Fock space context but we do assume the existence of a non-degenerate inner product on the space of states with respect to which the BRST operator is (anti-)hermitian. This assumption, we believe, is well founded on physical grounds. From this duality we then prove also a duality on operators. In **[95]**, Marc Henneaux obtained similar duality theorems but with weaker assumptions. In particular he dispensed with the notion of an inner product to prove the operator duality; but in order to turn this around and obtain a duality for the states he needs a non-degenerate inner product. The finiteness assumption, moreover, is still key in his proofs.

There is a peculiar difference between the classical and quantum versions of the duality in BRST cohomology. In the quantum case there are states with both negative and positive ghost numbers and the duality is a symmetry between the cohomologies at $+n$ and $-n$ ghost numbers:

$$H^n_{\text{quantum}} \cong H^{-n}_{\text{quantum}} \ ; \tag{IV.4.1}$$

whereas in the classical case—due to the vanishing of BRST cohomology for negative ghost number—the duality is a symmetry between complementary dimensions

$$H^n_{\text{classical}} \cong H^{k-n}_{\text{classical}} \ . \tag{IV.4.2}$$

We also saw in the last section how the quantum BRST cohomology (at least for our choice of polarization) had no states of negative ghost number. Hence it seems at first sight that we will not be able to obtain a duality similar to that in (IV.4.1). The resolution of this apparent paradox is that the classical and quantum ghost numbers don't quite correspond. We will see that for the ghost number operator to be antihermitian—a fact we will assume in Chapter V and which seems to occur in "nature"—we must shift the ghost number by $-\frac{k}{2}$, *i.e.*, we must normal order. In this case and assuming that the polarization is positive definite (notice that this already implies that it is totally complex) and modulo some conditions on the constraints we will recover an isomorphism à la (IV.4.1). However shifting the ghost number operator by a constant affects the ghost number of the states but not the one of the operators. In particular there will be operators which are BRST invariant and which have negative ghost number. These do not come from quantizing BRST invariant observables, since classical BRST cohomology is zero for negative ghost number. In fact, as remarked in **[95]**, they seem to come from distributions rather than from smooth functions.

From now on we assume that the polarization is positive definite. As remarked in the last section this forces a polarization on the ghost part in such a way that only the ∇_1 part of the BRST operator remains. The idea behind the proof of the duality is very simple and it is similar in spirit to that in Chapter V. We will define a positive definite inner product on the complex $K_F(E)$. This will allow us to define a formal adjoint of the BRST operator and a BRST laplacian. Then we will prove a Hodge decomposition for the BRST complex and from this the duality will follow.

We now define the inner product. The inner product splits as a product of two inner products: the one on the "matter" part and the one on the "ghost" part. We will choose the one on the matter part to be the prequantum inner product: the integral of the pointwise inner product with respect to the Liouville measure. For the ghost part we will take the following. Fix a non-zero top form $\varepsilon \in \bigwedge^k \mathbb{V}^*$. Since $\bigwedge^k \mathbb{V}^*$ is one dimensional any other k-form is proportional to ε. We define the Berezin integral on $\bigwedge \mathbb{V}^*$ by

$$\int\limits_{\text{Ber}} \omega = 0 \ , \qquad \forall\, \omega \in \textstyle\bigwedge^{p\neq k} \mathbb{V}^* \tag{IV.4.3}$$

$$\int\limits_{\text{Ber}} \lambda\varepsilon = \lambda \ , \qquad \forall\, \lambda \in \mathbb{R} \ . \tag{IV.4.4}$$

We then define the inner product of forms by

$$\langle \omega, \theta \rangle = (-i)^{p(k-p)} \int\limits_{\text{Ber}} \omega \wedge \theta \ , \tag{IV.4.5}$$

for $\omega \in \bigwedge^p \mathbb{V}^*$ and $i = \sqrt{-1}$. The reason for the factors of i is the hermiticity of the inner product on $K_F(E)$:

$$(\omega \otimes \sigma, \theta \otimes \tau) = (-i)^{p(k-p)} \int_M \langle \sigma, \tau \rangle \, d\mu_L \cdot \int\limits_{\text{Ber}} \omega \wedge \theta \ , \tag{IV.4.6}$$

for $\omega \in \bigwedge^p \mathbb{V}^*$. Notice that this is zero unless $\theta \in \bigwedge^{k-p} \mathbb{V}^*$. Let us define an element of $K_F(E)$ to be **integrable** if it belongs to $\bigwedge \mathbb{V}^* \otimes \Gamma_F(E)_{\text{fin}}$, where $\Gamma_F(E)_{\text{fin}}$ denotes the subspace of the polarized sections of finite norm.

In order for the ghost number operator to be skew-adjoint we will have to redefine it by shifting it by an appropriate constant. The ghost number operator is given by $N = \sum_{i=1}^{k} \varepsilon(c^i)\imath(b_i)$, where ε and $\imath$ are the exterior and interior product operations on $\bigwedge \mathbb{V}^*$. In particular notice that $\imath(b_i)$ is a derivation which can be "integrated by parts" in the Berezin integral. We shift N by a constant c and notice that, if $N_c \equiv N - c$, then for all $\omega \otimes \sigma, \theta \otimes \tau \in K_F(E)$ integrable, it is easy to show that

$$(N_c\, \omega \otimes \sigma, \theta \otimes \tau) = (k - 2c)(\omega \otimes \sigma, \theta \otimes \tau) - (\omega \otimes \sigma, N_c\, \theta \otimes \tau) \ ; \tag{IV.4.7}$$

from where it follows that for $c = \frac{k}{2}$ we obtain a skew-adjoint ghost number operator.

The BRST operator for this choice of polarization is given by the ∇_1 piece:

$$\sum_i \varepsilon(c^i)\, O(\phi_i) - \frac{1}{2} \sum_{i,j,k} f_{ij}{}^k \varepsilon(c^i)\varepsilon(c^j)\imath(b_k) \ . \tag{IV.4.8}$$

Let $\omega \otimes \sigma \in K_F^p(E)$, $\theta \otimes \tau \in K_F(E)$ be integrable. Then after a straight-forward calculation we find

$$(\nabla_1\, \omega \otimes \sigma, \theta \otimes \tau) = \pm \left[(\omega \otimes \sigma, \nabla_1\, \theta \otimes \tau) + \sum_{i,j} (\omega \otimes \sigma, f_{ij}{}^j \varepsilon(c^i)\theta \otimes \tau) \right] \ , \tag{IV.4.9}$$

where the explicit dependence of $\pm$ on p, k is of no consequence. We therefore see that if $\sum_{i,j} f_{ij}{}^j = 0$ for all i, then $\nabla_1^\dagger = \pm\nabla_1$, and therefore an inner product is induced in cohomology. To see this notice that the only condition needed for the inner product to descend to cohomology is that the cobundaries be orthogonal to the cocycles, that is, $\ker\ \nabla_1 \subseteq \ker\ \nabla_1^\dagger$. For the case of a group action, the condition $\sum_{i,j} f_{ij}{}^j = 0$ is precisely that $\mathrm{tr}\ \mathrm{ad}\ (b_i) = 0$, that is, that $\mathfrak{g}$ be unimodular. This condition is equivalent to the existence of a bi-invariant metric on the group manifold. The condition that the inner product descends to cohomology is, of course, very desirable on physical grounds since gauge related states should be physically equivalent. In this case, however, this inner product is not the physical inner product, and we make this choice, not on physical grounds, but to be able to prove duality.

Therefore from now on we assume that for all i, $\sum_{i,j} f_{ij}{}^j = 0$. In this case, $\nabla_1^\dagger = \pm\nabla_1$ acting on integrable elements of $K_F(E)$. We now define a Hodge-type operator. For definiteness and with no loss in generality, let us fix the following volume form on $\bigwedge \mathbb{V}^*$:

$\varepsilon = c^1 \wedge \cdots \wedge c^k$. We define the operator $\star : \bigwedge^p \mathbb{V}^* \to \bigwedge^{k-p} \mathbb{V}^*$ as follows:

$$\star(1) = \varepsilon \ ; \tag{IV.4.10}$$

$$\star(\varepsilon) = 1 \ ; \tag{IV.4.11}$$

$$\star(c^{i_1} \wedge \cdots \wedge c^{i_p}) = \pm c^{i_{p+1}} \wedge \cdots \wedge c^{i_k} \ ; \tag{IV.4.12}$$

where $(i_1, \ldots, i_k)$ is a permutation of $(1, \ldots, k)$ and $\pm$ refers to the sign of the permutation. It is trivially verified that $\star^2 = (-1)^{p(k-p)}$ on $\bigwedge^p \mathbb{V}^*$. Let us then define $\overline{\star} = (i)^{p(k-p)} \star$ on $\bigwedge^p \mathbb{V}^*$. It clearly satisfies $\overline{\star}^2 = 1$. This allows to redefine the inner product $(,)$ in such a way that it is now positive definite. In fact, let's define a new inner product

$$(\Psi, \Xi)_\star \equiv (\Psi, \overline{\star} \Xi) \ , \tag{IV.4.13}$$

for $\Psi, \Xi \in K_F(E)$. If $\omega \otimes \sigma, \theta \otimes \tau \in K_F(E)$ the new inner product becomes

$$(\omega \otimes \sigma, \theta \otimes \tau)_\star = \int_M \langle \sigma, \tau \rangle \, d\mu_L \cdot \int_{\mathrm{Ber}} \omega \wedge \star \theta \ . \tag{IV.4.14}$$

It follows that this inner product is positive definite. To see this notice that the inner product on the sections is positive definite by construction. And to show that the other part is positive definite, we need only exhibit an orthonormal basis. Let $I = (i_1, \ldots, i_p)$ denote a sequence $1 \leq i_1 < \cdots < i_n \leq k$. Then if we define $c^I = c^{i_1} \wedge \cdots \wedge c^{i_p}$, the collection of all such $\{c^I\}$ forms a basis for $\bigwedge \mathbb{V}^*$. On this basis, it is easy to see that

$$\int_{\mathrm{Ber}} c^I \wedge \star c^J = \delta^{IJ} = \begin{cases} 1 & \text{for } I = J \\ 0 & \text{otherwise} \end{cases} \ . \tag{IV.4.15}$$

Moreover we see that the integrable elements of $K_F(E)$ are precisely those elements of $K_F(E)$ which have finite norm with respect to this new inner product. We let $\Gamma_{F,L^2}(E)$ denote the Hilbert space completion of this space with respect to this new inner product.

The BRST operator ∇_1 is generally unbounded so we have to be careful and specify its domain. From the definition of $\Gamma_{F,L^2}(E)$, it is clear that the BRST operator is densely defined, since it is defined on the integrable elements and the $\Gamma_{F,L^2}(E)$ is their closure.[14]

[14] We are assuming here that the constraints are such that the BRST operator stabilizes $\Gamma_F(E)_{\mathrm{fin}}$. This is always the case, for instance, if M admits a finite holomorphic atlas.

Now let ∇_1^* denote the adjoint of ∇_1 with respect to the new inner product. Notice that on $\bigwedge \mathbb{V}^* \otimes \Gamma_F(E)_{\text{fin}}$

$$\nabla_1^* = \pm \bar{\star} \nabla_1 \bar{\star} \ . \tag{IV.4.16}$$

In particular, notice that ∇_1^* is also densely defined. Therefore ∇_1 is a closeable operator with minimal closed extension $\overline{\nabla_1} \equiv (\nabla_1^*)^*$. We can therefore define the BRST operator to be the closure of ∇_1. To avoid cumbersome notation we will denote this also by ∇_1.

Since ∇_1^* is closed, ker ∇_1^* is a closed subspace of $\Gamma_{F,L^2}(E)$ and hence we have the following orthogonal decomposition

$$\Gamma_{F,L^2}(E) = \ker\, \nabla_1^* \oplus (\ker\, \nabla_1^*)^{\perp} \ . \tag{IV.4.17}$$

Now, since ∇_1 is closed, $(\ker\, \nabla_1^*)^{\perp}$ is precisely the closure of im ∇_1. Thus we have the following orthogonal decomposition

$$\Gamma_{F,L^2}(E) = \ker\, \nabla_1^* \oplus \overline{\text{im}\, \nabla_1} \ . \tag{IV.4.18}$$

Therefore we can decompose ker ∇_1 as an orthogonal direct sum

$$\ker\, \nabla_1 = \ker\, \nabla_1 \cap \ker\, \nabla_1^* \oplus \overline{\text{im}\, \nabla_1} \ . \tag{IV.4.19}$$

Therefore we have the following isomorphism

$$H_{\text{quantum BRST}} \equiv \ker\, \nabla_1 / \overline{\text{im}\, \nabla_1} \cong \ker\, \nabla_1 \cap \ker\, \nabla_1^* \ , \tag{IV.4.20}$$

where we have defined the BRST cohomology in this way for convenience.[15]

Moreover this isomorphism is an isomorphism of graded (by ghost number) vector spaces. Because of (IV.4.16), $\bar{\star}$ stabilizes ker $\nabla_1 \cap \ker\, \nabla_1^*$. Since it connects opposite ghost numbers and $\bar{\star}^2 = 1$, it is an isomorphism between the BRST cohomology spaces at opposite ghost number. Thus we have proven the following duality theorem.

[15] Since im ∇_1 is, in general, infinite dimensional it is not necessarily closed. Factoring out by a non-closed subspace results in the natural surjection not being continuous and in the factor object not being complete.

Theorem IV.4.21. *If the constraints satisfy the condition* $\sum_{i,j} f_{ij}{}^j = 0$ *for all* i*, there is an isomorphism*

$$\boxed{H^g_{\text{quantum BRST}} \cong H^{-g}_{\text{quantum BRST}}} \; . \tag{IV.4.22}$$

Notice that the problems of closure as well as the condition that the constraints stabilize the integrable subspace is unnecessary if M is assumed compact, since in this case the space of holomorphic sections is finite.

Chapter Five:

Quantum BRST Cohomology

In this chapter we explore the basic general properties of quantum BRST complexes. In searching for a working definition of a typical quantum BRST complex we have not been led by a desire to achieve utmost generality. Instead we have tried to come up with a definition which crystallizes the main features of the BRST complexes appearing in "nature." We have therefore chosen to work in a Fock space framework; although this is not strictly necessary and, in fact, the bulk of the results in this chapter (decomposition theorem, "Poincaré" duality, ...) go through if we drop from the definition of a BRST complex given in Section 1 the condition that $\mathcal{F}$ be a Fock space and we demand in its place that it be endowed with a non-degenerate scalar product.

The typical model of a BRST complex as defined in this chapter is the Fock space of a BRST quantized field theory in which there is no BRST anomaly. In fact, we never address the issue of "quantization" in this chapter and our starting assumption is the existence of the quantum BRST complex. It is well known that this requirement is non-trivial and that not all gauge theories can be successfully (BRST) quantized. The BRST anomaly can be analyzed using Lie algebra cohomology.[81] Roughly this cohomology class is the class associated to the central extension of the quantum constraint algebra defined by the Schwinger term in the commutator of the constraints. Hence it is independent to a large extent from the particular physical system one is analyzing but depends only on the algebraic structure of the constraint algebra. The quantizable physical systems are precisely those whose algebra of observables affords a projective representation of the opposite class. We are familiar with this in string theory where, for instance, in the open bosonic string, the energy momentum tensor of the reparametrization ghosts has a central term in its operator product expansion which only depends on the conformal character of the ghost fields themselves. This constraints the allowed matter fields by requiring that the central term in the operator product expansion of the combined energy momentum tensor vanishes.

Another possible model is the complex computing the semi-infinite cohomology of an infinite-dimensional Lie algebra $\mathfrak{g}$ with coefficients in a (Fock) module in the category $\mathcal{O}_o$

(see Chapter VI). This will be the main theme of the next (and last) three chapters and, in particular, of Chapter VI.

Since BRST quantization introduces unphysical degrees of freedom it is not guaranteed that the BRST quantized theory makes sense. Certain things have to be checked—the most obvious being unitarity. In fact, unitarity is so deeply linked to BRST quantization that in modern physics folklore BRST is invoked whenever one encounters an indefinite Fock space. There is, of course, good reason. Already in elementary systems, unitarity and gauge invariance (hence BRST) seem to be related. For example, consider the construction of the unitary irreducible representation of the Poincaré group associated to a massless particle of helicity ± 1 as fields on Minkowski spacetime. If one follows the Mackey algorithm to obtain this representation one finds out that it cannot be carried by the Lorentz vector A_μ as one would have naïvely suspeceted. In fact, the simplest (*i.e.*, lowest spin) field that can carry it is the field strength $F_{\mu\nu}$. What saves the day in the case of A_μ is the gauge invariance.

However even when it is our experience that a consistent BRST quantization leads to a unitary theory there are no general theorems and it does not seem to be guaranteed by the quantization procedure that the physical space inherits a positive definite inner product. This property, unfortunately misnamed the **no-ghost theorem** has to be checked for consistency. It turns out that when the BRST cohomology happens to be concentrated at zero ghost number (a fact known as the **vanishing of BRST cohomology**) one can effectively reduce the verification of the no-ghost theorem to a mere calculation.

The key concept (and one we touched upon in Section II.1) is that of a resolution. Since the physical states are defined as the cohomology of the BRST complex at zero ghost number, the vanishing of cohomology implies that the BRST complex provides a (Fock) resolution for the physical space. In particular, as we mentioned in Section II.1, we can "count" the physical states by computing the Euler character of the complex. By definition, the no-ghost theorem holds if and only if there are no physical states of negative norm. So if we could count the number of negative norm physical states in an economical way (*i.e.*, without constructing them) we could verify if there are any. Interestingly enough, in some cases (*e.g.*, if the BRST complex is a Fock space) it is quite easy to compute the signature of the complex, *i.e.*, the difference between the number of positive norm states and the number of negative norm states in the cohomology. The signature is clearly bounded from above by how many states there are in the cohomology. Comparing these two numbers and checking to see if the bound is saturated allows us to verify if the no-ghost theorem is true.

Of course, in practice these spaces are infinite-dimensional so the counting needs to be regularized. A simple way to do this is to decompose the BRST complex into direct sums (or direct integrals) of independent finite-dimensional subcomplexes indexed by the eigenvalues of some commuting family of self-adjoints. We then compute everything on each finite-dimensional piece and later collate the results into a formal power series. This motivates the second part of the definition of a BRST complex given in Section 1.

Whereas the Euler character is relatively trivial to compute (given the vanishing of cohomology) the signature is not. In fact, to my knowledge, it can be done only for Fock complexes. Still many interesting examples (in particular some to be found among two-dimensional conformal quantum field theories) fall into this category and these BRST complexes are very useful for the study of these theories.

One cannot overemphasize the rôle played by the vanishing of cohomology in the reformulation of the no-ghost theorem, as well as providing a (Fock) resolution for the physical space. It is natural to ask to what extent the vanishing of cohomology is not just a calculational convenience but whether it is somehow tied in to the consistency of the quantization. Although there is no full proof argument it certainly is the back of the mind of the people working in this field. In their seminal work on BRST for Yang-Mills theory[25], Kugo and Ojima already gave arguments for the necessity of the vanishing for consistency. However some of their arguments have holes and some seem to be very peculiar to local quantum field theory. In this chapter we also attempt to give arguments for the necessity of the vanishing for consistency but they are not full proof either. Recently, in some work of a more axiomatic nature, A. Schwarz[96] defines BRST complexes with the vanishing included. Also recent work by Felder[97] and even more recent work by Distler & Qiu[98], and Bouwgknet, McCarthy, & Pilch[99] constructing Fock resolutions for irreducible modules of chiral algebras construct "BRST" operators whose main property is the vanishing of their cohomology except at zero ghost number; whence it seems that it is this property which is taking a dominant rôle. It is my conviction that the question of the importance of the vanishing of BRST cohomology is fundamental for our complete understanding of the BRST formalism.

This chapter is organized as follows. In Section 1 we define the notion of a quantum BRST complex and motivate it. We then define a second complex where the BRST operator acts on operators. Section 2 introduces the fundamental tool in our analysis, which is the existence of a positive definite inner product in the Fock space. Of course, cohomology is an algebraic construction and as such does not depend on the particular inner product. However the extra structure makes it very easy to obtain interesting results. The inner product we consider has the added benefit that it coincides with the "true" inner product on the physical states if the theory is free of "ghosts". In fact, the no-ghost theorem is reformulated in precisely these terms. This was first done in **[33]** although see also

[34] for an explicit application to the open bosonic string. Using this inner product we then prove the analogue of the Hodge decomposition theorem for BRST cohomology. This decomposition is applied in Section 3 to characterize the operator cohomology in terms of the ordinary BRST cohomology. We also argue the desirability of the vanishing theorem for a consistent BRST quantization and that a vanishing theorem in BRST cohomology implies a vanishing theorem in operator cohomology and vice versa. Finally in Section 4 we use the decomposition theorem to reformulate the no-ghost theorem in this language, and under the assumption that a vanishing theorem holds for BRST cohomology we reduce the proof of the no-ghost theorem to a direct computation. We also relate the partition function of the theory to a character-valued index of a Dirac-type operator made out of the BRST operator.

1. General Properties of BRST Complexes

For the purposes of this chapter a (quantum) **BRST complex** is a triple $(\mathcal{F}, Q, N_{\mathrm{gh}})$ consisting of a Fock space $\mathcal{F}$ graded by the eigenspaces of a skew-self-adjoint operator N_{gh} (the **ghost number operator**) with integer eigenvalues

$$\mathcal{F} = \bigoplus_{g\in\mathbb{Z}} \mathcal{F}_g \ ; \tag{V.1.1}$$

and a self-adjoint operator Q (the **BRST operator**) which has degree 1 with respect to this grading and obeys $Q^2 = 0$.

Moreover we shall assume that there is a family $\{\Lambda\}$ of self-adjoint mutually commuting operators which in turn commute with Q and with N_{gh} and which provide a decomposition (as orthogonal direct sums or more generally as direct integrals) of $\mathcal{F}$ into finite dimensional subspaces. This assumption is fulfilled by most of the cases of recent physical interest: in the case of the bosonic string, for instance, this family consists of the center of mass momentum of the string and the level operator in the Hamiltonian. In the context of the semi-infinite cohomology of Feigin (see Chapter VI) this is precisely the restriction to Lie algebra modules in the category $\mathcal{O}_o$. For definiteness of notation we will assume that their spectrum is discrete, and let $\{\lambda\}$ denote their eigenvalues. We can thus write the decomposition of $\mathcal{F}$ as

$$\mathcal{F} = \bigoplus_{\lambda} \mathcal{F}(\lambda) \qquad \text{and} \qquad \dim \mathcal{F}(\lambda) < \infty \ . \tag{V.1.2}$$

It is clear that for the BRST operator to be non-trivial the norm of $\mathcal{F}$ must be indefinite. Otherwise for all vectors ψ in $\mathcal{F}$, $\|Q\psi\|^2 = \langle Q\psi,\, Q\psi\rangle = \langle\psi,\, Q^2\psi\rangle = 0$ and hence Q would be identically zero. Therefore $\mathcal{F}$ must have null vectors (*i.e.*, vectors of zero norm), but because the norm is non-degenerate it must also contain negative norm vectors. Hence indefinite Fock spaces are inherent to BRST quantization. We will also assume that the Fock vacuum is BRST invariant. Notice also that the antihermiticity of the ghost number operator implies that only inner products between vectors of opposite ghost numbers are non-zero.

Because $\mathcal{F}$ is a Fock space we can assign to every vector ψ an operator $\mathcal{O}_\psi$ which creates it when acting on the Fock vacuum. This operator will be a polynomial in the creation operators. Of particular importance are the vectors created by monomials. These generate the entire Fock space and will hereafter be referred to as basis vectors. When acting on a basis vector ψ, the ghost number operator counts the number of ghost oscillators in $\mathcal{O}_\psi$ minus the number of antighost oscillators.

We will often denote by Q_g the map

$$Q_g\colon \mathcal{F}_g \longrightarrow \mathcal{F}_{g+1} \ , \tag{V.1.3}$$

induced by Q. Then, $Q_{g+1} \circ Q_g = 0$ for all g and the following sequence defines a graded differential complex

$$\cdots \longrightarrow \mathcal{F}_{g-1} \stackrel{Q_{g-1}}{\longrightarrow} \mathcal{F}_g \stackrel{Q_g}{\longrightarrow} \mathcal{F}_{g+1} \longrightarrow \cdots \tag{V.1.4}$$

For every g define the following subspaces of $\mathcal{F}_g$

$$\ker\, Q_g = \{\psi \in \mathcal{F}_g \,|\, Q\psi = 0\} \tag{V.1.5}$$

$$\mathrm{im}\, Q_{g-1} = \{Q\psi \,|\, \psi \in \mathcal{F}_{g-1}\} \ . \tag{V.1.6}$$

Elements of $\ker\, Q_g$ are called **BRST cocycles** and elements of $\mathrm{im}\, Q_{g-1}$ are called **BRST coboundaries**. The **BRST cohomology** is defined as

$$H(\mathcal{F}) = \bigoplus_g H^g(\mathcal{F}) \ , \tag{V.1.7}$$

where

$$H^g(\mathcal{F}) = \frac{\ker\, Q_g}{\mathrm{im}\, Q_{g-1}} \ . \tag{V.1.8}$$

The **physical space** $\mathcal{H}_{\text{phys}}$ is defined as the BRST cohomology at zero ghost number $H^0(\mathcal{F})$.

Using the decomposition in (V.1.2) we can decompose the BRST complex into subcomplexes indexed by $\{\lambda\}$:

$$\cdots \longrightarrow \mathcal{F}_{g-1}(\lambda) \stackrel{Q^\lambda_{g-1}}{\longrightarrow} \mathcal{F}_g(\lambda) \stackrel{Q^\lambda_g}{\longrightarrow} \mathcal{F}_{g+1}(\lambda) \longrightarrow \cdots \tag{V.1.9}$$

and we can equally well decompose the cohomology space $H^g(\mathcal{F})$ as follows

$$H^g(\mathcal{F}) = \bigoplus_\lambda H^g_\lambda(\mathcal{F}) \ , \tag{V.1.10}$$

where $H^g_\lambda(\mathcal{F}) = H^g(\mathcal{F}(\lambda))$ is the cohomology of the restricted operator.

The assumption of self-adjointness of the BRST operator is motivated by our need to induce a well-defined defined inner product on the BRST cohomology. BRST cocycles which are cohomologous are supposed to be physically equivalent and therefore they should have the same inner product with any other BRST cocycle. For this to hold it is necessary and sufficient that the BRST coboundaries be perpendicular to the BRST cocycles. In other words, for all ψ obeying $Q\psi = 0$ and for all $\xi \in \mathcal{F}$

$$\langle Q\xi,\, \psi\rangle = \langle \xi,\, Q^\dagger\psi\rangle = 0 \ , \tag{V.1.11}$$

which merely says that $\ker Q \subseteq \ker Q^\dagger$. In particular, if $Q = Q^\dagger$ this is true. Encouraged by the known examples—in which Q is indeed self-adjoint—we do not feel that demanding hermiticity of Q is unreasonably strong.

Let us denote by $\operatorname{End}\mathcal{F}$ the algebra of operators on $\mathcal{F}$. The cohomology of $\operatorname{ad}\,Q$ on $\operatorname{End}\mathcal{F}$ (*cf.* Section II.1) is called the **operator BRST cohomology**. We will see in Section 3 that it is isomorphic to the endomorphisms of the BRST cohomology. In particular, BRST invariant vectors will be created by BRST invariant operators acting on the vacuum. Therefore the BRST cohomology can be given a multiplication induced from the one on operators. Since physical vectors are defined as BRST cohomology classes at zero ghost number, this may be an interesting way to define interactions, thought of as maps

$$\mathcal{H}_{\text{phys}} \otimes \mathcal{H}_{\text{phys}} \longrightarrow \mathcal{H}_{\text{phys}} \ . \tag{V.1.12}$$

2. The Decomposition Theorem

In this section we prove the decomposition theorem. This allows us to identify the BRST cohomology—which is a subquotient—as a particular subspace of the kernel of the BRST operator. In other words, the decomposition theorem picks out a privileged representative from each cohomology class. This theorem is very powerful and we present in the next two sections two immediate consequences. The first one is the characterization of the operator cohomology introduced in the last section in terms of the BRST cohomology. The second one is the reformulation of the no-ghost theorem which is reduced to the computation of two weighted traces, given the vanishing theorem for the BRST cohomology.

Since cohomology is an algebraic construction which is independent of the inner product, we are free to choose a convenient inner product different, in principle, from the inner product induced by the quantization procedure. In particular it is very convenient, as we will now see, to have a positive definite inner product. We achieve this via the introduction of a self-adjoint involution $\mathcal{C}$ in $\mathcal{F}$—its sole purpose being to redefine the inner product so that it be both positive definite and hermitian.

Unless otherwise stated, let us restrict ourselves to the finite dimensional eigenspaces $\mathcal{F}(\lambda)$ of the family $\{\Lambda\}$ of operators introduced in the previous section. To simplify the notation we drop all mention of λ.

In order to define $\mathcal{C}$ we now choose an pseudo-orthonormal basis in $\mathcal{F}$, *i.e.*, a basis whose elements are mutually orthogonal and of norm ± 1. In this basis the metric will be diagonal with entries equal to ± 1. We now define $\mathcal{C}$ to be the identity when restricted to the subspace of positive norm and minus the identity when restricted to its complement. $\mathcal{C}$ defined this way is not unique because there is no unique split of $\mathcal{F}$ into a positive definite and a negative definite subspace, however any two such choices will be related by a pseudo-unitary transformation and hence anything that we shall infer from this will be independent of this choice. Finally, we remark that because we take the Fock vacuum to have unit norm, $\mathcal{C}$ leaves the Fock vacuum invariant.

Equipped with such an operator we now introduce a new inner product in $\mathcal{F}$ as follows:

$$\langle\psi,\,\phi\rangle_{\mathcal{C}} \stackrel{\text{def}}{=} \langle\psi,\,\mathcal{C}\phi\rangle = \langle\mathcal{C}\psi,\,\phi\rangle\ , \tag{V.2.1}$$

for all ψ and ϕ in $\mathcal{F}$. The positive definiteness of this new inner product implies that $\mathcal{C}$ must map $\mathcal{F}_g$ to $\mathcal{F}_{-g}$, since the old inner product only coupled vectors of opposite ghost number. Other properties of $\mathcal{C}$ are particular to the actual theory we are quantizing: in the open bosonic string, for instance, $\mathcal{C}$ will turn out to involve time reversal as well.

Under this new inner product Q is no longer self-adjoint. In fact we denote its adjoint by Q^*. That is, for any two vectors ψ and ϕ in $\mathcal{F}$, $\langle Q\psi,\, \phi\rangle_{\mathcal{C}} = \langle \psi,\, Q^*\phi\rangle_{\mathcal{C}}$. It is easy to give an explicit expression for Q^*. In fact let $\mathcal{O}$ be any operator, self-adjoint or not. Then,

$$\begin{aligned}\langle \psi,\, \mathcal{O}^*\phi\rangle_{\mathcal{C}} &= \langle \mathcal{O}\psi,\, \phi\rangle_{\mathcal{C}} \\ &= \langle \mathcal{O}\psi,\, \mathcal{C}\phi\rangle \\ &= \langle \psi,\, \mathcal{O}^\dagger\mathcal{C}\phi\rangle \\ &= \langle \psi,\, \mathcal{C}\mathcal{O}^\dagger\mathcal{C}\phi\rangle_{\mathcal{C}} \qquad \text{since } \mathcal{C} \text{ is an involution.}\end{aligned}$$

Therefore we see that $\mathcal{O}^* = \mathcal{C}\,\mathcal{O}^\dagger\,\mathcal{C}$; and, in particular, $Q^* = \mathcal{C}\,Q\,\mathcal{C}$.

This new operator Q^* has similar properties to Q. In particular it obeys ${Q^*}^2 = 0$ and it has ghost number -1. Therefore we have the following differential complex dual to the BRST complex:

$$\cdots \longrightarrow \mathcal{F}_{g+1} \xrightarrow{Q^*_{g+1}} \mathcal{F}_g \xrightarrow{Q^*_g} \mathcal{F}_{g-1} \longrightarrow \cdots \tag{V.2.2}$$

We can define its cohomology as the direct sum of vector spaces

$$H(\mathcal{F}, Q^*) = \bigoplus_g H^g(\mathcal{F}, Q^*) \ , \tag{V.2.3}$$

where the definition of $H^g(\mathcal{F}, Q^*)$ parallels (V.1.8).

These cohomologies are not unrelated. Indeed, we claim that $H^{-g}(\mathcal{F}, Q^*)$ is isomorphic to $H^g(\mathcal{F}, Q)$. Consider the isomorphism $\mathcal{C}\colon \mathcal{F}_g \longrightarrow \mathcal{F}_{-g}$. It follows from the explicit expression for Q^* that the squares in the following diagram commute:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{F}_{g-1} & \xrightarrow{Q_{g-1}} & \mathcal{F}_g & \xrightarrow{Q_g} & \mathcal{F}_{g+1} & \longrightarrow & \cdots \\ & & \big\downarrow \mathcal{C} & & \big\downarrow \mathcal{C} & & \big\downarrow \mathcal{C} & & \\ \cdots & \longrightarrow & \mathcal{F}_{-g+1} & \xrightarrow{Q^*_{-g+1}} & \mathcal{F}_{-g} & \xrightarrow{Q^*_g} & \mathcal{F}_{-g-1} & \longrightarrow & \cdots \end{array} \tag{V.2.4}$$

That is, $\mathcal{C}$ is a chain map and thus induces a well-defined map in cohomology sending $H^g(\mathcal{F}, Q) \longrightarrow H^{-g}(\mathcal{F}, Q^*)$ which, abusing the notation, will also be referred to as $\mathcal{C}$ and which is defined to map $[\psi] \mapsto [\mathcal{C}\psi]$. Since $\mathcal{C}$ is an isomorphism, the claim follows.

Now we come to the decomposition theorem. Because the new inner product is positive definite we can split $\mathcal{F}$ as the orthogonal direct sum of vector spaces

$$\mathcal{F} = \mathrm{im}\ Q \oplus (\mathrm{im}\ Q)^\perp \ . \tag{V.2.5}$$

Notice, however, that $(\mathrm{im}\ Q)^\perp = \ker Q^*$ since Q and Q^* are adjoints under this inner product. Now let ψ be a vector in $\ker Q$. Under the above decomposition of $\mathcal{F}$ we can write ψ uniquely as a sum of two vectors $\phi + Q\chi$, where ϕ is in ker Q^*. Let $\mathcal{H}$ stand for the intersection ker $Q \cap$ ker Q^*. Then $\mathcal{H}$ is a direct sum

$$\mathcal{H} = \bigoplus_g \mathcal{H}^g \ , \tag{V.2.6}$$

where $\mathcal{H}^g = \mathcal{H} \cap \mathcal{F}_g$. Let h denote the projection onto $\mathcal{H}$. Then $h(\psi) = \phi$, where ψ and ϕ are as above. This projection induces a map in cohomology, which we also call h, and which maps $[\psi] \mapsto h(\psi)$. It is clearly independent of the particular representative we choose and moreover it is injective since $h(\psi) = 0$ if and only if ψ is cohomologous to zero. This provides us with an isomorphism between $H^g(\mathcal{F}, Q)$ and $\mathcal{H}^g$.

We could have done exactly the same construction with Q^* and thus obtain an isomorphism $H^g(\mathcal{F}, Q^*) \cong \mathcal{H}^g$. This gives us an isomorphism $H^g(\mathcal{F}, Q) \cong H^g(\mathcal{F}, Q^*)$ which, together with the isomorphism $H^g(\mathcal{F}, Q) \cong H^{-g}(\mathcal{F}, Q^*)$ induced by $\mathcal{C}$, gives the first important result about the BRST cohomology; namely

$$\boxed{H^g(\mathcal{F}, Q) \cong H^{-g}(\mathcal{F}, Q)} \ . \tag{V.2.7}$$

In analogy with the similar result for the de Rham cohomology of a compact oriented manifold we will refer to the above isomorphism as **Poincaré (⊡) duality**.

Notice that the above construction for both Q and Q^* gives a decomposition of $\mathcal{F}_g$ as the orthogonal direct sum

$$\boxed{\mathcal{F}_g = \text{im } Q_{g-1} \oplus \text{im } Q^*_{g+1} \oplus \mathcal{H}^g} \ . \tag{V.2.8}$$

We may further identify the space $\mathcal{H}^g$ with the kernel of a new operator. Define the **BRST laplacian** as

$$\triangle \stackrel{\text{def}}{=} QQ^* + Q^*Q \ . \tag{V.2.9}$$

It is a self-adjoint operator which satisfies the following properties:

$$\triangle\, \mathcal{C} = \mathcal{C} \triangle \tag{V.2.10}$$

$$\triangle\, Q = Q \triangle \tag{V.2.11}$$

$$\triangle\, Q^* = Q^* \triangle \tag{V.2.12}$$

$$\mathcal{H} = \ker\, \triangle \tag{V.2.13}$$

The first three properties are trivially verified and are left as exercises for the reader. We prove the last one. Let ψ be in ker $\triangle$. Then, in particular $\langle \triangle\psi,\, \psi\rangle_{\mathcal{C}} = 0$. But by definition,

$\langle \triangle\psi, \psi\rangle_{\mathcal{C}} = \|Q\psi\|^2 + \|Q^*\psi\|^2$ which, being a sum of non-negative quantities, must vanish termwise. Therefore, since the norm is positive definite, ψ must be annihilated by both Q and Q^* and hence be an element of $\mathcal{H}$. Conversely, if $\psi \in \mathcal{H}$ it is trivially in $\ker \triangle$. This proves the assertion. States in $\mathcal{H}$ will be referred to as **harmonic**, in analogy with the Hodge decomposition for de Rham cohomology. It is worth remarking that it follows from the definition of $\triangle$ that it commutes with any operator commuting with Q and Q^* or, equivalently, with Q and $\mathcal{C}$. Therefore, in particular, it maps $\mathcal{F}_g(\lambda) \longrightarrow \mathcal{F}_g(\lambda)$.

We now define the **Green's operator** to be an inverse to the BRST laplacian away from its kernel. In fact let $h : \mathcal{F} \to \mathcal{H}$ denote the projection onto the harmonic vectors. Then letting $\mathcal{H}^\perp$ stand for $\operatorname{im} Q \oplus \operatorname{im} Q^*$ we define the Green's operator to be a map $G : \mathcal{F} \to \mathcal{H}^\perp$ such that $G\psi = \omega$, where ω is the unique solution of $\triangle\omega = \psi - h(\psi)$ in $\mathcal{H}^\perp$. That such a solution is indeed unique is easy to verify.

The most important property of the Green's operator is that it commutes with every operator which commutes with the laplacian. In fact let T be any operator commuting with the BRST laplacian. Then T stabilizes both the image and kernel of the BRST laplacian. However the image of the BRST laplacian is just $\mathcal{H}^\perp$. Therefore let $\psi \in \mathcal{H}$. Then $G\psi = 0$, hence $T G\psi = 0$. But also $T\psi \in \mathcal{H}$ and hence $G T\psi = 0$. Now let $\psi \in \mathcal{H}^\perp$. Then by definition $G\psi = \omega$ where ω is the unique solution to $\triangle\omega = \psi$, since $h(\psi) = 0$. Therefore, $T G\psi = T\omega$. Now, $G T\psi = \phi$, where ϕ is the unique solution to $\triangle\phi = T\psi$, since $h(T\psi) = T h(\psi) = 0$. But $T\omega$ also satisfies $\triangle T\omega = T \triangle \omega = T\psi$. Hence by uniqueness $T\omega = \phi$ and G and T commute.

As a corollary of the above result we have that G commutes with $\mathcal{C}$, Q, Q^*, N_{gh} and the family $\{\Lambda\}$ of commuting self-adjoint operators. In particular G stabilizes each $\mathcal{F}_g(\lambda)$. It also stabilizes $\operatorname{im} Q$ and $\operatorname{im} Q^*$.

It is worth remarking that (V.2.10) together with (V.2.13) imply the isomorphism (V.2.7) and therefore, comparing this to the proof of Poincaré duality from the Hodge decomposition theorem, we see that $\mathcal{C}$ plays an analogous rôle to the Hodge star operator.

In this language we see that the space of physical vectors $\mathcal{H}_{\mathrm{phys}}$ is isomorphic to $\mathcal{H}^0$, the harmonic vectors of zero ghost number. We shall often use this as a model for the physical vectors.

3. The Operator BRST Cohomology

We now come to the first application of the decomposition theorem proven in the previous section. Here we will prove that the operator cohomology $H(\mathrm{End}\,\mathcal{F})$ is isomorphic to the algebra of operators $\mathrm{End}\,H(\mathcal{F})$. Recall that we have a well-defined map

$$_* : H(\mathrm{End}\,\mathcal{F}) \longrightarrow \mathrm{End}\,H(\mathcal{F})$$

defined by

$$[\varphi] \mapsto \varphi_* \ .$$

We show that this map is an isomorphism. That is, we show that every map in cohomology (*i.e.*, every element of $\mathrm{End}\,H(\mathcal{F})$) is induced by a chain map and hence by a class in $H(\mathrm{End}\,\mathcal{F})$, thus proving surjectivity. Then we show that if two chain maps induce the same map in cohomology they are necessarily chain homotopic, thus proving injectivity.

It will be very convenient for both steps to introduce an auxiliary concept. Let us denote by $\pi : H(\mathcal{F}) \xrightarrow{\cong} \mathcal{H}$ the isomorphism between the BRST cohomology and the BRST harmonic vectors which the decomposition theorem yields. Given any map $\psi \in \mathrm{End}\,H(\mathcal{F})$ let's denote by $\widehat{\psi} \in \mathrm{End}\,\mathcal{F}$ the map $\pi\,\psi\,\pi^{-1}$ extended trivially to all of $\mathcal{F}$. In other words, $\pi\,\psi\,\pi^{-1}$ as it stands is a map in $\mathrm{End}\,\mathcal{H}$. The trivial extension consists in having it vanish identically in $\mathrm{im}\ Q \oplus \mathrm{im}\ Q^*$. We call this the **minimal extension** of ψ and it is easily checked that it is a chain map with respect to both Q and Q^*. Moreover it is also easy to see that $\widehat{\psi}_* = \psi$. Hence this already proves surjectivity.

To prove injectivity all we have to show that if φ is any chain map then it is chain homotopic to the minimal extension $\widehat{\varphi}_*$ of φ_*. Given the decomposition $\mathcal{F} = \mathcal{H} \oplus \mathrm{im}\ Q \oplus \mathrm{im}\ Q^*$ we find it convenient to express all endomorphisms as 3×3 matrices of endomorphisms. Thus, for example, Q is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q \\ 0 & 0 & 0 \end{pmatrix} \ ; \tag{V.3.1}$$

and the minimal extension $\widehat{\psi}$ of ψ is represented by

$$\begin{pmatrix} \widehat{\psi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{V.3.2}$$

Now let $\varphi \in \mathrm{End}_g\mathcal{F}$ be a chain map. Because it must map $\mathrm{im}\ Q \to \mathrm{im}\ Q$ and $\ker\ Q \to \ker\ Q$

it has the following matrix representation

$$\varphi = \begin{pmatrix} F & 0 & A \\ B & C & D \\ 0 & 0 & E \end{pmatrix} . \tag{V.3.3}$$

First of all, it is obvious that $F : \mathcal{H} \to \mathcal{H}$ must coincide with the minimal extension of φ_*. Also because it is a chain map, $Q\,\varphi = (-1)^g\,\varphi\,Q$ and hence C and E are not independent but rather

$$Q\,E = (-1)^g\,C\,Q\ . \tag{V.3.4}$$

Therefore the difference between the chain map φ and the minimal extension $\widehat{\varphi}_*$ can be represented by

$$\varphi - \widehat{\varphi}_* = \begin{pmatrix} 0 & 0 & A \\ B & C & D \\ 0 & 0 & E \end{pmatrix} , \tag{V.3.5}$$

where E and C obey equation (V.3.4). We proceed to show that this is chain homotopic to zero. Indeed, consider the endomorphism $K \in \mathrm{End}_{g-1}\,\mathcal{F}$ given by

$$K = \begin{pmatrix} 0 & W & 0 \\ 0 & 0 & 0 \\ X & Y & Z \end{pmatrix} , \tag{V.3.6}$$

where

$$\begin{aligned} W &: \mathrm{im}\ Q \to \mathcal{H} \\ X &: \mathcal{H} \to \mathrm{im}\ Q^* \\ Y &: \mathrm{im}\ Q \to \mathrm{im}\ Q^* \\ Z &: \mathrm{im}\ Q^* \to \mathrm{im}\ Q^*\ . \end{aligned}$$

After a straight-forward calculation we see that

$$Q\,K + (-1)^g\,K\,Q = \begin{pmatrix} 0 & 0 & (-1)^g\,W\,Q \\ Q\,X & Q\,Y & Q\,Z \\ 0 & 0 & (-1)^g\,Y\,Q \end{pmatrix} . \tag{V.3.7}$$

Equating this with 4.5 we find that the following identities must be satisfied

$$
\begin{aligned}
A &= (-1)^g\, W\, Q \quad \text{mapping} \quad \text{im } Q \to \mathcal{H} \\
B &= Q\, X \quad \text{mapping} \quad \mathcal{H} \to \text{im } Q \\
C &= Q\, Y \quad \text{mapping} \quad \text{im } Q \to \text{im } Q \\
D &= Q\, Z \quad \text{mapping} \quad \text{im } Q^* \to \text{im } Q \\
E &= (-1)^g\, Y\, Q \quad \text{mapping} \quad \text{im } Q^* \to \text{im } Q^* \ .
\end{aligned}
$$

First of all we notice that since $\ker Q \cap \text{im } Q^* = 0$, $Q \restriction \text{im } Q^*$ is invertible and its inverse is given by $G\, Q^*$, where G is the Green's operator. Therefore we can indeed solve for X, W, Y, and Z in terms of A, B, C, D, and E as follows

$$
\begin{aligned}
W &= (-1)^g\, A\, G\, Q^* \\
X &= G\, Q^*\, B \\
Y &= G\, Q^*\, C \\
Z &= G\, Q^*\, D \\
Y &= (-1)^g\, E\, G\, Q^* \ .
\end{aligned}
$$

We must, of course, satisfy a consistency condition: namely that the two expressions for Y are really the same. But this can be trivially seen to follow from (V.3.4).

Therefore we have shown that every chain map is chain homotopic to the minimal extension of the map it induces in cohomology. But this is clearly equivalent to injectivity. For let φ and ϑ be two chain maps which induce the same map in cohomology, *i.e.*, such that $\varphi_* = \vartheta_*$. This implies that $\widehat{\varphi}_* = \widehat{\vartheta}_*$. Hence φ and ϑ are both chain homotopic to $\widehat{\varphi}_* = \widehat{\vartheta}_*$, and hence they are mutually chain homotopic.

Now suppose that a vanishing theorem holds for BRST cohomology, *i.e.*,

$$
H^{g\neq 0}(\mathcal{F}) = 0 \ . \tag{V.3.8}
$$

Then it is clear that a vanishing theorem holds for the operator BRST cohomology since the only non-trivial endomorphisms of BRST cohomology consists of maps taking $H^0(\mathcal{F})$ to $H^0(\mathcal{F})$.

Conversely suppose that a vanishing theorem holds for the operator cohomology. Every physical vector can be obtained from the vacuum by a suitable BRST invariant operator; just think of both the vacuum and the vector as harmonic vectors and then find an endomorphism which takes one to the other. Then we see that all physical vectors have the same ghost number as the vacuum which is the vanishing theorem for BRST cohomology.

With this result in mind it is easy to justify why the vanishing theorem for BRST cohomology is physically desirable. Suppose that there is a BRST harmonic vector ψ with ghost number g different from zero. Then since the operator cohomology coincides with the endomorphisms on $H(\mathcal{F})$ there is certainly at least one BRST invariant operator $\mathcal{O}$ which creates ψ from the vacuum. Moreover and without loss of generality we can choose $\mathcal{O}$ to be a chain map with respect Q^* as well. Let $\mathcal{O}^*$ denote its adjoint under the positive definite inner product. It follows from the definition of this inner product that $\mathcal{O}^*$ has ghost number $-g$ and moreover that it is a chain map as well with respect to both Q and Q^*.

Consider the vector $\mathcal{O}^*\,\psi$. This vector cannot be zero because of positivity of the inner product: just take the inner product with the vacuum; and, furthermore, it is a BRST harmonic vector of zero ghost number, *i.e.*, a physical vector. If this operator contained ghost excitations it would not be present in the spectrum of the theory had we quantized the physical phase space directly, however non-covariantly, without the introduction of the ghost and anti-ghost degrees of freedom. Hence the quantization procedure would be inconsistent. Also, if the Fock space arises from a local quantum field theory, if ψ has odd ghost number one can show[25] that the norm of $\mathcal{O}^*\,\psi$ is negative. This would mean that the theory would not be unitary. Hence it seems that the vanishing theorem is tied in rather closely to the consistency of the BRST quantization. To what extent the connection is fundamental and not just a curiosity is an interesting open problem and probably one of the fundamental pieces in the BRST puzzle.

4. The Reformulation of the No-Ghost Theorem

Throughout this section and unless otherwise stated we are restricting ourselves to one of the finite dimensional eigenspaces $\mathcal{F}(\lambda)$ of the family $\{\Lambda\}$. For clarity of notation we drop all reference to λ; hence, in particular, Q is to be understood as Q^λ.

The No-Ghost Theorem

In order to reformulate the no-ghost theorem we will analyze how $\mathcal{C}$ acts on the physical space. Recall that the physical space $\mathcal{H}_{\rm phys}$ is defined to be the harmonic vectors at zero ghost number $\mathcal{H}^0$. Because $\mathcal{C}$ maps $\mathcal{H}^g$ isomorphically to $\mathcal{H}^{-g}$, we see that it leaves $\mathcal{H}^0$ invariant and because $\mathcal{C}^2 = \mathbf{1}$, we can break $\mathcal{H}^0$ into eigenspaces corresponding to its eigenvalues ± 1. We denote by $\mathcal{H}^0_\pm$ the subspaces of $\mathcal{H}^0$ on which $\mathcal{C}$ acts as $\pm\mathbf{1}$. The definition of $\mathcal{C}$ was such that it was the identity when restricted to the positive definite subspace of $\mathcal{F}$ and minus the identity when restricted to the negative definite one. If the physical subspace is to be free of negative norm vectors then $\mathcal{C}$ must be the identity when restricted to it. That is, the no-ghost theorem is true if $\mathcal{H}^0 = \mathcal{H}^0_+$. Notice, however, that

$$\begin{aligned} \operatorname{Tr}_{\mathcal{H}^0} \mathcal{C} &= \dim\, \mathcal{H}^0_+ - \dim\, \mathcal{H}^0_- \\ &\leq \dim\, \mathcal{H}^0 \ . \end{aligned} \tag{V.4.1}$$

Thus it is precisely when this bound is saturated that the physical space is free of negative norm vectors. Let us define the **signature** $\operatorname{sgn} \mathcal{F}$ of the BRST complex as

$$\operatorname{sgn} \mathcal{F} \equiv \operatorname{Tr}_{\mathcal{H}^0} \mathcal{C} \ . \tag{V.4.2}$$

In practice the computation of $\operatorname{sgn} \mathcal{F}$ can be quite non-trivial, because the definition of $\mathcal{H}^0$ is not directly amenable to computations. However we can use the decomposition theorem to make this calculation easier. Recall that from the decomposition theorem, $\mathcal{F}_0$ breaks up as

$$\mathcal{F}_0 = \text{im}\ \ Q_{-1} \oplus \text{im}\ \ Q_1^* \oplus \mathcal{H}^0 \ , \tag{V.4.3}$$

and that $\mathcal{C}$ maps im Q_{-1} isomorphically to im Q_1^* because it is a chain map. Therefore if we took the trace of $\mathcal{C}$ over all of $\mathcal{F}_0$ it would only pick a contribution from $\mathcal{H}^0$. Therefore we have

$$\operatorname{Tr}_{\mathcal{H}^0} \mathcal{C} = \operatorname{Tr}_{\mathcal{F}_0} \mathcal{C} \ . \tag{V.4.4}$$

In fact, since $\mathcal{C}$ takes $\mathcal{F}_g$ to $\mathcal{F}_{-g}$ we may extend the trace to the whole Fock space $\mathcal{F}$ and conclude that (*cf.* (II.1.24))

$$\operatorname{sgn} \mathcal{F} = \operatorname{Tr}_{\mathcal{F}} \mathcal{C} \ . \tag{V.4.5}$$

Now assume that the vanishing theorem for BRST cohomology holds, *i.e.*, $H^{g\neq 0}(Q) = 0$. Then we have the following equality

$$\dim\, \mathcal{H}^0 = \sum_g (-1)^g \dim\, \mathcal{H}^g \ . \tag{V.4.6}$$

The right hand side of this equation is the **Euler character** of this differential complex and will be denoted by $\operatorname{ch} \mathcal{F}$. Again the calculation of $\operatorname{ch} \mathcal{F}$ may be non-trivial to perform.

We rewrite it in a suitable way using the following standard fact from linear algebra:

$$\mathcal{F}_g \cong \ker\ Q_g \oplus \mathrm{im}\ Q_g \ . \tag{V.4.7}$$

However ker Q_g splits into $\mathcal{H}^g \oplus \mathrm{im}\ Q_{g-1}$ which implies the following

$$\dim \mathcal{H}^g = \dim \mathcal{F}_g - \dim \mathrm{im}\ Q_g - \dim \mathrm{im}\ Q_{g-1} \ . \tag{V.4.8}$$

Performing the alternating sum we see that the last two terms of the right hand side cancel pairwise and we are left with the identity known in homological algebra as the **Euler-Poincaré principle**

$$\mathrm{ch}\,\mathcal{F} = \sum_g (-1)^g \dim \mathcal{F}_g = \mathrm{Tr}_{\mathcal{F}}\,(-1)^{N_{\mathrm{gh}}} \ . \tag{V.4.9}$$

Therefore we can express succinctly the condition for the absence of negative norm vectors from our physical space—under the assumption that the vanishing theorem holds—as

$$\boxed{\mathrm{ch}\,\mathcal{F} = \mathrm{sgn}\,\mathcal{F}} \ . \tag{V.4.10}$$

Collating the contributions to (V.4.10) from each $\mathcal{F}(\lambda)$, obtain

$$\boxed{\sum_\lambda q^\lambda\,\mathrm{ch}\,\mathcal{F}(\lambda) = \sum_\lambda q^\lambda\,\mathrm{sgn}\,\mathcal{F}(\lambda)} \ , \tag{V.4.11}$$

where q^λ is shorthand for $\prod_{i=1}^N q_i^{\lambda_i}$ (N is the number of mutually commuting operators providing this decomposition) and this is to be understood as a formal power sum. The LHS of (V.4.11) is called the **formal q-character** and the RHS is the **formal q-signature** and are denoted respectively by $\mathrm{ch}_q\,\mathcal{F}$ and $\mathrm{sgn}_q\,\mathcal{F}$.

<u>The Character as an Index</u>

We can rewrite these results in terms of an index theorem in much the same way that the Euler characteristic of a compact manifold can be expressed as the index of a suitable elliptic operator acting on the space of differential forms.

To this end we introduce a new grading in the BRST complex. We define the following "even" and "odd" subspaces

$$\mathcal{F}^e = \bigoplus_n \mathcal{F}_{2n} \qquad\qquad \mathcal{F}^o = \bigoplus_n \mathcal{F}_{2n+1} \; . \tag{V.4.12}$$

Then define the operator $D = Q + Q^*$ mapping $\mathcal{F}^e \to \mathcal{F}^o$. Its adjoint is $D^* = Q + Q^*$ mapping $\mathcal{F}^o \to \mathcal{F}^e$. In this way we turn the BRST complex into a two-space complex. With respect to the family $\{\Lambda\}$ of mutually commuting operators we can furthermore grade each space as follows:

$$\mathcal{F}^e = \bigoplus_\lambda \mathcal{F}^e(\lambda) \qquad\qquad \mathcal{F}^o = \bigoplus_\lambda \mathcal{F}^o(\lambda) \; . \tag{V.4.13}$$

We can think of $\{\Lambda\}$ as providing a toral action on the Fock space. This action commutes with the operator D and its adjoint and therefore one can define its character-valued index. Therefore we define

$$\mathrm{index}_q D \equiv \sum_\lambda q^\lambda \, \mathrm{index}_\lambda D \; , \tag{V.4.14}$$

where $\mathrm{index}_\lambda D$ is the index of the operator D restricted to the eigenspace with eigenvalue λ. This index is finite because of the finite dimensionality of the eigenspaces $\mathcal{F}_g(\lambda)$.

Restricting ourselves to $\mathcal{F}(\lambda)$ we can compute $\mathrm{index}_\lambda D$ very easily using the relation between the cohomology classes and the harmonic vectors provided by Hodge decomposition. First of all notice that because of the positive definiteness of the inner product $D^*\,D\,\psi = 0 \iff D\,\psi = 0$ and $D\,D^*\,\psi = 0 \iff D^*\,\psi = 0$, or equivalently, ker $D^*\,D =$ ker D and ker $D\,D^* =$ ker D^*. Notice also that $D^*\,D$ is nothing but the BRST laplacian restricted to the "even" subspace: $\triangle_e$, and $D\,D^*$ is the BRST laplacian restricted to the "odd" subspace: $\triangle_o$. Therefore,

$$\begin{aligned}
\mathrm{index}_\lambda D &= \dim\ker\ D - \dim\ker\ D^* \\
&= \dim\ker\ D^*\,D - \dim\ker\ D\,D^* \\
&= \dim\ker\ \triangle_e - \dim\, ker\, \triangle_o \\
&= \sum_g (-1)^g \dim\, H^g_\lambda(\mathcal{F}) \\
&= \mathrm{ch}_\lambda\, \mathcal{F} \; .
\end{aligned} \tag{V.4.15}$$

And therefore the character-valued index is nothing but the weighted trace of the Euler characteristic of the BRST complex

$$\boxed{\text{index}_q D = \text{ch}_q\, \mathcal{F}}\ . \tag{V.4.16}$$

Chapter Six:

BRST and Lie Algebra Cohomology

Perhaps the most fundamental example of a BRST complex as defined in Chapter V—and certainly one of the simplest to analyze—is the complex of semi-infinite forms on a graded Lie algebra $\mathfrak{g}$. This is a variant of the complex computing the ordinary Lie algebra cohomology introduced in Section II.1. Unlike ordinary Lie algebra cohomology, semi-infinite cohomology groups exists in both positive and negative dimensions. For finite dimensional Lie algebras, though, both complexes compute the same cohomology except that one must shift the dimension. In other words, an n-dimensional Lie algebra has cohomology groups in dimensions $0, \ldots, n$ whereas the "semi-infinite" cohomology would exist in dimensions $-\frac{n}{2}, \ldots, \frac{n}{2}$. Of course in finite dimensions we can always shift by a finite number, but in infinite dimensions we cannot. This is just one of the peculiar characteristics of the semi-infinite cohomology for infinite dimensional algebras. Another peculiar characteristic is that whether the ordinary cohomology makes sense with coefficients in any module, this is not generally the case for semi-infinite cohomology.

There are two conditions on the modules. First we must demand that the modules satisfy a finiteness hypothesis. But this is not all. In ordinary Lie algebra cohomology the fact that the differential d obeys $d^2 = 0$ is a direct consequence of the space of forms affording a representation of the algebra. We will see that in general the semi-infinite forms only admit a projective representation[16] of $\mathfrak{g}$ whose associated cocycle represents a cohomology class in $H^2(\mathfrak{g})$ which is none other than the Kac-Peterson class[**101**]. In practice, the reason for $d^2 \neq 0$ is essentially a renormalization effect: the naïve expression for the differential is not well defined since it involves infinite sums and hence it must be regularized; and there is no

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[16] This is generally true only for Lie algebras of finite growth. For more general Lie algebras (*cf. e.g.*, the very interesting paper of Figueirido & Ramos on diffeomorphism algebras [**100**]) it may be that the semi-infinite forms do not even admit a projective representation but only a "pseudo-representation" in terms of bilinear forms.

regularization which satisfies $d^2 = 0$, since this would mean that the semi-infinite forms do indeed afford an honest representation of the Lie algebra.

There are two possible ways out of this problem. The least interesting one is to pass to the central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ which kills the Kac-Peterson cocycle. This amounts to adding a 1-form dual to the central element. The interesting way is to tensor the semi-infinite forms with a module $\mathfrak{M}$ affording a projective representation of the opposite class. Then the semi-infinite forms of $\mathfrak{g}$ with with coefficients in $\mathfrak{M}$ have a well defined differential d obeying $d^2 = 0$ whose cohomology can be computed. This, of course, restricts the possible modules $\mathfrak{M}$ and this is how the critical dimensionality gets fixed in string theory.

It turns out that for a certain kind of Lie algebras and modules, one can prove[**33**] a vanishing theorem stating that the semi-infinite cohomology of the subcomplex relative to a particular subalgebra is concentrated at zero dimension. This subcomplex is the one appearing in the BRST quantization of the bosonic string so that this proves the vanishing theorem for the BRST cohomology of the bosonic string (away from zero center of mass momentum). This fact allows one, as was done in [**33**] and [**34**], to give a conceptually and computationally simple proof of the no-ghost theorem. Therefore, in Section 1 we review the construction of the semi-infinite cohomology and prove the vanishing theorem for the relative subcomplex; and in Section 2 we then apply this to the BRST cohomology of the bosonic string and prove the no-ghost theorem.

Although semi-infinite cohomology occurs naturally in many interesting 2-dimensional conformal field theories (CFTs): bosonic string (*cf.* Section 2), NSR string (*cf.* Chapter VII), and gauged WZNW models (*cf.* Chapter VIII); these do not form the bulk of the interesting CFTs. In CFT the Lie algebras (affine and Virasoro) are but very special cases of the relevant operator algebras (chiral algebras). An interesting field of research—and one in which very few results have been obtained—is in the BRST treatment of these more general chiral algebras. A particularly simple kind of chiral algebras are comprised by “quadratically non-linear Lie algebras” like, for instance, the Zamolodchikov spin 3 algebra. These are associative algebras generated by some fields, the commutators of whose modes almost close: *i.e.*, they close up to terms quadratic in modes.

BRST cohomology theories for these algebras have been constructed in certain cases: Thierry-Mieg[**102**] for the Zamolodchikov algebra, and Schoutens, Sevrin, & van Nieuwenhuizen[**103**] for affine-type quadratically non-linear Lie algebras, for instance; although the cohomologies have not been analyzed. This is an interesting open problem which perhaps would become relevant if an interesting example of a CFT were found whose “gauge” algebra is one of these chiral algebras.

1. Semi-Infinite Cohomology of Graded Lie Algebras

In this section we describe the complex of semi-infinite forms associated to a graded Lie algebra of finite growth. The formal definition of this complex is due to Feigin[104]; although, as he also shows in the cited paper, his construction is nothing but a generalization to arbitrary graded Lie algebras of the Virasoro ghost Fock space appearing, for example, in the BRST quantization of the bosonic string[29]. This alternate construction in terms of a ghost Fock space turns out to be much more fruitful for it trivially generalizes both to Lie superalgebras and to Lie algebras which are not necessarily of finite growth. We will follow the approach of Frenkel, Garland, & Zuckerman[33] for the most part. We present the main definitions, we prove a duality theorem using the results of Chapter V and we also explore the subcomplex relative to a subalgebra, proving the celebrated vanishing theorem of **[33]**.

The Complex of Semi-Infinite Forms

Let $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ be a graded complex Lie algebra of finite growth: $\left[\mathfrak{g}_m\, ,\, \mathfrak{g}_n\right] \subset \mathfrak{g}_{m+n}$ and $\dim \mathfrak{g}_m < \infty$. Let $\mathfrak{n}_\pm$ be the two subalgebras $\mathfrak{n}_\pm = \bigoplus_{\pm n>0} \mathfrak{g}_n$. Let $\{e_i\}$ be a basis for $\mathfrak{g}$ which is adapted to the grading: $e_i \in \mathfrak{g}_n$ for some n and if $e_i \in \mathfrak{g}_m$ then $e_{i+1} \in \mathfrak{g}_m$ or $e_{i+1} \in \mathfrak{g}_{m+1}$. Let $\mathfrak{g}'_n$ denote the dual space of $\mathfrak{g}_n$ and let $\mathfrak{g}' = \bigoplus_n \mathfrak{g}'_n$ be the **restricted dual**. Notice that it is much smaller than the full dual for it only consists of finite linear combinations of elements of the $\mathfrak{g}'_n$. Let $\{e'_i\}$ denote the basis for $\mathfrak{g}'$ which is canonically dual to the $\{e_i\}$. We define $\mathfrak{n}'_\pm$ analogously.

The space $\bigwedge_\infty \mathfrak{g}'$ of **semi-infinite forms** is spanned by the monomials

$$\omega = e'_{i_1} \wedge e'_{i_2} \wedge \cdots \tag{VI.1.1}$$

such that $i_1 > i_2 > \cdots$ and with the property that for k large enough $i_{k+1} = i_k - 1$. In other words, is is spanned by monomials with the property that only a finite number of the basis elements of $\mathfrak{n}'_-$ are missing. This space will turn out to have a nice "physical" interpretation as the ghost Fock space. The semi-infinite condition just says that we are allowed only a finite number of holes in the associated Dirac sea. We shall often abbreviate $\bigwedge_\infty \mathfrak{g}'$ merely to $\bigwedge_\infty$ when there is no possibility of confusion.

Let $\mathrm{Cl}(\mathfrak{g}\oplus\mathfrak{g}')$ denote the Clifford algebra associated to the bilinear form $\langle,\rangle$ on $\mathfrak{g}\oplus\mathfrak{g}'$ induced by the bilinear pairing $\mathfrak{g}\otimes\mathfrak{g}' \to \mathbb{C}$ sending $x\otimes x' \mapsto \langle x, x'\rangle$. Then $\bigwedge_\infty$ has the structure of a Clifford module over $\mathrm{Cl}(\mathfrak{g}\oplus\mathfrak{g}')$. For $x\oplus x' \in \mathfrak{g}\oplus\mathfrak{g}'$ define $\ell_{x\oplus x'} \equiv \imath(x) + \varepsilon(x')$

where $\imath$ and ε stand for interior and exterior products, respectively:

$$\varepsilon(x')\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots = x' \wedge e'_{i_1} \wedge e'_{i_2} \wedge \cdots \tag{VI.1.2}$$

$$\imath(x)\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots = \sum_{k \geq 1} (-1)^{k+1} \langle x, e'_{i_k} \rangle e'_{i_1} \wedge \cdots \wedge \widehat{e'_{i_k}} \wedge \cdots \tag{VI.1.3}$$

Notice that the above sum is actually finite. The following are easy to verify that for $x, y \in \mathfrak{g}$ and $x', y' \in \mathfrak{g}'$

$$\left\{ \imath(x)\, ,\, \varepsilon(y') \right\} = \langle x, y' \rangle \tag{VI.1.4}$$

$$\left\{ \imath(x)\, ,\, \imath(y) \right\} = 0 \tag{VI.1.5}$$

$$\left\{ \varepsilon(x')\, ,\, \varepsilon(y') \right\} = 0 \tag{VI.1.6}$$

whence

$$\ell_{x \oplus x'}{}^2 = \langle x, x' \rangle \tag{VI.1.7}$$

yielding, as claimed, a representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{g} \oplus \mathfrak{g}')$.

Whereas $\bigwedge \mathfrak{g}'$—the exterior algebra spanned by finite linear combinations of monomials like $e'_{i_1} \wedge \cdots \wedge e'_{i_k}$—affords naturally a representation of $\mathfrak{g}$, the semi-infinite forms afford, in general, only a projective representation. Let ad x and ad$'\, x$ denote the adjoint and coadjoint representations of $x \in \mathfrak{g}$. For $x \in \mathfrak{g}_{n \neq 0}$ we can define the natural representation $\rho(x)$ on $\bigwedge_\infty$ as

$$\rho(x) e'_{i_1} \wedge e'_{i_2} \wedge \cdots = \sum_{k \geq 1} e'_{i_1} \wedge \cdots \wedge \mathrm{ad}'\, x \cdot e'_{i_k} \wedge \cdots \ , \tag{VI.1.8}$$

where the sum is actually finite because the degree of x is non-zero. One can verify, in fact, that for $x \in \mathfrak{g}_{n \neq 0}$, $y \in \mathfrak{g}$ and $y' \in \mathfrak{g}'$,

$$\left[\rho(x)\, ,\, \imath(y) \right] = \imath(\mathrm{ad}\ x \cdot y) \tag{VI.1.9}$$

$$\left[\rho(x)\, ,\, \varepsilon(y') \right] = \varepsilon(\mathrm{ad}'\, x \cdot y')\ . \tag{VI.1.10}$$

When $x \in \mathfrak{g}_0$ the sum in (VI.1.8) is not finite and we must essentially renormalize $\rho(x)$ in this case. In this case normal ordering will do, since the algebras are of finite growth.

To define the renormalization let us introduce a **vacuum semi-infinite form** ω_0 satisfying $\left[\rho(x)\, ,\, \rho(y) \right] \omega_0 = \lambda(x, y) \omega_0$ for some antisymmetric bilinear form $\lambda : \bigwedge^2 \mathfrak{g} \to \mathbb{C}$ and for all $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_{-n}$, for any $n \neq 0$. A typical example is obtained by choosing i_0 such that $e'_{i_0} \in \mathfrak{g}'_m$ and $e'_{i_0+1} \in \mathfrak{g}'_{m+1}$ and defining

$$\omega_0 = e'_{i_0} \wedge e'_{i_0-1} \wedge \cdots \ . \tag{VI.1.11}$$

For a fixed vacuum ω_0 and a fixed element $\beta \in \mathfrak{g}'_0$ such that it is zero on $\left[\mathfrak{g}_0\,,\,\mathfrak{g}_0\right]$ define $\rho(x)$ for $x \in \mathfrak{g}_0$ on the vacuum by $\rho(x)\omega_0 = \langle\beta, x\rangle\omega_0$. We then extend it to all of $\bigwedge_\infty$ via (VI.1.9) and (VI.1.10). If ω_0 is of the form (VI.1.11), we can write $\rho(x)$ explicitly as

$$\rho(x) = \sum_{i\in\mathbb{Z}} :\varepsilon(\mathrm{ad}'\,x \cdot e'_i)\imath(e_i): + \langle\beta, x\rangle \ , \tag{VI.1.12}$$

where

$$:\imath(e_i)\varepsilon(e'_j): = \begin{cases} \imath(e_i)\varepsilon(e'_j) & \text{if } i \leq i_0 \\ -\varepsilon(e'_j)\imath(e_i) & \text{if } i > i_0 \end{cases} \ . \tag{VI.1.13}$$

A slightly tedious but straightforward calculation shows that ρ defines a projective representation of $\mathfrak{g}$ on $\bigwedge_\infty$

$$\left[\rho(x)\,,\,\rho(y)\right] = \rho(\left[x\,,\,y\right]) + \gamma(x,y) \ , \tag{VI.1.14}$$

where $\gamma(x,y) = \lambda(x,y) - \langle\beta, \left[x\,,\,y\right]\rangle$. In other words, $\gamma = \lambda + d\beta$. Clearly it is the class of λ in $H^2(\mathfrak{g})$ which determines whether ρ can be turned into an honest representation by a suitable choice of β. If this is the case, β is moreover unique if and only if $H^1(\mathfrak{g})$. From here on we assume that ω_0 and β are fixed in such a way that $\gamma \equiv 0$ so that (VI.1.12) defines an honest representation of $\mathfrak{g}$. This may require going to the universal central extension of $\mathfrak{g}$, *i.e.*, tossing in new generators to kill whatever $H^2(\mathfrak{g})$ there is.

We now introduce several gradings on $\bigwedge_\infty$. Fix $\mathrm{Deg}\,\omega_0 \in \mathbb{Z}$ and define

$$\mathrm{Deg}\,\varepsilon(x') = 1 \qquad \mathrm{Deg}\,\imath(x) = -1 \ . \tag{VI.1.15}$$

This allows to define Deg on $\bigwedge_\infty$ by extension. It will turn out that Deg will agree with the ghost number. Define $\bigwedge_\infty^m \equiv \{\omega \in \bigwedge_\infty \mid \mathrm{Deg}\,\omega = m\}$. Then it is plain that since $\mathrm{Deg}\,\rho(x) = 0$, $\bigwedge_\infty^m$ is a $\mathfrak{g}$-submodule for each m. We can also define the structure of a graded $\mathfrak{g}$-module for every $\bigwedge_\infty^m$ as follows. Fix $\deg \omega_0 \in \mathbb{Z}$ and for $x \in \mathfrak{g}_n$ and $x' \in \mathfrak{g}'_n$, define

$$\deg \imath(x) = n \qquad \deg \varepsilon(x') = -n \ . \tag{VI.1.16}$$

This defines it on $\bigwedge_\infty^m$ by extension. We can therefore define

$$\bigwedge\nolimits_\infty^{m,n} = \left\{\omega \in \bigwedge\nolimits_\infty^m \mid \deg \omega = n\right\} \ . \tag{VI.1.17}$$

Notice that for Virasoro or affine Lie algebras, deg corresponds to the negative of the L_0 level. We shall use the following abbreviated notation $\bigwedge_\infty^{\bullet;n} = \bigoplus_m \bigwedge_\infty^{m,n}$ whenever possible.

It is easy to see that $\dim \bigwedge_\infty^{\bullet;n} < \infty$ and that there is some integer n_0 such that $\bigwedge_\infty^{\bullet;n} = 0$ for all $n > n_0$. Graded $\mathfrak{g}$-modules $\mathfrak{M}$ obeying these properties; *i.e.*, that $\dim \mathfrak{M}^n < 0$ and that there is a n_0 such that $\mathfrak{M}^n = 0$ for $n > n_0$, form a category known as $\mathcal{O}_o$. This is a subcategory of the celebrated category $\mathcal{O}$, which consists of graded $\mathfrak{g}$-modules such that the $\mathfrak{n}_+$-submodule generated by any vector is finite dimensional. For any graded $\mathfrak{g}$-module $\mathfrak{M} \in \mathcal{O}_o$ we can define its **formal character** as

$$\mathrm{ch}_q\, \mathfrak{M} \equiv \sum_n \dim \mathfrak{M}^{-n} q^n \ . \tag{VI.1.18}$$

This is well defined because $\dim \mathfrak{M}^n < \infty$ and because there is a maximum power n_0 such that q^{-n_0} appears in the series. For $\bigwedge_\infty$ this is quite easy to compute. In fact, it is not difficult to convince oneself that

$$\sum_{m\in\mathbb{Z}} \mathrm{ch}_q \ \bigwedge\nolimits_\infty^m = q^{-n_0} \prod_{n\in\mathbb{Z}} (1 + q^{|n|})^{\dim \mathfrak{g}_n} \ . \tag{VI.1.19}$$

Let $\mathfrak{M}$ be any module in the category $\mathcal{O}$ and let $x \mapsto \pi(x)$ denote the representation of $\mathfrak{g}$ it carries. We define a differential d on $\mathfrak{M} \otimes \bigwedge_\infty$ by analogy with finite dimensional Lie algebras as

$$\boxed{d \equiv \sum_i \pi(e_i)\varepsilon(e'_i) + \sum_{i<j} :\imath([e_i\, ,\, e_j])\varepsilon(e'_i)\varepsilon(e'_j): + \varepsilon(\beta)} \ . \tag{VI.1.20}$$

Notice that this makes sense because both sums are actually finite. The first sum is finite because $\mathfrak{M} \in \mathcal{O}$ if and only $\pi(e_i)m = 0$ for i large enough for any $m \in \mathfrak{M}$; whereas $\varepsilon(e'_i)\omega$ is zero for i small enough and for ω any semi-infinite form. The second sum is also finite because of the definition of normal order and also of the semi-infinite forms. The all-important property of d is that it is a differential, *i.e.*, $d^2 = 0$—which can be checked after a slightly tedious calculation. This property of d only depends on the fact that $\mathfrak{M} \otimes \bigwedge_\infty$ is a $\mathfrak{g}$-module. Define $C_\infty(\mathfrak{g}; \mathfrak{M}) \equiv \bigoplus_m C_\infty^m(\mathfrak{g}; \mathfrak{M})$, where $C_\infty^m(\mathfrak{g}; \mathfrak{M}) \equiv \mathfrak{M} \otimes \bigwedge_\infty^m$. Then we have a graded differential complex

$$\cdots \longrightarrow C_\infty^m(\mathfrak{g}; \mathfrak{M}) \xrightarrow{d} C_\infty^{m+1}(\mathfrak{g}; \mathfrak{M}) \longrightarrow \cdots \tag{VI.1.21}$$

since, as can be seen from (VI.1.20), $\operatorname{Deg} d = 1$. Its cohomology is called the **semi-infinite cohomology** of the graded Lie algebra $\mathfrak{g}$ with coefficients in the graded $\mathfrak{g}$-module $\mathfrak{M}$ in the category $\mathcal{O}$; and will be denoted by $H_\infty(\mathfrak{g}; \mathfrak{M})$.

Now assume that $\mathfrak{M} \in \mathcal{O}_o$ and extend deg by defining $\deg(\omega \otimes m) = \deg \omega + \deg m$. This turns $C_\infty(\mathfrak{g}; \mathfrak{M})$ into a graded $\mathfrak{g}$-module in the category $\mathcal{O}_0$. In fact, this in nothing but the fact that $\mathcal{O}_o$ is closed under tensor products. We can see from (VI.1.20) that $\deg d = 0$. Hence it preserves the graded structure and therefore

$$H_\infty^m(\mathfrak{g}; \mathfrak{M}) = \prod_n H_\infty^{m,n}(\mathfrak{g}; \mathfrak{M}) \ , \tag{VI.1.22}$$

where $\dim H_\infty^{m,n} < \infty$. Define

$$\operatorname{ch}_q H_\infty^m(\mathfrak{g}; \mathfrak{M}) \equiv \sum_n \dim H_\infty^{m,-n}(\mathfrak{g}; \mathfrak{M})\, q^n . \tag{VI.1.23}$$

Then applying the Euler-Poincaré principle at each graded level and collating the contributions we find that for $\mathfrak{M} \in \mathcal{O}_o$

$$\boxed{\sum_m (-1)^m \operatorname{ch}_q C_\infty^m(\mathfrak{g}; \mathfrak{M}) = \sum_m (-1)^m \operatorname{ch}_q H_\infty^m(\mathfrak{g}; \mathfrak{M})} \ . \tag{VI.1.24}$$

Let $x \in \mathfrak{g}$ and define $\theta(x) \equiv \pi(x) + \rho(x)$. Then it is easy to prove the following formulas (analogous to the ones for finite $\mathfrak{g}$) for all $x \in \mathfrak{g}$ and $x' \in \mathfrak{g}'$

$$\big[d\,,\,\theta(x)\big] = 0 \tag{VI.1.25}$$

$$\big\{d\,,\,\imath(x)\big\} = \theta(x) \tag{VI.1.26}$$

$$\big\{d\,,\,\varepsilon(x')\big\} = \varepsilon(dx') \ , \tag{VI.1.27}$$

where on the RHS of the (VI.1.27), d refers to the usual Lie algebra differential $d : \bigwedge^m \mathfrak{g}' \to \bigwedge^{m+1} \mathfrak{g}'$ computing the Lie algebra cohomology $H(\mathfrak{g})$. As a consequence of (VI.1.27) we can extend d as a derivation on $\bigwedge \mathfrak{g}' \otimes C_\infty(\mathfrak{g};\mathfrak{M})$; that is, if $\xi \in \bigwedge^m \mathfrak{g}'$ and $\omega \in C_\infty(\mathfrak{g};\mathfrak{M})$ then

$$d(\xi\omega) = (d\xi)\omega + (-1)^m \xi(d\omega) \ . \tag{VI.1.28}$$

Duality for Hermitian Modules

Let $\mathfrak{g}$ admit an antilinear involution σ (*i.e.*, a Lie algebra automorphism of order 2) such that $\sigma(\mathfrak{g}_n) = \mathfrak{g}_{-n}$. This automorphism induces an antilinear map $\mathfrak{g}' \to \mathfrak{g}'$ also denoted by σ via

$$\langle \sigma(x), \sigma(x')\rangle = \overline{\langle x, x'\rangle} \ . \tag{VI.1.29}$$

This allows us to introduce a hermitian form on $\bigwedge_\infty$ denoted by $(,)$ relative to which

$$\varepsilon(x')^\dagger = -\varepsilon(\sigma(x')) \qquad \imath(x)^\dagger = -\imath(\sigma(x)) \ . \tag{VI.1.30}$$

To fix $(,)$ uniquely we choose two semi-infinite monomials $\omega_1 = e'_{i_1} \wedge e'_{i_2} \wedge \cdots$ and $\omega_2 = e'_{j_1} \wedge e'_{j_2} \wedge \cdots$ such that $\{e'_{i_k}\}_{k\geq 1} \cup \{\sigma(e'_{j_k})\}_{k\geq 1}$ forms a basis for $\mathfrak{g}'$. Then we set $(\omega_1, \omega_2) = 1$. Notice that

$$\epsilon = \cdots \wedge \sigma(e'_{j_2}) \wedge \sigma(e'_{j_1}) \wedge e'_{i_1} \wedge e'_{i_2} \wedge \cdots \tag{VI.1.31}$$

is a "volume" form. Since any two volume forms are unique up to a constant complex multiple we find that for all $\omega_1, \omega_2 \in \bigwedge_\infty$

$$\overline{(\omega_2, \omega_1)} = z\,(\omega_1, \omega_2) \ , \tag{VI.1.32}$$

for some complex number z. It may seem surprising that this holds for all ω_1, ω_2 but notice that once this is the case for the original pair wich sets the normalization the rest follows since we can obtain every other scalar product from this one by use of (VI.1.30). Iterating (VI.1.32) we find

$$(\omega_1, \omega_2) = \overline{z}\overline{(\omega_2, \omega 1)} = z\overline{z}\,(\omega_1, \omega_2) \ ; \tag{VI.1.33}$$

whence z is a phase $e^{i\theta}$. Redefining $(,)$ by $(,)\, e^{i\theta/2}$ makes it hermitian:

$$(\omega_1, \omega_2) = \overline{(\omega_2, \omega_1)} \ . \tag{VI.1.34}$$

Let us further assume that $\mathfrak{M}$ has a non-degenerate hermitian form such that

$$\pi(x)^\dagger = -\pi(\sigma(x)) \qquad \forall\, x \in \mathfrak{g} \ . \tag{VI.1.35}$$

In this case, we call $\mathfrak{M}$ a **hermitian module**. We can define a hermitian form in $C_\infty(\mathfrak{g};\mathfrak{M})$ by tensoring the form on $\mathfrak{M}$ with the one on $\bigwedge_\infty$ given by (VI.1.34). Notice that since we chose β to cancel the central extension in the representation ρ, we have that $\sigma(\beta) = -\beta$. This then implies that $\rho(x)^\dagger = -\rho(\sigma(x))$ and $d^\dagger = d$.

We therefore see that $(C_\infty(\mathfrak{g};\mathfrak{M}), d)$ defines a BRST complex as defined in Chapter V, except that it is not a Fock space although it possesses a non-degenerate inner product under which d is self-adjoint. Still, following the remarks at the beginning of Chapter V, we can make use of the results in that chapter to prove the following duality

$$\boxed{H^m_\infty(\mathfrak{g};\mathfrak{M}) \cong H^{-m}_\infty(\mathfrak{g};\mathfrak{M})} \ , \tag{VI.1.36}$$

for $\mathfrak{M}$ a hermitian module in $\mathcal{O}_o$.

Vanishing Theorem for the Relative Subcomplex

Let $\mathfrak{h} \subset \mathfrak{g}_0$ be a subalgebra. We define the **semi-infinite forms relative to** $\mathfrak{h}$ as

$$C_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}) \equiv \{\omega \in C_\infty(\mathfrak{g};\mathfrak{M}) \mid \imath(x)\omega = \theta(x)\omega = 0 \ \forall\, x \in \mathfrak{h}\} \ . \tag{VI.1.37}$$

This is stabilized by d and hence it is a subcomplex of $C_\infty(\mathfrak{g};\mathfrak{M})$ whose cohomology—the **relative semi-infinite cohomology**—we will denote by $H_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M})$. Notice that when $\mathfrak{M}$ is hermitian we have an analogous theorem to (VI.1.36) for an appropriate choice of grading for the relative subcomplex; which we will assume hereafter. Notice, in particular, that this grading could be half-integral.

If $\mathfrak{h}$ contains the center $\mathfrak{z}$ of $\mathfrak{g}$ and if $\mathfrak{z}$ acts as scalars on $\mathfrak{M}$, the condition $\theta(z)\omega = 0$ for $z \in \mathfrak{z}$ implies that $\pi(z) = -\langle\beta, z\rangle \cdot \mathbf{1}_{\mathfrak{M}}$. In particular, if z is the central element tossed in to kill the cocycle associated to the projective representation ρ, π must be also a projective representation of the opposite class for there to be any relative cohomology at all. This is precisely what happens in string theory with $\mathfrak{g}$ the Virasoro algebra and $\mathfrak{h}$ the center. In particular, this fixes the critical dimensionality. We will see this in more detail in the next section.

We take $\mathfrak{h}$ to be all of $\mathfrak{g}_0$. Let $C^m \equiv C^m_\infty(\mathfrak{g},\mathfrak{g}_0;\mathfrak{M})$ and $C \equiv \bigoplus_m C^m$. Then C^m is naturally bigraded as

$$C^m = \bigoplus_{c-b=m} C^{b,c} \ , \tag{VI.1.38}$$

where $C^{b,c}$ is spanned by those $\omega = e'_{i_1} \wedge e'_{i_2} \wedge \cdots$ such that b is the number of missing basis elements from $\mathfrak{n}'_-$ and c is the number of vectors of $\mathfrak{n}'_+$ and $\operatorname{Deg}\omega_0$ is chosen consistently so that for $\omega \in C^{b,c}$, $\operatorname{Deg}\omega = c - b$ as induced from $\operatorname{Deg}\omega_0$. In other words, we can decompose $C_\infty(\mathfrak{g},\mathfrak{g}_0;\mathfrak{M})$ as follows

$$C_\infty(\mathfrak{g},\mathfrak{g}_0;\mathfrak{M}) = \left(\mathfrak{M} \otimes \bigwedge \mathfrak{n}'_+ \otimes \bigwedge\nolimits_\infty \mathfrak{n}'_-\right)^{\mathfrak{g}_0} \quad . \tag{VI.1.39}$$

And, under this decomposition,

$$C^{b,c} = \left(\mathfrak{M} \otimes \bigwedge\nolimits^c \mathfrak{n}'_+ \otimes \bigwedge\nolimits^b_\infty \mathfrak{n}'_-\right)^{\mathfrak{g}_0} \quad . \tag{VI.1.40}$$

We now come to the fundamental theorem of this section: the vanishing theorem of **[33]**. This theorem states that for $\mathfrak{M}$ a free $\mathfrak{n}_-$-module the relative cohomology is concentrated at zero Degree. To prove the vanishing theorem we shall first filter the relative subcomplex. This filtration will give rise to a spectral sequence for whose E_1 term we shall be able to prove a half-vanishing theorem. This will propagate to a half-vanishing theorem for the relative cohomology. Together with (VI.1.36) we will get the vanishing theorem.

So let $\mathfrak{M}$ be a free $\mathfrak{n}_-$-module. Without loss of generality we can let it have rank one. That is, $\mathfrak{M} \cong \mathfrak{U}(\mathfrak{n}_-)$, the universal enveloping algebra of $\mathfrak{n}_-$. Explicitly, $\mathfrak{M}$ is the span of monomials obtained from a cyclic vector m_o by the action of elements of $\mathfrak{U}(\mathfrak{n}_-)$:

$$\left\{\pi(e_{i_1})\cdots\pi(e_{i_k})\cdot m_o \mid i_1 \leq i_2 \leq \cdots \text{ and } e_{i_j} \in \mathfrak{n}_- \ \forall j,k\right\} \quad . \tag{VI.1.41}$$

For example, if $\mathfrak{M}$ is a Verma module of $\mathfrak{g}$ then $\mathfrak{M}$ is a free $\mathfrak{n}_-$-module of rank ℓ where ℓ is the dimension of the irreducible $\mathfrak{g}_0$ representation from which we induce. If $\mathfrak{g}_0$ is abelian, like in affine and Virasoro Lie algebras, then $\ell = 1$ and $\mathfrak{M}$ is a free $\mathfrak{n}_-$-module of rank 1.

Let ω_o be a vacuum semi-infinite form and choose $\deg\omega_0 \in \mathbb{Z}$. In order to define the filtration we introduce the following filtration degree:

$$\operatorname{fdeg}(m\otimes\omega) = \deg m - \deg\omega_+ + \deg\omega_- \ , \tag{VI.1.42}$$

where $m \in \mathfrak{M}$, $\omega = \omega_+ \wedge \omega_-$, $\omega_+ \in \bigwedge \mathfrak{n}'_+$ and $\omega_- \in \bigwedge_\infty \mathfrak{n}'_-$ are homogeneous elements in the decomposition of C given by (VI.1.39). We then define $F^pC = \{\omega \in C \mid \operatorname{fdeg}\omega \geq p\}$. It is clear, after a little thought, that FC is a bounded filtration of C. To verify that it is a filtered complex we need to look at the filtration degrees of the homogeneous pieces in d.

The following are the filtration degrees of the relevant objects

$$
\begin{array}{|c|c|}
\hline
\text{Object} & \text{Filtration Degree} \\
\hline
\pi(\mathfrak{g}_n) & n \\
\varepsilon(\mathfrak{g}_n) & |n| \\
\imath(\mathfrak{g}_n) & -|n| \\
\hline
\end{array}
\tag{VI.1.43}
$$

for $n \neq 0$ and where, for example, $\varepsilon(\mathfrak{g}_n)$ means $\varepsilon(x)$ for any $x \in \mathfrak{g}_n$. From this table we can read off the filtration degrees of all the homogeneous pieces of d:

$$\operatorname{fdeg}\left(\pi(\mathfrak{g}_n)\varepsilon(\mathfrak{g}'_n)\right) = n + |n| \tag{VI.1.44}$$

$$\operatorname{fdeg}\left(:\imath(\mathfrak{g}_{n+m})\varepsilon(\mathfrak{g}'_n)\varepsilon(\mathfrak{g}'_m):\right) = |n| + |m| - |n+m| \tag{VI.1.45}$$

whence we see that all terms in d have non-negative filtration degree. Therefore (FC, d) is a bounded filtered complex and Theorem II.1.32 guarantees the existence of a spectral sequence converging finitely to the cohomology of the relative subcomplex whose E_1 term is the cohomology of the associated graded complex with the induced differential. The induced differential, which we denote by d_0 is the part of d with zero filtration degree. A look at (VI.1.44) and (VI.1.45) allows us to identify the fdeg $= 0$ part of d as

$$
\begin{aligned}
d_0 = \sum_{i\in I_-} \pi(e_i)\varepsilon(e'_i) &+ \sum_{i<j\in I_-} :\imath(\left[e_i\,,\,e_j\right])\varepsilon(e'_i)\varepsilon(e'_j): \\
&+ \sum_{i<j\in I_+} :\imath(\left[e_i\,,\,e_j\right])\varepsilon(e'_i)\varepsilon(e'_j): \ ,
\end{aligned}
\tag{VI.1.46}
$$

where we have defined the index sets $I_\pm$ in such a way that $i \in I_\pm \iff e_i \in \mathfrak{n}_\pm$. The first two terms in d_0 correspond to the differential in the complex $\mathfrak{M} \otimes \bigwedge_\infty \mathfrak{n}'_-$ computing the semi-infinite cohomology $H_\infty(\mathfrak{n}_-; \mathfrak{M})$ of $\mathfrak{n}_-$ with coefficients in the free module $\mathfrak{M}$. The third term corresponds to the differential in $\bigwedge \mathfrak{n}_+$ computing the ordinary cohomology $H(\mathfrak{n}_+)$ of the Lie algebra $\mathfrak{n}_+$. Therefore the E_1 term is the cohomology of the complex $K^{\mathfrak{g}_0}$ where ${}^{\mathfrak{g}_0}$ stands for $\mathfrak{g}_0$ invariants and K is short for

$$K \equiv \bigwedge \mathfrak{n}'_+ \otimes C_\infty(\mathfrak{n}_-; \mathfrak{M}) \ . \tag{VI.1.47}$$

Let $\mathfrak{g}_0$ act reducibly on K, so that K breaks up as $K^{\mathfrak{g}_0} \oplus (\mathfrak{g}_0\, K)$, where by $\mathfrak{g}_0\, K$ we mean the image of K under the action of $\mathfrak{g}_0$. Then the cohomology of $K^{\mathfrak{g}_0}$ is precisely the $\mathfrak{g}_0$ invariant elements of the cohomology of K. In other words, if $\mathfrak{g}_0$ acts reducibly on K, then $H(K^{\mathfrak{g}_0}) = H(K)^{\mathfrak{g}_0}$. But we can compute $H(K)$ using the Künneth formula (II.1.56). Putting it all together and keeping track of the bigrading we get

$$E_1^{c,b} = \left(H^c(\mathfrak{n}_+) \otimes H^b_\infty(\mathfrak{n}_-;\mathfrak{M})\right)^{\mathfrak{g}_0} \ . \tag{VI.1.48}$$

In the appendix we prove that for $\mathfrak{M}$ a free $\mathfrak{n}_-$-module,

$$\boxed{H^b_\infty(\mathfrak{n}_-;\mathfrak{M}) \cong \begin{cases} \mathbb{C} & \text{for } b=0 \\ 0 & \text{otherwise} \end{cases}} \ . \tag{VI.1.49}$$

Since $E_1^n = \bigoplus_{c-b=n} E_1^{c,b}$, (VI.1.49) already implies that $E_1^{m<0} = 0$. This vanishing propagates to the limit term and therefore

$$H^{m<0}_\infty(\mathfrak{g},\mathfrak{g}_0;\mathfrak{M}) = 0 \ . \tag{VI.1.50}$$

Using the duality theorem (VI.1.36) we can extend this vanishing for $m > 0$. Therefore, if $\mathfrak{M}$ is a free $\mathfrak{n}_-$-module,

$$\boxed{H^{m\neq 0}_\infty(\mathfrak{g},\mathfrak{g}_0;\mathfrak{M}) = 0} \ . \tag{VI.1.51}$$

2. Semi-Infinite Cohomology of the Virasoro Algebra

In this section we define the relative subcomplex for the Virasoro algebra appearing in bosonic string theory and we identify it with the ghost Fock space. We prove a vanishing theorem for its cohomology using the results of the previous section and as a consequence we can give a very simple proof of the no-ghost theorem. This section follows **[33]**. The proof of the no-ghost theorem also appeared in **[34]**.

Let $\mathfrak{V} = \bigoplus_n \mathfrak{V}_n$ denote the centrally extended complexified Virasoro algebra, where $\mathfrak{V}_n$ is spanned by ℓ_n for n different from zero and $\mathfrak{V}_0$ is spanned by ℓ_0 and c. This algebra is defined by

$$\left[\ell_m \, , \, \ell_n\right] = (m-n)\ell_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \ , \tag{VI.2.1}$$

and by the fact that c is central. Let $\mathfrak{V}_\pm = \bigoplus_{\pm n>0} \mathfrak{V}_n$. Let $\mathfrak{V}' = \bigoplus_n \mathfrak{V}'_n$ denote the restricted dual space to $\mathfrak{V}$. It therefore consists of linear functionals of finite rank. Let $\{\ell'_n, c'\}$ denote the canonical dual basis for $\mathfrak{V}'$. We define $\mathfrak{V}'_\pm$ in the obvious manner.

As discussed in the previous section, the space of semi-infinite forms $\bigwedge_\infty \mathfrak{V}'$ is spanned by monomials $\ell'_{i_1} \wedge \ell'_{i_2} \wedge \cdots$ with $i_1 > i_2 > \cdots$ with the property that only a finite number of the basis elements for $\mathfrak{V}'_-$ are missing. Notice that it also allows monomials containing c'. This space can be characterized in a way which allows its immediate generalization to Lie superalgebras and which is very familiar to string theorists[17]. Choose as vacuum the $\mathfrak{sl}(2,\mathbb{C})$-invariant semi-infinite form ω_o

$$\omega_o = \ell'_1 \wedge \ell'_0 \wedge \ell'_{-1} \wedge \cdots \ . \tag{VI.2.2}$$

To see that it is indeed $\mathfrak{sl}(2,\mathbb{C})$ invariant just use the expressions for the operators representing the $\mathfrak{sl}(2,\mathbb{C})$ generators $\{\ell_{\pm 1}, \ell_0\}$. The correspondence between the semi-infinite forms and the ghost Fock space starts by identifying the ghost oscillators:

$$b_n \leftrightarrow \imath(\ell_n) \qquad c_n \leftrightarrow \varepsilon(\ell'_{-n}) \ . \tag{VI.2.3}$$

It then follows from (VI.1.4), (VI.1.5), and (VI.1.5) that the usual anticommutation rules apply. Notice that the Degree corresponds to the ghost number. It is well known from string theory that the semi-infinite forms afford a representation of (VI.2.1) in which $\rho(c) = -26\,\mathbf{1}_{\bigwedge_\infty}$. Now let $\mathfrak{M}$ be a Virasoro module affording a representation π with the property that $\pi(c) = 26\,\mathbf{1}_{\mathfrak{M}}$. Then we can go to the subcomplex relative to the center. We choose to work in this subcomplex from now on. Therefore for all intents and purposes we can assume we are working with the Virasoro algebra without the central extension. The differential d turns out to be expressible very simply in terms of conformal fields.

Let

$$b(z) = \sum_n b_n z^{-n-2} \tag{VI.2.4}$$

$$c(z) = \sum_n c_n z^{-n+1} \tag{VI.2.5}$$

$$T^{\mathrm{gh}}(z) = \sum_n \rho(\ell_n) z^{-n-2} \ , \tag{VI.2.6}$$

where $\rho(\ell_n)$ is given by the specialization to the Virasoro algebra of (VI.1.12). Following the physics literature let us denote $\rho(\ell_n)$ by L_n^{gh}. From (VI.2.6) we find that $T^{\mathrm{gh}}(z)$ is simply

[17] Actually what is familiar to string theorists is the subcomplex relative to the center which corresponds, as we shall see, to the ghost Fock space of the open bosonic string. The full complex would also contain a ghost/antighost pair for the central generator c.

given by

$$T^{\mathrm{gh}}(z) =: c\partial b\colon (z) + 2\colon (\partial c) b\colon (z) \ , \tag{VI.2.7}$$

where, if $A(z)$, $B(z)$ are two quantum fields, their **normal ordered product** is defined as

$$: AB\colon (z) = \oint_{C_z} \frac{dw}{2\pi i} \frac{1}{w - z} A(w) B(z) \ , \tag{VI.2.8}$$

where C_z is a positively oriented contour around z in the complex w plane and the operator product inside the integrand is the radial ordered product. Similarly, defining

$$T_{\mathfrak{M}}(z) = \sum_n \pi(\ell_n) z^{-n-2} \ , \tag{VI.2.9}$$

we find that the differential d is given simply by

$$d = \oint_{C_0} \frac{dz}{2\pi i} j(z) \tag{VI.2.10}$$

where

$$\begin{aligned} j(z) &=: T_{\mathfrak{M}} c\colon (z) + : bc\partial c\colon (z) \\ &=: (T_{\mathfrak{M}} + \frac{1}{2} T^{\mathrm{gh}}) c\colon (z) \ . \end{aligned} \tag{VI.2.11}$$

The case of interest in string theory is the case where $\mathfrak{M}$ is the Fock space associated to the propagation of string in (25 + 1)-dimensional Minkowski spacetime. Define, for $\mu = 0, \ldots, 25$ the following fields

$$X^\mu(z) = x^\mu - i a_0^\mu \log z + i \sum_{n \neq 0} \frac{1}{n} a_n^\mu z^{-n} \ , \tag{VI.2.12}$$

with operator product expansions

$$X^\mu(z) X^\nu(w) \sim -\eta^{\mu\nu} \log(z - w) \tag{VI.2.13}$$

from which we can read off the commutator relations for the modes. Here, $\eta^{\mu\nu} = \mathrm{diag}\,[- + \cdots +]$ is the Minkowski metric. The Fock space is constructed as follows. Pick a vector k in Minkowski spacetime and define the Fock vacuum $|k\rangle$ by

$$a_0^\mu \,|k\rangle = k^\mu \,|k\rangle \qquad a_n^\mu \,|k\rangle = 0 \;\forall\, n > 0 \ . \tag{VI.2.14}$$

Then the Fock space $\mathfrak{M}(k)$ is obtained from $|k\rangle$ by successively applying the creation operators a_{-n}^μ on $|k\rangle$. One defines an inner product by fixing $\langle k|k\rangle = 1$ and then demanding that $a_n^{\mu\dagger} = a_{-n}^{\mu}$. Notice that this Fock space has an indefinite scalar product due to the indefiniteness of the Minkowski metric. The Virasoro generators are represented by the modes of the energy momentum tensor $T^{(X)}(z)$ given by

$$T^{(X)}(z) = -\frac{1}{2} {:}\, \partial X \cdot \partial X(z) {:} \ , \tag{VI.2.15}$$

where $\cdot$ means Minkowski inner product. Explicitly,

$$\pi(\ell_n) = \frac{1}{2} \sum_m {:}\, a_m \cdot a_{n-m} {:} \equiv L_n^{(X)} \ . \tag{VI.2.16}$$

The degree is minus the eigenvalue of $\theta(\ell_0) = L_0^{\rm gh} + L_o^{(X)} \equiv L_0$, which is the sum of the degrees in $\bigwedge_\infty$ and $\mathfrak{M}(k)$. A typical monomial in the Fock space $\mathfrak{M}(k)$ has degree

$$\deg \prod_{i=1}^{N} a_{-m_i}^{\mu_i} \left|k\right\rangle = -\sum_{i=1}^{N} m_i - \frac{1}{2} k \cdot k \ . \tag{VI.2.17}$$

For $\mathfrak{M}(k)$ a Fock module the differential d given by (VI.2.10) and (VI.2.11) is the BRST operator for the open bosonic string[29].

Brower[105] proved that for $k \neq 0$ the Fock module $\mathfrak{M}(k)$ is a free $\mathfrak{V}_-$-module. The proof consisted in showing that $\mathfrak{M}(k)$ is obtained by successive application of the $L_{-n}^{(X)}$ for $n > 0$ on the states obtained from the vacuum via the operators in the full spectrum generating algebra. Using the Kac determinant formula for the Šapovalov form of the Virasoro algebra we can understand this from a more Lie algebraic point of view.

Let $\mathbb{V}$ denote the $(25+1)$-dimensional Minkowski space and let us decompose it as an orthogonal direct sum (with respect to the Minkowski metric) $\mathbb{V}' \oplus \mathbb{V}''$, where $\dim \mathbb{V}' = 25$ and $\dim \mathbb{V}'' = 1$, in such a way that this split induces a decomposition of $k = k' + k''$ such that $k' \cdot k' > 0$ and $k'' \cdot k'' < 0$. Clearly this is always possible as long as $k \neq 0$. Since the split is orthogonal and the Virasoro generators involve the Minkowski metric it follows that as Virasoro module $\mathfrak{M}(k)$ breaks up as

$$\mathfrak{M}(k) \cong \mathfrak{M}'(k') \otimes \mathfrak{M}''(k'') \ , \tag{VI.2.18}$$

where $\mathfrak{M}'$ (resp. $\mathfrak{M}''$) is the Fock module corresponding to those fields X^μ such that $X^\mu(z) \in \mathbb{V}'$ (resp. $X^\mu \in \mathbb{V}''$). First let us decompose $\mathfrak{M}''(k'')$ into irreducible modules. The Fock vacuum $\left|k''\right\rangle$ is a Virasoro highest weight vector with highest weight $(h, c) = (\frac{1}{2} k'' \cdot k'', 1)$. From the Kac determinant formula[106],[107] we know that Verma modules with $h < 0$ and

$c = 1$ are irreducible and non-unitary. Therefore we have an inclusion of the Verma module $M(\frac{1}{2}k'' \cdot k'', 1)$ inside $\mathfrak{M}''(k'')$. But computing the characters for both sides (or noticing that the combinatorics of a Virasoro Verma module is exactly that of the Fock space of the modes of a scalar field) we find that, in fact, $\mathfrak{M}''(k'') \cong M(\frac{1}{2}k'' \cdot k'', 1)$ and hence it is irreducible.

Similarly we can decompose $\mathfrak{M}'(k')$ into irreducibles as follows. Notice that in this Fock module $c = 25$ and that the Fock vacuum $|k'\rangle$ is a Virasoro highest weight with $L_0^{(X)}$-weight $h = \frac{1}{2}k' \cdot k'$. Moreover the spectrum of $L_0^{(X)}$ is $k' \cdot k' + n$ for n a non-negative integer. But again the Kac determinant formula implies that the Verma modules $M(h, c)$ for $h > 0$ and $c > 1$ are unitary and irreducible. Therefore $\mathfrak{M}'(k')$ decomposes into a direct sum of Verma modules. Computing the Virasoro character of $\mathfrak{M}'(k')$ and comparing with the Virasoro character of $M(h, c)$ allows us to conclude that

$$\mathfrak{M}'(k') \cong \bigoplus_{n \geq 0} p_{(25)}(n)\, M(\frac{1}{2}k' \cdot k' + n, 25) \;, \tag{VI.2.19}$$

where $p_{(D)}(n)$ is the coefficient of q^n in the partition function $\prod_{n=1}^{\infty}(1-q^n)^{-D}$ for D bosons.

Therefore $\mathfrak{M}(k)$, for $k \neq 0$, is written as

$$\mathfrak{M}(k) \cong \bigoplus_{n \geq 0} p_{(25)}(n)\, M(\frac{1}{2}k' \cdot k' + n, 25) \otimes M(\frac{1}{2}k'' \cdot k'', 1) \;, \tag{VI.2.20}$$

which shows that it is $\mathfrak{V}_-$-free. Therefore, using (VI.1.51), we deduce that the BRST cohomology of the open bosonic string, which is the cohomology of the subcomplex relative to $\mathfrak{V}_0$, vanishes for ghost number different from zero:

$$\boxed{H_\infty^m(\mathfrak{V}, \mathfrak{V}_0; \mathfrak{M}(k)) = 0 \text{ for } m \neq 0} \;. \tag{VI.2.21}$$

For $k = 0$ we can compute the BRST cohomology explicitly and we find that there are exactly 28 cocycles which are not coboundaries: $a_{-1}^\mu\, |0\rangle \otimes b_0 b_{-1}\, |0\rangle_{\text{gh}}$, $|0\rangle \otimes b_0 b_{-1} b_1\, |0\rangle_{\text{gh}}$ and $|0\rangle \otimes b_0\, |0\rangle_{\text{gh}}$, which correspond to the semi-infinite forms $a_{-1}^\mu \otimes \ell'_{-1} \wedge \ell'_{-2} \wedge \cdots$, $1 \otimes \ell'_{-2} \wedge \ell'_{-3} \wedge \cdots$, and $1 \otimes \ell'_1 \wedge \ell'_{-1} \wedge \ell'_{-2} \wedge \cdots$ respectively. This, of course, does not contradict the vanishing theorem because $\mathfrak{M}(0)$ is not $\mathfrak{V}_-$ free, since $L_{-1}^{(X)}$ annihilates the Fock vacuum.

There is a similar "vanishing" theorem for the full complex $C_\infty(\mathfrak{V}; \mathfrak{M})$ which can be obtained from the vanishing theorem for the relative subcomplex. We will not comment on this here but rather wait until we prove the analogous theorems for the NSR string in Chapter VII. The proof for the bosonic string can be read off from that one.

The No-Ghost Theorem

Finally we come to perhaps the most important application of the vanishing theorem. In this subsection we shall prove the no-ghost theorem for the open bosonic string as long as the center of mass momentum is different from zero. The method we shall follow was explained in Section V.4 and consists of comparing the signature and the character of the relative complex.

The characters are easy to compute. We first compute the Virasoro character of $\mathfrak{M}(k)$. As a Fock space,

$$\mathfrak{M}(k) \cong \bigotimes_{\mu=0}^{25} \bigotimes_{n=1}^{\infty} S_n^\mu \ , \tag{VI.2.22}$$

where S_n^μ is the one particle Hilbert space corresponding to the oscillator a_{-n}^μ which is isomorphic to the polynomial algebra in one variable: a_{-n}^μ. Therefore using the fact that the trace is multiplicative over tensor products we obtain

$$\begin{aligned} \mathrm{ch}_q\ \mathfrak{M}(k) &\equiv \mathrm{Tr}_{\mathfrak{M}(k)}\, q^{L_0^{(X)}} \\ &= q^{\frac{1}{2}k^2} \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n^\mu}\, q^{a_{-n}^\mu a_{n\mu}} \\ &= q^{\frac{1}{2}k^2} \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} q^{mn} \\ &= q^{\frac{1}{2}k^2} \prod_{n=1}^{\infty} (1-q^n)^{-26} \ . \end{aligned} \tag{VI.2.23}$$

The character of the semi-infinite forms can be read from (VI.1.19) or computed from the combinatorics of two fermions and one gets

$$\mathrm{ch}_q \bigwedge\nolimits_{\infty} = q^{-1} \prod_{n=1}^{\infty} (1-q^n)^2 \ ; \tag{VI.2.24}$$

whence the total character is given by

$$\boxed{\mathrm{ch}_q\ \mathfrak{M}(k) \otimes \bigwedge\nolimits_{\infty} = q^{\frac{1}{2}k^2-1} \prod_{n=1}^{\infty} (1-q^n)^{-24}} \ , \tag{VI.2.25}$$

which shows the same combinatorics as the light cone count: 24 bosonic oscillators.

We now come to the computation of the signature of the complex. As explained in Section V.4 all we need is to construct the self-adjoint involution $\mathcal{C}$. Since our complex is a Fock space it is very easy to determine $\mathcal{C}$. The inner product on a Fock space is completely specified by the norm of the Fock vacuum and the hermiticity properties of the operators. Besides in a Fock space it is very easy to identify where the negative norm states come from. In $\mathfrak{M}(k)$ the negative norm states come from the fact that the Minkowski metric is indefinite. Therefore we define $\mathcal{C}$ such that the adjoint of a time-like oscillator develops a sign which cancels the sign in the Minkowski metric: in other words,

$$\mathcal{C} a_n^\mu \mathcal{C} = (-1)^{\delta_{\mu 0}} a_n^\mu \ . \tag{VI.2.26}$$

For the semi-infinite forms clearly the negative norms arise from the fact that the adjoint of a ghost creation operator is not a ghost annihilation operator but an antighost annihilation operator. Therefore $\mathcal{C}$ must correspond to ghost conjugation. Actually, $\mathcal{C}$ is defined as ghost conjugation on the ghost and antighost modes of [**40**], which are unitarily related to the usual ones (*cf. e.g.*, [**108**]). On the usual ghost modes we must not only conjugate but also perform a unitary transformation. The details are given in Appendix VII.C which defines $\mathcal{C}$ for all modes appearing in the NSR string.

With this definition of $\mathcal{C}$ is straight forward to compute the signatures of the complexes. For $\mathfrak{M}(k)$ we have

$$\begin{aligned} \operatorname{sgn}_q \mathfrak{M}(k) &= \operatorname{Tr}_{\mathfrak{M}(k)} \mathcal{C}\, q^{L_0^{(X)}} \\ &= q^{\frac{1}{2}k^2} \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \operatorname{Tr}_{S_n^\mu} \mathcal{C}\, q^{a_{-n}^\mu a_{n\mu}} \\ &= q^{\frac{1}{2}k^2} \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} ((-1)^{\delta_{\mu,0}} q^n)^m \\ &= q^{\frac{1}{2}k^2} \prod_{n=1}^{\infty} (1+q^n)^{-1} \cdot (1-q^n)^{-25} \ . \end{aligned} \tag{VI.2.27}$$

The signature of the semi-infinite forms is also easy to compute noticing that, as a Fock space, $\bigwedge_\infty \cong \bigotimes_{n=1}^\infty A_n$, where A_n is the Hilbert space corresponding to the ghost creation modes $\{b_{-n}, c_{-n}\}$; *i.e.*, A_n is isomorphic to the exterior algebra on two generators: b_{-n} and c_{-n}. With these remarks behind us we can easily compute

$$\operatorname{sgn}_q \bigwedge_\infty = \operatorname{Tr}_{\bigwedge_\infty} \mathcal{C}\, q^{L_0^{\text{gh}}}$$

$$
\begin{aligned}
&= q^{-1} \prod_{n=1}^{\infty} \mathrm{Tr}_{A_n} \mathcal{C}\, q^{n(c_{-n}b_n + b_{-n}c_n)} \\
&= q^{-1} \prod_{n=1}^{\infty} (1-q^{2n}) \\
&= q^{-1} \prod_{n=1}^{\infty} (1-q^n)\cdot(1+q^n)\ . \qquad \text{(VI.2.28)}
\end{aligned}
$$

The total signature is then

$$
\mathrm{sgn}_q\ \mathfrak{M}(k)\otimes \bigwedge\nolimits_\infty = q^{\frac{1}{2}k^2-1}\prod_{n=1}^{\infty}(1+q^n)^{-1}\cdot(1-q^n)^{-25}\cdot(1-q^n)\cdot(1+q^n)\ ; \qquad \text{(VI.2.29)}
$$

which clearly agrees with the computation of the character. Therefore, as long as $k \neq 0$ (for which there is no vanishing theorem) there are no negative norm states in the physical space.

Appendix A. Computation of $H_\infty(\mathfrak{n}_-;\mathfrak{M})$

This is a technical appendix where we prove (VI.1.49). This is the analog for the semi-infinite cohomology of the classical theorem[63] that the cohomology $H(\mathfrak{g};\mathfrak{U}(\mathfrak{g}))$ of any Lie algebra with coefficients in a free module is zero except in zero dimension where it is one-dimensional. In fact, this theorem is also true for Lie superalgebras. We provide a proof in Appendix VII.A for the case of the $N=1$ superconformal algebras; although it is equally easy to generalize this proof for any Lie superalgebra.

Theorem VI.A.1. *Let $\mathfrak{M}\cong\mathfrak{U}(\mathfrak{n}_-)$ be a free $\mathfrak{n}_-$-module. Then*

$$
H^m_\infty(\mathfrak{n}_-;\mathfrak{M}) \cong \begin{cases} \mathbb{C} & \textit{for } b=0 \\ 0 & \textit{otherwise} \end{cases} . \qquad \text{(VI.A.2)}
$$

Proof: Notice that the universal enveloping algebra—being a quotient of the (graded) tensor algebra $\mathfrak{T}(\mathfrak{n}_-)$—inherits a filtration from the canonical filtration of $\mathfrak{T}(\mathfrak{n}_-)$. This, in turn, induces a filtration on $\mathfrak{M}$ via the isomorphism $\mathfrak{M}\cong\mathfrak{U}(\mathfrak{n}_-)$. This allows us to filter the complex $C\equiv\mathfrak{M}\otimes\bigwedge_\infty\mathfrak{n}'_-$ as follows:

$$
\mathrm{fdeg}\,(m\otimes\omega) = \mathrm{fdeg}\,m + \mathrm{fdeg}\,\omega\ , \qquad \text{(VI.A.3)}
$$

where, if $\omega = e'_{i_1}\wedge e'_{i_2}\wedge\cdots$, its filtration degree is defined to be the number of missing $e'_i\in\mathfrak{n}'_-$ in ω. Define $F^pC\equiv\{w\in C\mid \mathrm{fdeg}\,w\le p\}$. Then $F^pC\subseteq F^{p+1}C$, $F^{-1}C=0$ and $\cup_p F^pC =$

C. Although notice that the filtration is not bounded. This is fine, since we will not make use of a spectral sequence. Consider the cohomology of the associated graded complex $\mathrm{Gr}\, C$ with the induced differential. The differential on C has two terms, each homogeneous of different filtration degree: $\pi(e_i)\varepsilon(e'_i)$ which has fdeg $= 0$; and $\imath(\big[e_i\,,\, e_j\big])\varepsilon(e'_i)\varepsilon(e'_j)$ which has fdeg $= 1$. Therefore only the first term survives upon going to the graded object. Therefore, if $\underbrace{e'_{i_1} \wedge e'_{i_2} \wedge \cdots}_{\text{fdeg } = p} \otimes \underbrace{e_{k_1} \cdots e_{k_m}}_{\text{fdeg } = m}$ defines a class in $\mathrm{Gr}^{p+m} C$,

$$d\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots \otimes e_{k_1} \cdots e_{k_m} = \sum_n e'_n \wedge e'_{i_1} \wedge e'_{i_2} \wedge \cdots \otimes e_n \cdot e_{k_1} \cdots e_{k_m} \ , \tag{VI.A.4}$$

where the sum is actually finite and where all expressions (here and in the rest of the proof) are modulo $F^{p+m-1}C$. We know define a chain homotopy Γ as follows

$$\Gamma\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots \otimes e_{k_1} \cdots e_{k_m} = \sum_{j=1}^{m} \imath(e_{k_j})\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots \otimes e_{k_1} \cdots \widehat{e_{k_j}} \cdots e_{k_m} \ , \tag{VI.A.5}$$

where a $\widehat{\ }$ adorning a symbol denotes its omission. A short calculation shows that

$$(d\Gamma + \Gamma d)\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots \otimes e_{k_1} \cdots e_{k_m} = (p+m)\, e'_{i_1} \wedge e'_{i_2} \wedge \cdots \otimes e_{k_1} \cdots e_{k_m} \ . \tag{VI.A.6}$$

Therefore unless $p + m = 0$, the cohomology is trivial, and if it $p + m = 0$ then it is one-dimensional since there is only one cochain and it is automatically a cocycle. From the long exact sequence in cohomology associated to the exact sequence of complexes for $p \geq 1$

$$0 \longrightarrow F^{p-1}C \longrightarrow F^p C \longrightarrow \mathrm{Gr}^p C \longrightarrow 0 \tag{VI.A.7}$$

we find that $H^n(F^pC) \cong H^n(F^{p-1}C)$ for $n \neq 0$. Since $H^n(F^0C) = 0$ for $n \neq 0$ we find that $H^n(C) = 0$ for $n \neq 0$. The zeroth dimensional cohomology $H^0(C)$ can be computed explicitly: there is only one cochain, it is a cocycle, and there are no coboundaries. ■

Chapter Seven:

The BRST Cohomology of the NSR String

In Chapter VI, we identified the BRST cohomology of the open bosonic string with the subcomplex of semi-infinite forms of the Virasoro algebra relative to the "Cartan subalgebra" spanned by ℓ_0 and c. In this chapter we extend that construction to the representations of the super-Virasoro algebras appearing in the NSR string.

This chapter is organized as follows. Section 1 discusses in detail the cohomology of the relative BRST subcomplex of the Neveu-Schwarz sector of the NSR string. Using a straight-forward generalization of the method used in Chapter VI, we prove a vanishing theorem for this cohomology. As before, the vanishing is induced from the vanishing of cohomology at the E_1 term of the spectral sequence associated to a certain filtration of this complex. In order to prove the vanishing theorem for the E_1 term of the spectral sequence, we use a basic result from the semi-infinite cohomology theory of Lie superalgebras. We have never seen a published proof of this theorem, although it is the semi-infinite analog of the Lie superalgebra version of the theorem which states that the cohomology of any Lie superalgebra with coefficients in a free module is trivial except in dimension zero, where it is one-dimensional. Fuks[67] hints that this latter theorem is a straight-forward generalization of the similar theorem for Lie algebras and, in fact, it is. We prove the semi-infinite version of the theorem in Appendix A for the special cases we need in this paper. It is obvious, however, that the proof goes through unmodified for the general case.

In Section 2 we consider the Ramond sector. This is somewhat more complicated because of the existence of the superconformal ghosts' zero modes. In fact there has not appeared in the literature a unique treatment of these zero modes and thus we treat them in two different ways. The cohomologies turn out to be isomorphic although one of them does not admit a grading by ghost number. Therefore for this case the vanishing theorem does not make sense. The proof of the vanishing theorem in the case where it does make sense is slightly more complicated than the Neveu-Schwarz sector. Matters are complicated by the fact that the superconformal generator f_0 does not act reducibly in the complex. What we do is cook up a spectral sequence converging to the first term of a spectral sequence converging to what we want. It is for this first term of the first spectral sequence that we can prove a vanishing theorem. Then the vanishing propagates through all spectral sequences to produce the desired vanishing in the final limit term.

In Section 3 we use spectral sequences again to infer the vanishing theorem for the full BRST complex. This complex is half-integrally graded and what we show is that its cohomology is trivial except at ghost number $\pm\frac{1}{2}$. The spectral sequence used in this case is the one associated to one of the two canonical filtrations of a double complex. A remarkable result of this section is that the cohomology of the full complex in the Ramond sector is only finitely degenerate even though the superconformal zero modes make the cochains themselves infinitely degenerate. The degeneracy is a two-fold degeneracy, just like in the Neveu-Schwarz sector.

Finally in Section 4 we prove the no-ghost theorems for the NSR string using the vanishing theorems proven earlier. Specifically what we prove is that the inherited norm on the BRST cohomology of the relative subcomplex (ignoring the ghosts' zero modes) is positive definite. This is a straight-forward application of the methods introduced in Chapter V, for which we need to show that we can find a positive-definite inner product for the Fock space where the BRST operator acts. Appendix B briefly describes this inner product. In order to prove the no-ghost theorem for both representations of the superconformal zero modes it is necessary to show a special isomorphism between the cohomologies arising from these complexes. For this we need a technical result which we leave for Appendix C. This result is a straight-forward generalization to the NSR string of the similar result found in **[60]** for the open bosonic string.

1. The Neveu-Schwarz Sector

In this section we define the relative subcomplexes for the super-Virasoro algebra appearing in the Neveu-Schwarz sector of the NSR string and we prove a vanishing theorem for its cohomology. We use the Poincaré duality proven in Chapter V for the BRST cohomology of a Fock space possessing a positive definite inner product. In Appendix C we construct this inner product for the Fock space of the NSR string.

Let $\mathfrak{N}$ (my apologies to John Schwarz) denote the centrally extended complexified super-Virasoro algebra appearing in the Neveu-Schwarz sector of the NSR string. This is a Lie superalgebra whose even part is the Virasoro algebra $\mathfrak{V}$ defined by (VI.2.1). The odd part of $\mathfrak{N}$ is graded according to $\mathfrak{G} = \bigoplus_{r\in\mathbb{Z}+\frac{1}{2}} \mathfrak{G}_r$, where $\mathfrak{G}_r$ is spanned by g_r. These generators obey

$$\{g_r\,,\,g_s\} = 2\ell_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r,-s} \tag{VII.1.1}$$

and

$$[\ell_n\,,\,g_r] = (\frac{n}{2} - r)g_{r+n}\ . \tag{VII.1.2}$$

Supplementing these relations by the assertion that c is central, fully defines the super-Virasoro algebra in this sector. Again we define $\mathfrak{N}_\pm = \mathfrak{V}_\pm \oplus \bigoplus_{\pm r>0} \mathfrak{G}_r$.

As is well known the ghost Fock space of the Neveu-Schwarz sector carries a representation of $\mathfrak{N}$ where $c \mapsto -15\,\mathbf{1}$ and $\ell_n \mapsto L_n^{\text{gh}}$, $g_r \mapsto G_r^{\text{gh}}$. The Fock space of the string oscillators also carries a representation of $\mathfrak{N}$ with the opposite central charge—in the critical dimension—and where $\ell_n \mapsto L_n^{\text{mat}}$ and $g_r \mapsto G_r^{\text{mat}}$. Let us denote by L_n and G_r the operators representing ℓ_n and g_r respectively in the full Fock space (including ghosts). The formulas for these generators are standard and can be found for instance in **[108]**.

It was proven by Brower and Friedman[**109**] that this representation is fully reducible into Verma modules. That is, it can be written as an infinite direct sum of Verma modules whose highest weight vectors are obtained by repeated application of the creation operators in the full spectrum-generating algebra[18]. Since the BRST operator commutes with the $\{L_n\}$ and the $\{G_r\}$ it respects this decomposition and hence we may restrict our attention to one such Verma module at a time when computing the BRST cohomology. Let $\mathfrak{M}$ denote one such Verma module. As in Chapter VI, we denote the BRST (or semi-infinite) cohomology[19] of the $\mathfrak{N}$ superalgebra with coefficients in $\mathfrak{M}$ by $H_\infty(\mathfrak{N};\mathfrak{M})$. This is the cohomology of the BRST operator Q acting on the graded complex $C_\infty(\mathfrak{N};\mathfrak{M}) = \bigoplus_n C_\infty^n(\mathfrak{N};\mathfrak{M})$ where

$$C_\infty^n(\mathfrak{N};\mathfrak{M}) = C_\infty^n(\mathfrak{N}) \otimes \mathfrak{M}\ , \tag{VII.1.3}$$

where $C_\infty^n(\mathfrak{N})$ is the subspace of the ghost Fock space at ghost number n.

---

18 Strictly speaking, this is not true for the case of zero center of mass momentum. In this case the highest weight vector is also annihilated by $G_{-\frac{1}{2}}^{\text{mat}}$ and hence does not generate a Verma module. For this case the theorem in Appendix A does not hold and neither does our proof of the vanishing theorem. Here, however, the BRST cohomology is easy to compute explicitly.

19 To be precise, this is the semi-infinite cohomology relative to the center. In other words, from now on $\mathfrak{N}$ denotes the unextended Neveu-Schwarz algebra.

Let us define the subcomplex $C_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{M})$ relative to $\mathfrak{V}_0$ by

$$C_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{M}) = \{\omega \in C_\infty(\mathfrak{N}; \mathfrak{M}) \mid L_0\omega = b_0\omega = 0\} \ . \tag{VII.1.4}$$

For the sake of notation let us abbreviate $C_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{M})$ by C_∞. Notice that C_∞ is finite dimensional. From the identity $\left\{Q \, , \, b_0\right\} = L_0$ we notice that this indeed defines a subcomplex. That is, $QC_\infty \subseteq C_\infty$. We denote its cohomology by $H_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{M})$.

Let

$$|i,j,k,l,m,q\rangle = \prod_{r\geq\frac{1}{2}} \gamma_{-r}^{i_r} \prod_{r\geq\frac{1}{2}} \beta_{-r}^{j_r} \prod_{n>0} c_{-n}^{k_n} \prod_{n>0} b_{-n}^{l_n} \, |0\rangle \otimes \prod_{r\geq\frac{1}{2}} G_{-r}^{q_r} \prod_{n>0} L_{-n}^{m_n} \, |p\rangle \tag{VII.1.5}$$

denote a vector in C_∞ with $|p\rangle$ a highest weight vector of momentum p such that

$$\frac{1}{2}(p^2-1) = -\left(\sum_n (k_n+l_n+m_n)n + \sum_r (i_r+j_r+q_r)r\right) = -\frac{N}{2} \ , \tag{VII.1.6}$$

for some non-negative integer N. Define the filtration degree as

$$\text{fdeg}\ |i,j,k,l,m,q\rangle = \sum_n (k_n-l_n-m_n)n + \sum_r (i_r-j_r-q_r)r \ . \tag{VII.1.7}$$

This allows us to define a half-integral filtration of C_∞ by

$$F^pC_\infty = \{\omega \in C_\infty \mid \text{fdeg}\,\omega \geq p\} \ . \tag{VII.1.8}$$

First of all notice that $F^pC_\infty \supseteq F^{p+\frac{1}{2}}C_\infty$ and that the filtration is bounded. Finally we must check that this indeed defines a filtered complex, that is, $QF^pC_\infty \subseteq F^pC_\infty$. This is done by examining the filtration degree of the homogeneous terms in Q and making sure they are all non-negative. From (VII.1.7) we can read off the filtration degree of all the oscillators which make up Q and we find them to be the following:

Operator	Filtration Degree
c_n	$\lvert n\rvert$
b_n	$-\lvert n\rvert$
γ_r	$\lvert r\rvert$
β_r	$-\lvert r\rvert$
L_n^{mat}	n
G_r^{mat}	r

Therefore it is trivial to verify that all terms in Q have zero filtration degree except for the terms $L_n^{\text{mat}}c_{-n}$ for $n>0$ which have filtration degree $2n$; the terms $G_r^{\text{mat}}\gamma_{-r}$ for $r>0$

which have filtration degrees $2r$; the terms $c_m\, c_n\, b_{-(m+n)}$ for $\operatorname{sign}(m) \neq \operatorname{sign}(n)$ which have filtration degree $|m|+|n|-|m+n|$; the terms $\gamma_r\, \gamma_s\, b_{-(r+s)}$ for $\operatorname{sign}(r) \neq \operatorname{sign}(s)$ which have filtration degree $|r|+|s|-|r+s|$; and finally the terms $\gamma_r\, c_n\, \beta_{-(r+n)}$ for $\operatorname{sign}(r) \neq \operatorname{sign}(n)$ which have filtration degree $|r|+|n|-|r+n|$. Hence all terms have non-negative filtration degrees and $\{F^p C_\infty\}$ indeed defines a bounded filtered complex.

By Theorem II.1.32 there exists a spectral sequence converging finitely to $H_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{M})$ whose E_1 term is the cohomology of the associated graded complex $\operatorname{Gr} C_\infty = \bigoplus_p \operatorname{Gr}^p C_\infty$, where $\operatorname{Gr}^p C_\infty = F^p C_\infty / F^{p+\frac{1}{2}} C_\infty$. The differential in this complex is precisely the part of Q with zero filtration degree since the terms with positive filtration degree will automatically map to zero in $\operatorname{Gr} C_\infty$. By the above discussion the induced differential can be seen to be the differential on the complex

$$C^{L_0} = \Big(C(\mathfrak{N}_+) \otimes C_\infty(\mathfrak{N}_-; \mathfrak{M}) \Big)^{L_0} , \tag{VII.1.9}$$

where $C(\mathfrak{N}_+)$ denotes the Lie superalgebra cochains[20] of $\mathfrak{N}_+$ with coefficients in the trivial representation, ^{L_0} denotes the L_0 invariant subspace and $\mathfrak{M}$ is to be thought of as a representation of only $\mathfrak{N}_-$. We remark that this particular expression makes it very easy to keep track of ghosts and antighosts separately. In fact, the subspace of C^{L_0} with c ghosts and b antighosts is just

$$(C^{L_0})^{b,c} = \Big(C^c(\mathfrak{N}_+) \otimes C^b_\infty(\mathfrak{N}_-; \mathfrak{M}) \Big)^{L_0} . \tag{VII.1.10}$$

We now compute this cohomology. Since L_0 is diagonalizable in C

$$C = C^{L_0} \oplus L_0(C) , \tag{VII.1.11}$$

where $L_0(C)$ denotes the image of C under L_0. Since L_0 commutes with Q we deduce that

$$QC^{L_0} \subseteq C^{L_0} \tag{VII.1.12}$$

and

[20] Strictly speaking we mean here cochains of finite support. That is, super-symmetric linear functionals of finite rank. They correspond to polynomials in the ghost creation operators.

$$QL_0(C) \subseteq L_0(C) \ . \tag{VII.1.13}$$

Now suppose that ω is an L_0 invariant cocycle. If $\omega = Q\phi$ then we can chose ϕ to be L_0 invariant as well. To see this notice that if ϕ is not L_0 invariant already then by (VII.1.11) $\phi = \phi_0 + \psi$ where $\phi_0 \in C^{L_0}$ and $\psi \in L_0(C)$. Then $\omega = Q\phi = Q\phi_0 + Q\psi$. By (VII.1.12) and (VII.1.13) $Q\psi = 0$ and therefore $\omega = Q\phi_0$. Hence we have proven the inclusion

$$H(C^{L_0}) \subseteq H(C)^{L_0} \ . \tag{VII.1.14}$$

The reverse inclusion is easier. If $[\omega] \in H(C)^{L_0}$ then $L_0\omega = (Q\,b_0 + b_0\,Q)\,\omega = 0$ since $Q\omega = 0$ and $b_0\omega = 0$. Therefore $\omega \in C^{L_0}$ defines a class in $H(C^{L_0})$ which, if trivial, is trivial also in $H(C)^{L_0}$. Therefore we conclude that

$$H(C)^{L_0} \cong H(C^{L_0}) \ . \tag{VII.1.15}$$

But by the Künneth formula (II.1.56)

$$H(C) \cong H(\mathfrak{N}_+) \otimes H_\infty(\mathfrak{N}_-;\mathfrak{M}) \ ; \tag{VII.1.16}$$

whence, keeping track of ghosts and antighosts separately, the E_1 term in the spectral sequence is

$$E_1^{b,c} = \left(H^c(\mathfrak{N}_+) \otimes H^b_\infty(\mathfrak{N}_-;\mathfrak{M})\right)^{L_0} \ . \tag{VII.1.17}$$

In Appendix A we prove that $H^b_\infty(\mathfrak{N}_-;\mathfrak{M}) = 0$ for $b \neq 0$ and $H^0_\infty(\mathfrak{N}_-;\mathfrak{M}) \cong \mathbb{C}$. Thus,

$$\begin{aligned} E_1^m =& \bigoplus_{c-b=m} E_1^{b,c} \\ =& E_1^{0,m} \\ =& \left(H^m(\mathfrak{N}_+)\right)^{L_0} \\ \therefore \quad E_1^m =& 0 \quad \text{for } m < 0 \ . \end{aligned}$$

But $(E_r^m) \Rightarrow H^m_\infty(\mathfrak{N},\mathfrak{V}_0;\mathfrak{M})$, thus $H^m_\infty(\mathfrak{N},\mathfrak{V}_0;\mathfrak{M}) = 0$ for $m < 0$. Taking into account all the Verma modules $\mathfrak{M}$ we find that $H^m_\infty(\mathfrak{N},\mathfrak{V}_0;\mathfrak{H}) = 0$ for $m < 0$, where $\mathfrak{H}$ is the full Fock space (including ghosts) of the Neveu-Schwarz string.

Now in Appendix C we show that there exists a positive definite inner product in $\mathfrak{H}$. This and the obvious fact that $\mathfrak{H}$ breaks up into finite dimensional subspaces stabilized by Q allow us to use (V.2.7) or (VI.1.36) to deduce that

$$H^m_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{H}) \cong H^{-m}_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{H}) \; ; \qquad \text{(VII.1.18)}$$

which gives the vanishing theorem for the relative subcomplex

$$\boxed{H^{m\neq 0}_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{H}) = 0} \; . \qquad \text{(VII.1.19)}$$

In Section 3 we will prove that this induces a vanishing theorem in the full complex $H_\infty(\mathfrak{N}; \mathfrak{H})$ as well.

2. The Ramond Sector

Let $\mathfrak{R}$ denote the centrally extended complexified super-Virasoro algebra appearing in the Ramond sector of the NSR string. This algebra is very similar to the Neveu-Schwarz algebra except that the odd part $\mathfrak{F} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{F}_n$ is integrally graded, where $\mathfrak{F}_n$ is spanned by f_n. The even subalgebra is still given by (VI.2.1). The rest of the algebra obeys

$$\left\{f_m \, , \, f_n\right\} = 2\ell_{n+m} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m,-n} \qquad \text{(VII.2.1)}$$

and

$$\left[\ell_m \, , \, f_n\right] = (\frac{m}{2} - n) f_{m+n} \; . \qquad \text{(VII.2.2)}$$

Again we impose that c is central and as before we define $\mathfrak{R}_\pm = \mathfrak{V}_\pm \oplus \bigoplus_{\pm n>0} \mathfrak{F}_n$.

The ghost Fock space of the Ramond sector carries a representation of $\mathfrak{R}$ with $c \mapsto -15\,\mathbf{1}$, $\ell_n \mapsto L^{\text{gh}}_n$, $f_n \mapsto F^{\text{gh}}_n$. The Fock space of the string oscillators also carries a representation of $\mathfrak{R}$ with the opposite central charge (in the critical dimension) and where $\ell_n \mapsto L^{\text{mat}}_n$ and $f_n \mapsto F^{\text{mat}}_n$. Finally, let us denote by L_n and F_n the operators representing ℓ_n and f_n respectively in the full Fock space (including ghosts). Again the formulas for these generators are standard and we refer the reader to **[108]**.

In **[109]** Brower and Friedman claim to have proven full reducibility of this representation, although they do not write down the explicit spectrum generating algebra. Therefore, just as in the Neveu-Schwarz case we can decompose the string Fock space into Verma modules and thus restrict our attention to one such Verma module at a time when computing the BRST cohomology.[21]

Let $\mathfrak{M}$ be one such Verma module and let $H_\infty(\mathfrak{R};\mathfrak{M})$ denote the BRST cohomology[22] on the graded complex $C_\infty(\mathfrak{R};\mathfrak{M}) = \bigoplus_n C^n_\infty(\mathfrak{R};\mathfrak{M})$ where again

$$C^n_\infty(\mathfrak{R};\mathfrak{M}) = C^n_\infty(\mathfrak{R}) \otimes \mathfrak{M} \ , \tag{VII.2.3}$$

where $C^n_\infty(\mathfrak{R})$ is the subspace of the ghost Fock space at ghost number n.

There are two natural subcomplexes to consider. One could consider the subcomplex relative to the zeroth subalgebra $\mathfrak{R}_0 = \mathfrak{V}_0 \oplus \mathfrak{F}_0$ or relative to just the even part $\mathfrak{V}_0$. The choice of subcomplex has to do with the choice of Hilbert space $\mathcal{H}$ for the zero modes of the superconformal ghosts. The reason is the following. In order to consider the subcomplex relative to the full zeroth subalgebra we have to be able to impose the condition $\beta_0\,\omega = 0$. This may or may not be possible as we shall now see.

The algebra obeyed by the ghost zero modes is the Heisenberg algebra

$$\left[\gamma_0\,,\,\beta_0\right] = 1 \ , \tag{VII.2.4}$$

and the hermiticity conditions are such that γ_0 is hermitian and β_0 is anti-hermitian. The unique[23] representation of this algebra as operators in a Hilbert space (*i.e.*, with a positive definite inner product) is the Schrödinger representation in which $\mathcal{H}$ is isomorphic with $\mathfrak{L}^2(\mathbb{R}, dx)$ and where β_0 is represented by i times the multiplication operator: $(\beta_0\,h)(x) = i\,x\,h(x)$ and γ_0 is the momentum operator: $(\gamma_0\,h)(x) = -i\,h'(x)$. If this is the case we cannot impose the equation $\beta_0\,\omega = 0$ because the multiplication operator has no eigenvalues in $\mathfrak{L}^2(\mathbb{R}, dx)$. In this case we would look at the subcomplex relative to $\mathfrak{V}_0$.

If on the other hand—like many other authors, notably Henneaux[**37**]—we treat γ_0 and β_0 as creation and annihilation operators (respectively) the Hilbert space is now (the

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[21] Just as before the vanishing theorem as it stands does not apply to the case where the center of mass momentum is zero. In this case the cohomology is again easy to compute explicitly.

[22] Again this should be relative to the center. Therefore from now on $\mathfrak{R}$ denotes the unextended Ramond algebra.

[23] Strictly speaking the uniqueness is proven for the Weyl form of the Heisenberg algebra.

completion of) the polynomial algebra in one variable $\mathbb{C}[\gamma_0]$. In this case the hermiticity conditions that induce a positive definite inner product are such that γ_0 and β_0 are mutually adjoint. In this case we can consider the subcomplex relative to the full zeroth subalgebra $\mathfrak{R}_0$. It may seem unnatural to alter the hermiticity properties inherited from the classical fields, but for operators which do not correspond to physical observables the hermiticity properties are not too crucial. There is however a major drawback. Changing the hermiticity properties of γ_0 and β_0 changes the hermiticity properties of the BRST operator: it is no longer hermitian. This means that it is no longer guaranteed that the cohomology space inherits a well-defined (*i.e.*, independent of the representative) inner product. In fact, a necessary and sufficient condition is $\ker\ Q \subset \ker\ Q^\dagger$. In particular, since $\mathrm{im}\ Q \subset \ker\ Q$, it is necessary that $Q^\dagger\, Q \equiv 0$. In this case it can be checked explicitly that this does not hold.

On the other hand keeping the original hermiticity conditions has one major inconvenience: the cohomology is not graded by ghost number and hence the vanishing theorem makes no sense. This is due essentially to the fact that the ghost number operator has no eigenvalues in $\mathfrak{L}^2(\mathbb{R}, dx)$. Still, we can find a particular class of representatives which does admit a grading. In this case the cohomology agrees with the one obtained by altering the hermiticity properties of γ_0 and β_0, for which we can prove a vanishing theorem.

Therefore we will consider both choices of hermiticity properties. We will see that both cohomologies are isomorphic as ungraded vector spaces; and we will prove a vanishing theorem for the graded case.

<u>The Henneaux Representation</u>

Let us first assume that $\mathcal{H} = \mathbb{C}[\gamma_0]$. It is then possible to consider the relative subcomplex $C_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$. This complex, which we abbreviate by C_∞, is given by

$$C_\infty = \{\omega \in C_\infty(\mathfrak{R}; \mathfrak{M}) \mid F_0\omega = b_0\omega = \beta_0\omega = 0\}\ . \tag{VII.2.5}$$

Just as in the Neveu-Schwarz case, it is finite dimensional. Hence a typical vector in C_∞ is a linear combination of monomials

$$|i, j, k, l, m, q\rangle = \prod_{n>0} \gamma_{-n}^{i_n} \prod_{n>0} \beta_{-n}^{j_n} \prod_{n>0} c_{-n}^{k_n} \prod_{n>0} b_{-n}^{l_n}\ |0\rangle \otimes \prod_{n>0} F_{-n}^{q_n} \prod_{n>0} L_{-n}^{m_n}\ |p\rangle\ , \tag{VII.2.6}$$

where $|p\rangle$ a highest weight vector of momentum p such that

$$\frac{1}{2}p^2 = -\sum_n (i_n + j_n + q_n + k_n + l_n + m_n)n = -N\ , \tag{VII.2.7}$$

for some non-negative integer N. Define the filtration degree as

$$\text{fdeg } |i,j,k,l,m,q\rangle = \sum_n (i_n - j_n - q_n + k_n - l_n - m_n)n \ . \tag{VII.2.8}$$

Just as in the Neveu-Schwarz case the filtration defined by this degree is bounded and defines a filtered complex. Therefore the theorem in Section 2 applies, yielding the existence of a spectral sequence which converges finitely to $H_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$; and whose E_1 term is the differential for the complex

$$C^{F_0} = \Big(C(\mathfrak{R}_+) \otimes C_\infty(\mathfrak{R}_-; \mathfrak{M}) \Big)^{F_0} \ . \tag{VII.2.9}$$

In this case, however, we cannot use the arguments used for the Neveu-Schwarz case because F_0 does not act reducibly. In fact, in the subspace left invariant by L_0, F_0 is nilpotent and not identically zero. Therefore ker $F_0 \cap$ im $F_0 \neq 0$ and a decomposition à la (VII.1.11) is impossible. Therefore we follow a completely different line of approach. We find a spectral sequence converging to $H(C^{F_0})$ which preserves the grading by ghost number and for whose E_1 term we can prove a vanishing theorem.

The spectral sequence in question will be that associated to one of the canonical filtrations of a double complex. The double complex is constructed as follows. For any ghost number p the space $(C^p)^{L_0}$ naturally affords a representation of F_0. Moreover since $F_0^2 = L_0$ the action of F_0 is nilpotent and its cohomology may be defined. We define

$$\mathbb{K}^{p,q} = C^q(\mathfrak{F}_0; (C^p)^{L_0}) \ , \tag{VII.2.10}$$

the q-cochains of the $\mathfrak{F}_0$ with coefficients in $(C^p)^{L_0}$. Let $\delta : \mathbb{K}^{p,q} \to \mathbb{K}^{p,q+1}$ to be the coboundary operator for $\mathfrak{F}_0$ cochains. It is defined by

$$\delta\, (f_0')^q \otimes \omega = (f_0')^{q+1} \otimes F_0 \omega \ , \tag{VII.2.11}$$

for $\omega \in (C^p)^{L_0}$. Similarly define $d : \mathbb{K}^{p,q} \to \mathbb{K}^{p+1,q}$ to be the trivial extension of the differential Q for C^{L_0}:

$$d\, (f_0')^q \otimes \omega = (f_0')^q \otimes Q\omega \ , \tag{VII.2.12}$$

for $\omega \in (C^p)^{L_0}$. Therefore the double complex can be represented as follows:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathbb{K}^{p,1} & \xrightarrow{d} & \mathbb{K}^{p+1,1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\cdots & \longrightarrow & \mathbb{K}^{p,0} & \xrightarrow{d} & \mathbb{K}^{p+1,0} & \longrightarrow & \cdots
\end{array}
$$

Since Q and F_0 anticommute so do d and δ. Therefore $D = d + \delta$ is nilpotent and computes the cohomology of the total complex $\mathbb{K} = \bigoplus_m \mathbb{K}^m$ where $\mathbb{K}^m = \bigoplus_p \mathbb{K}^{p,m-p}$.

Because C^{L_0} is finite-dimensional its grading by ghost number is bounded and therefore the total complex is finite in each dimension. Therefore we can use the results of Section 2 and deduce that there exist two spectral sequences converging to the total cohomology in each dimension. We now compute the early terms. We first look at the vertical δ cohomology. The space $Z_\delta^{p,q}$ of (p,q)–cocycles of δ is just $(f_0')^q \otimes (C^p)^{F_0}$ whereas the (p,q)–coboundaries are $(f_0')^q \otimes F_0 (C^p)^{L_0}$ for $q > 0$ whereas for $q = 0$ there are no coboundaries since there are no -1 cochains. Therefore the vertical cohomology is

$$
H_\delta^{p,q} = \begin{cases} 1 \otimes (C^p)^{F_0} & \text{for } q = 0 \\ (f_0')^q \otimes H_{F_0}((C^p)^{L_0}) & \text{for } q \neq 0 \end{cases}, \tag{VII.2.13}
$$

where $H_{F_0}((C^p)^{L_0})$ is the cohomology of the nilpotent operator F_0 in $(C^p)^{L_0}$. This space, however, turns out to be trivial[24]. Therefore the vertical cohomology is zero except in dimension zero where it is isomorphic to C^{F_0}.

The spectral sequence associated to the horizontal filtration has as $'E_1$ term the vertical cohomology and as $'E_2$ term $H_d(H_\delta)$. Therefore this is zero everywhere but in dimension zero and there it is just $H(C^{F_0})$. Because d_2 maps already between different rows we see that it is identically zero and so are all the higher d_r's. Hence the spectral sequence collapses

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[24] This follows from the following fact[**37**]. Let q be any Minkowski vector and let $F_0(q)$ denote the operator obtained from F_0 by replacing a_0^μ with q^μ. On $(C^p)^{L_0}$, $\{F_0\,,\, F_0(q)\} = (q \cdot k - k^2)$, where k is the eigenvalue of a_0^μ. Therefore as long as $k \neq 0$ we can always choose q such that $\{F_0\,,\, F_0(q)\} = 1$ whence the cohomology of F_0 is trivial. In the Ramond sector any on-shell (L_0-invariant) state with $k = 0$ corresponds to one of the degenerate vacua and hence it has manifestly zero ghost number.

and we have that the total cohomology is

$$H_D^m \cong H^m(C^{F_0}) \ . \tag{VII.2.14}$$

If we take the vertical filtration the first term in the spectral sequence is the horizontal cohomology H_d. Therefore the $''E_1$ is precisely

$$''E_1^{q,p} = (f_0')^q \otimes H^p(C^{L_0}) \ , \tag{VII.2.15}$$

where by an argument identical to that in the in the Neveu-Schwarz case we can show that $H^p(C^{L_0}) \cong H^p(C)^{L_0}$. By arguments identical to the ones in the Neveu-Schwarz sector — *i.e.*, using the Künneth formula and the theorem in Appendix A — it follows that $H^p(C)^{L_0}$ is zero for $p < 0$. Therefore

$$''E_1^m \cong \bigoplus_{q \geq 0} H^{m-q}(C^{L_0}) \ . \tag{VII.2.16}$$

Since $H^p(C^{L_0}) = 0$ for $p < 0$ we have that $''E_1^m = 0$ for $m < 0$. Therefore $''E_\infty^m = 0$ for $m < 0$. But by the theorem in Section 2, this limit term is also the total cohomology. Therefore $H^m(C^{F_0}) = 0$ for $m < 0$. But this is the E_1 term in a spectral sequence converging to $H_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$. Therefore we conclude that $H_\infty^m(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M}) = 0$ for $m < 0$ and the same for $H_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{H})$. By (V.2.7) or (VI.1.36), this implies the vanishing theorem

$$\boxed{H_\infty^{m \neq 0}(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{H}) = 0} \ . \tag{VII.2.17}$$

We will see in the next section that this implies a vanishing theorem for the cohomology of the full complex $C_\infty(\mathfrak{R}; \mathfrak{H})$

<u>The Schrödinger Representation</u>

Now let us assume that $\mathcal{H} = \mathfrak{L}^2(\mathbb{R}, dx)$. We find it convenient to work in a dense domain in which γ_0 and β_0 are defined. To this end let us introduce the operators a and $a^\dagger$ defined by

$$\beta_0 = \frac{1}{\sqrt{2}}(a^\dagger - a) \qquad \gamma_0 = \frac{1}{\sqrt{2}}(a^\dagger + a) \ , \tag{VII.2.18}$$

and let $\mathcal{H}$ be the completion of the polynomial algebra $\mathbb{C}[a^\dagger]$. Combining (VII.2.4) and (VII.2.18) we find that a and $a^\dagger$ obey $\left[a \, , \, a^\dagger\right] = 1$.

Let us define the subcomplex

$$C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M}) = \{\omega \in C_\infty(\mathfrak{R};\mathfrak{M}) \mid L_0\omega = b_0\omega = 0\} \ . \tag{VII.2.19}$$

To study the cohomology of this complex it is convenient to discuss the differentials occurring in the various complexes under study. The differential in the full complex $C_\infty(\mathfrak{R};\mathfrak{M})$ is the BRST operator Q. Making the dependence on the ghosts' zero modes manifest we can write it as

$$Q = c_0\, L_0 - 2b_0\, T - \gamma_0^2 b_0 + \mathbb{Q} \ , \tag{VII.2.20}$$

where

$$\mathbb{Q} = \beta_0\, K + \gamma_0\, F_0 + \mathcal{Q} \ . \tag{VII.2.21}$$

We don't need the explicit expressions for these operators but only the following relations which follow from the nilpotency of Q:

$$\mathbb{Q}^2 = 0 \qquad F_0^2 = L_0 \qquad \big[F_0\,,\, T\big] = K \qquad \mathcal{Q}^2 = 2L_0\, T + F_0\, K \ , \tag{VII.2.22}$$

and all other (anti)commutators vanish; in particular, $\big[T\,,\, K\big] = 0$.

The differential in the relative subcomplex $C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M})$ is $\mathbb{Q}$. Isolating the representation space of the superghosts' zero modes, this subcomplex can be written as $C \otimes \mathbb{C}[a^\dagger]$ which defines C. According to this decomposition the differential becomes

$$\mathbb{Q} = \mathcal{Q} \otimes \mathbf{1} + \frac{1}{\sqrt{2}}(F_0 + K) \otimes a^\dagger + \frac{1}{\sqrt{2}}(F_0 - K) \otimes a \ . \tag{VII.2.23}$$

In this subcomplex the following identities are satisfied

$$F_0^2 = 0 \qquad \mathcal{Q}^2 = F_0\, K \ . \tag{VII.2.24}$$

Hence the space C^{F_0} is a differential complex with respect to $\mathcal{Q}$. Notice that this complex is isomorphic to $C_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{M})$ in the Henneaux representation. Therefore their cohomologies are isomorphic as well. We will now prove that the cohomology of this complex, denoted by $H_{\mathcal{Q}}(C^{F_0})$ is isomorphic to $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M})$. But first we need a preliminary result.

Because $\big[F_0\,,\,T\big]=K$ and $\big[T\,,\,K\big]=0$ we can write

$$F_0+K=e^{-T}\,F_0\,e^{T}\ , \tag{VII.2.25}$$

which is well defined as it stands because C is finite dimensional. Also because C is finite dimensional any operator with non-zero ghost number[25] is automatically nilpotent. In particular, since T has ghost number 2, it is nilpotent and therefore $\exp\,(\alpha T)$ is an isomorphism for any complex number α. Because F_0 is nilpotent, F_0+K is also nilpotent and its cohomology is isomorphic to that of F_0: $\exp\,(-T)$ gives the isomorphism by (VII.2.25). As shown above, the cohomology of F_0 is trivial and, thus, so is the cohomology of F_0+K.

We now proceed to prove the isomorphism of $H_{\mathcal{Q}}(C^{F_0})$ and $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M})$. Let Ψ be a cocycle in $C\otimes\mathbb{C}[a^\dagger]$. Then we can write it as a polynomial with coefficients in C as follows

$$\Psi=\sum_{n=0}^{N}\psi_n\otimes(a^\dagger)^n\ , \tag{VII.2.26}$$

where $\psi_n\in C$ for all n. Then the fact that it is a cocycle implies that $(F_0{+}K)\psi_N=0$. By the vanishing of the cohomology of $F_0{+}K$ there exists a cochain ϕ such that $\psi_N{+}(F_0{+}K)\phi=0$. Therefore adding the coboundary $\mathbb{Q}(\phi\otimes(a^\dagger)^{N-1})$ to Ψ we get rid of the N^{th} order term in Ψ. Continuing in this fashion we can reduce Ψ to a constant monomial $\psi\otimes 1$, which is still a cocycle cohomologous to Ψ. The fact that it is a cocycle implies that $\mathcal{Q}\psi=0$ and $(F_0+K)\psi=0$. Therefore, using the fact that $\big[T\,,\,\mathcal{Q}\big]=0$, we see that $\exp\,(T)\,\psi$ obeys

$$\mathcal{Q}\,e^{T}\,\psi=0\qquad F_0\,e^{T}\,\psi=0\ , \tag{VII.2.27}$$

hence it defines a class $[\exp\,(T)\,\psi]$ in $H_{\mathcal{Q}}(C^{F_0})$. It is straight-forward to verify that if this class is trivial then the class $[\Psi]$ in $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M})$ is also trivial. Therefore we have an injection $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M})\hookrightarrow H_{\mathcal{Q}}(C^{F_0})$.

We now prove the reverse injection. Let ψ define a class in $H_{\mathcal{Q}}(C^{F_0})$. Then $[e^{-T}\,\psi\otimes 1]$ defines a class in $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{M})$. Now suppose that this class is trivial; that is,

$$e^{-T}\,\psi\otimes 1=\mathbb{Q}\,\Xi\qquad\text{for some }\Xi\ . \tag{VII.2.28}$$

Just as before we may add coboundaries to Ξ in such a way that (VII.2.28) is still obeyed and such that Ξ gets reduced to a constant monomial $\xi\otimes 1$. In that case, $F_0\exp\,(T)\,\xi=0$ and $\psi=\mathcal{Q}\exp\,(T)\,\xi$; whence $[\psi]=0$. This gives the reverse injection and concludes the proof of the isomorphism.

[25] Here ghost number does not take into account the zero modes.

Notice that the isomorphism is only an isomorphism of ungraded vector spaces. In particular the cohomology space $H_\infty(\mathfrak{R}, \mathfrak{V}_0; \mathfrak{M})$ is not graded by ghost number since the ghost number operator on $\mathbb{C}[a^\dagger]$ is of the form $\frac{1}{2}((a^\dagger)^2 - a^2)$ and therefore has no eigenvalues. As a consequence, a vanishing theorem has no meaning in this representation. This is not a serious drawback when it comes to proving the no-ghost theorem as we shall see, although it takes away some of the structure.

One can also show that every cohomology class in $H_\infty(\mathfrak{R}, \mathfrak{V}_0; \mathfrak{M})$ has at least one representative of ghost number zero. This uses a straight-forward generalization for the NSR string of a result proven in **[60]** for the open bosonic string which states that every cohomology class in $H_Q(C^{F_0})$ has a representative annihilated by T. For completeness, we provide a proof in Appendix B. If this is the case then it is also annihilated by K and therefore it defines a class in $H_\infty(\mathfrak{R}, \mathfrak{V}_0; \mathfrak{M})$; and by the vanishing theorem for $H_Q(C^{F_0})$ it has ghost number zero.

3. Vanishing Theorems for the Full Complexes

In this section we prove vanishing theorems for the cohomology of the full complexes $C_\infty(\mathfrak{R}; \mathfrak{H})$ and $C_\infty(\mathfrak{N}; \mathfrak{H})$. For the Ramond sector we only work with the Henneaux representation since for the Schrödinger representation there is no vanishing theorem. First we will prove that

$$\boxed{H_\infty(\mathfrak{R}, \mathfrak{V}_0; \mathfrak{H}) \cong H_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{H})} \; . \tag{VII.3.1}$$

Then we will prove that

$$\boxed{H^n_\infty(\mathfrak{N}; \mathfrak{H}) \cong \begin{cases} H^0_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{H}) & \text{for } n = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}} \; , \tag{VII.3.2}$$

and

$$\boxed{H^n_\infty(\mathfrak{R}; \mathfrak{H}) \cong \begin{cases} H^0_\infty(\mathfrak{R}, \mathfrak{V}_0; \mathfrak{H}) & \text{for } n = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}} \; . \tag{VII.3.3}$$

Several remarks are in order before we start proving these results. The first is to notice the rather surprising fact that the BRST cohomology of the Ramond sector has the same finite degeneracy as the one of the Neveu-Schwarz sector despite the fact that at the level of cochains the Ramond sector is infinitely degenerate due to the existence of the zero modes for the superconformal ghosts. Secondly we notice that the grading of the full complex is half integral. This is the choice that makes the full ghost number operator hermitian. Thirdly, because the proofs of (VII.3.2) and (VII.3.3) are virtually identical we will only present the one for the Ramond sector: this being the more involved of the two. Finally, the proof of (VII.3.1) is similar to the proof of the isomorphisms of the relative BRST cohomology of the Ramond sector in the Henneaux and Schrödinger representations. In fact, part of the proof already appears in [**37**].

With these remarks behind us we proceed with the proofs.

Proof of (VII.3.1)**:** Let's isolate the space in which the zero modes of the superconformal ghosts act by writing $C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$ as $C^{L_0}\otimes\mathbb{C}[\gamma_0]$, which defines C. Then $C_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$ may be identified with C^{F_0} and embedded in $C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$ as $C^{F_0}\otimes 1$. That is, if $\psi\in C_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$, then $\psi\otimes 1\in C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$. Suppose that Ψ is a cocycle in $C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$. Then Ψ is a polynomial in γ_0 with coefficients in C^{L_0}

$$\Psi=\sum_{n=0}^{N}\psi_n\otimes\gamma_0^n \qquad \psi_n\in C^{L_0}\ \forall n\ , \tag{VII.3.4}$$

such that, in particular, $F_0\,\psi_N=0$. Since the cohomology of F_0 is trivial, there exists $\phi\in C^{L_0}$ such that $\psi_N+F_0\,\phi=0$. Therefore $\Psi+\mathbb{Q}(\phi\otimes\gamma_0^{N-1})$ is a cocycle cohomologous to Ψ but lacking the highest order term in γ_0. Continuing in this fashion we can reduce Ψ to a constant monomial $\psi\otimes 1$ still cohomologous to Ψ. The cocycle condition translates into

$$\mathcal{Q}\,\psi=0 \qquad F_0\,\psi=0\ ; \tag{VII.3.5}$$

hence it defines a class in $H_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$. Suppose that this class is trivial; that is, $\psi=\mathcal{Q}\,\zeta$ where $F_0\,\zeta=0$. Then $\psi\otimes 1=\mathbb{Q}(\zeta\otimes 1)$ and thus Ψ represents the trivial class. Therefore we have an injection $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})\hookrightarrow H_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$.

Conversely, let ψ be a cocycle in $H_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$. Then $[\psi\otimes 1]$ is a class in $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$ which, if trivial, implies that

$$\psi\otimes 1=\mathbb{Q}\,\Xi \tag{VII.3.6}$$

for some polynomial $\Xi=\sum_{n=0}^{N}\xi_n\otimes\gamma_0^n$. In particular, (VII.3.6) implies that $F_0\,\xi_N=0$. As before there exists λ such that $\xi_N+F_0\,\lambda=0$. Thus $\Xi+\mathbb{Q}(\lambda\otimes\gamma_0^{N-1})$ still obeys (VII.3.6) but has no order N term. Continuing in this way we can reduce Ξ to a constant monomial $\xi\otimes 1$ still obeying (VII.3.6). In particular, this implies that

$$\mathcal{Q}\,\xi=\psi \qquad \text{and} \qquad F_0\,\xi=0\ . \tag{VII.3.7}$$

Therefore ψ defines the trivial class in $H_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$. This proves the reverse injection and hence the isomorpshism (VII.3.1). ■

In order to prove (VII.3.3) and because $\mathfrak{V}_0$ acts diagonally in the relative subcomplex $C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$ we could appeal to a suitably generalized result of Koszul[77] which asserts the existence of a spectral sequence converging to $H_\infty(\mathfrak{R};\mathfrak{H})$ whose E_2 term is

$$H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H}) \otimes H(\mathfrak{V}_0)\ . \tag{VII.3.8}$$

This, together with the easily verifiable fact that

$$H^n(\mathfrak{V}_0) \cong \begin{cases} \mathbb{C} & \text{for } n = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{VII.3.9}$$

and the fact that—due to the vanishing theorem for $H_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$—the spectral sequence collapses at the E_2 term, yields (VII.3.3).

However we can arrive at the same result in a slightly more pedestrian way by using the spectral sequence associated to a particular double complex.

Proof of (VII.3.3)**:** The differential in the complex $C_\infty(\mathfrak{R};\mathfrak{H})$ is the full BRST operator given by (VII.2.20) where $\mathbb{Q}$ is the differential in the complex $C_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H})$ and is given by (VII.2.21). For notational convenience we define $\mathcal{T} = -2(T + \frac{1}{2}\gamma_0^2)$. Notice that since L_0 is diagonalizable and null homotopic: $L_0 = \{Q\,,\,b_0\}$, we can restrict ourselves to L_0–invariants. Therefore we write the differential in $C_\infty(\mathfrak{R};\mathfrak{H})$ as

$$Q = \mathbb{Q} + b_0\,\mathcal{T} \tag{VII.3.10}$$

where, due to the nilpotency of Q, $\mathbb{Q}$ and b_0, both terms anticommute. Abbreviating $C_\infty(\mathfrak{R};\mathfrak{H})^{L_0}$ to C, let us define a trigrading on this complex as follows:

$$C = \bigoplus_{m\in\mathbb{Z}} \bigoplus_{n=\pm\frac{1}{2}} \bigoplus_{p\in\mathbb{Z}+\frac{1}{2}} C^{m,n,p}\ , \tag{VII.3.11}$$

where $C^{m,n,p}$ consists of those cochains which are tensor products of homogeneous terms of relative ghost number m, (b_0,c_0)–ghost number n and (β_0,γ_0)–ghost number p; and where by relative ghost number we mean the ghost number which grades the relative subcomplex $C_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$.

According to this trigrading the relevant terms appearing in Q have the following tridegree:

Term	Tridegree
$\mathcal{Q}$	$(1,0,0)$
$\beta_0\,K$	$(2,0,-1)$
$\gamma_0\,F_0$	$(0,0,1)$
$b_0\,T$	$(2,-1,0)$
$b_0\,\gamma_0^2$	$(0,-1,2)$

Defining the bigraded complex $K = \bigoplus_{r,s} K^{r,s}$ by

$$K^{r,s} = \bigoplus_{m+p=r} C^{m,s,p} \ , \tag{VII.3.12}$$

we notice that $\mathbb{Q}$ has bidegree $(1,0)$ but that $b_0\,\mathcal{T}$ has bidegree $(2,-1)$. Hence the complex as it stands is slightly skewed. Making a last redefinition, let us introduce another bigraded complex $\mathbb{K}$ which is just a relabeling of K by $\mathbb{K} = \bigoplus_{p,q} \mathbb{K}^{p,q}$ where

$$\mathbb{K}^{p,q} = K^{p+2q,-q} \ . \tag{VII.3.13}$$

Then $\mathbb{Q}\colon \mathbb{K}^{p,q} \to \mathbb{K}^{p+1,q}$ and $b_0\,\mathcal{T}\colon \mathbb{K}^{p,q} \to \mathbb{K}^{p,q+1}$ yielding a double complex. Decomposing this double complex into eigenspaces of the level operator (the momentum independent part of L_0) we easily see that it yields an infinite direct sum of finite double complexes. Proving (VII.3.3) for each subcomplex and then collating all terms proves (VII.3.3) for the full complex. Hence from now on we are working in a given eigenspace of the level operator so that the double complex $\mathbb{K}$ is finite. Notice that the complex is only two rows high in any case, since q only takes $\pm\frac{1}{2}$ as values.

As discussed in Section 2 we have two canonical spectral sequences associated to this double complex. We use the $''$ filtration. Its E_1 term is the horizontal cohomology for which we have a vanishing theorem. Keeping track of the gradings we have

$$''E_1^{q,p} \cong \begin{cases} H^0_\infty(\mathfrak{R},\mathfrak{V}_0;\mathfrak{H}) & \text{for } (p,q) = (-1,\tfrac{1}{2}) \text{ and } (1,-\tfrac{1}{2}) \\ 0 & \text{otherwise} \end{cases} \ . \tag{VII.3.14}$$

Notice further that d_1 is identically zero since it maps vertically and by (VII.3.14) its domain or its range is zero in all cases. Furthermore all higher d_r are also zero because they skip at least one row and there are only two rows in the complex. Therefore the E_1 term is the limit term which is the cohomology of the full complex: $H_\infty(\mathfrak{R};\mathfrak{H})$. This proves (VII.3.3). ■

As remarked earlier the proof of (VII.3.2) follows the same steps as the proof of (VII.3.3), but without the complications arising from the superconformal ghosts.

4. No-Ghost Theorems

In this section we prove the no-ghost theorem for the NSR string along the lines suggested in Chapter V. That is, our proof of the no-ghost theorem will consist in proving that the character and the signature of the relative subcomplex for the NSR string agree. Since the relative subcomplex is graded by the level operator $\mathcal{L}$ (the momentum independence piece of L_0) and each level eigenspace is finite dimensional the following identities are well-defined

$$\mathrm{ch}_q\ H_\infty = \mathrm{Tr}_{C_\infty}\ q^{\mathcal{L}}\mathcal{C} \tag{VII.4.1}$$

$$\mathrm{sgn}_q\ H_\infty = \mathrm{Tr}_{C_\infty}\ q^{\mathcal{L}}(-1)^{N_{\mathrm{gh}}}\ , \tag{VII.4.2}$$

where C_∞ denotes generically the relevant relative subcomplex and H_∞ its cohomology. Because C_∞ splits as tensor products corresponding to the different oscillators and the trace is multiplicative over the tensor product, we compute each term separately and then multiply the results. There are two terms common to both sectors: the $\{a\}$ and $\{b,c\}$ oscillators;and we do these now. This calculation was done in Section VI.2 (for $D=26$) but we repeat it here (for $D=10$) for completeness.

The space over which we are taking the traces has the following structure

$$C = \bigotimes_{\mu=0}^{9}\bigotimes_{n=1}^{\infty} S_n^\mu \bigotimes_{n=1}^{\infty} A_n\ , \tag{VII.4.3}$$

where S_n^μ is the one particle Hilbert space corresponding to the oscillator $a_n^{\mu\dagger}$ and A_n is the Hilbert space corresponding to the oscillators $\{b_n^\dagger, c_n^\dagger\}$. The space S_n^μ is isomorphic to the polynomial algebra in one variable: $a_n^{\mu\dagger}$ whereas the space A_n is isomorphic to the exterior algebra on two generators: $b_n^\dagger$ and $c_n^\dagger$.

Therefore using the fact that the trace is multiplicative over tensor products the character of H_∞ becomes

$$\begin{aligned}
\mathrm{ch}_q\ H_\infty &= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty}\mathrm{Tr}_{S_n^\mu}\ q^{n a_n^{\mu\dagger}a_n^\mu} \times \prod_{n=1}^{\infty}\mathrm{Tr}_{A_n}\left[(-1)^{c_n^\dagger b_n - b_n^\dagger c_n}\ q^{n(c_n^\dagger b_n + b_n^\dagger c_n)}\right] \\
&= \left[\prod_{n=1}^{\infty}\left(\sum_{m=0}^{\infty} q^{nm}\right)\right]^{10} \times \prod_{n=1}^{\infty}\left(1 - q^n - q^n + q^{2n}\right) \\
&= \prod_{n=1}^{\infty}(1-q^n)^{-10}\cdot(1-q^n)^2 \\
&= \prod_{n=1}^{\infty}(1-q^n)^{-8}\ .
\end{aligned} \tag{VII.4.4}$$

As for the signature we have

$$\begin{aligned}
\operatorname{sgn}_q H_\infty &= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty} \operatorname{Tr}_{S_n^\mu} \mathcal{C}\, q^{n a_n^{\mu\,\dagger} a_n^\mu} \times \prod_{n=1}^{\infty} \operatorname{Tr}_{A_n} \mathcal{C}\, q^{n(c_n^\dagger b_n + b_n^\dagger c_n)} \\
&= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty}\sum_{m=0}^{\infty} \left((-1)^{\delta_{\mu,0}} q^n\right)^m \times \prod_{n=1}^{\infty} \left(1-q^{2n}\right) \\
&= \prod_{n=1}^{\infty} (1+q^n)^{-1} \cdot (1-q^n)^{-9} \cdot (1-q^n) \cdot (1+q^n) \\
&= \prod_{n=1}^{\infty} (1-q^n)^{-8} \quad . \qquad \text{(VII.4.5)}
\end{aligned}$$

We see already that the identity is satisfied. This is not surprising since this is essentially the no-ghost theorem for the bosonic string. Of course, in this case, the calculation has no cohomological significance since we are away from the critical dimension.

Having done the calculations common to both sectors we now do each sector separately.

The Neveu-Schwarz Sector

The relative subcomplex $C_\infty(\mathfrak{N}, \mathfrak{V}_0; \mathfrak{H})$, which we abbreviate to C_∞, has the following structure

$$C_\infty = \mathcal{F}^{(a)} \otimes \mathcal{F}^{(b,c)} \otimes \mathcal{F}^{(b)} \otimes \mathcal{F}^{(\beta,\gamma)} \qquad \text{(VII.4.6)}$$

where

$$\mathcal{F}^{(a)} = \bigotimes_{\mu=0}^{9} \bigotimes_{n=1}^{\infty} S_n^\mu \qquad \text{(VII.4.7)}$$

$$\mathcal{F}^{(b,c)} = \bigotimes_{n=1}^{\infty} A_n \qquad \text{(VII.4.8)}$$

$$\mathcal{F}^{(b)} = \bigotimes_{\mu=0}^{9} \bigotimes_{r=\frac{1}{2}}^{\infty} A_r^\mu \qquad \text{(VII.4.9)}$$

and

$$\mathcal{F}^{(\beta,\gamma)} = \bigotimes_{r=\frac{1}{2}}^{\infty} S_r \quad . \qquad \text{(VII.4.10)}$$

The first two terms are the ones over which we computed the relevant traces in the beginning of this section. Therefore we shall concentrate on the last two terms. Here A_r^μ is the Hilbert space of the $b_r^{\mu\,\dagger}$ oscillator and is isomorphic to the exterior algebra on one generator; and S_r is the Hilbert space of the $\{\beta_r^\dagger, \gamma_r^\dagger\}$ oscillators and is isomorphic to the polynomial algebra in two variables.

The contribution to the character coming from the first two terms in the above decomposition are $\prod_{n=1}^{\infty}(1-q^n)^{-8}$. The contribution coming from the Neveu-Schwarz oscillators can be computed as follows

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}^{(b)}}\, q^{\mathcal{L}^{(b)}} &= \prod_{\mu=0}^{9}\prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{A_r^\mu}\, q^{r\, b_r^{\mu\,\dagger} b_r^\mu} \\ &= \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^{10} \quad ,\end{aligned}$$

whereas the contribution from the superghosts is

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\,(-1)^{N_{\mathrm{gh}}}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{S_r}\,(-1)^{N_\gamma - N_\beta}\, q^{r(N_\gamma+N_\beta)} \\ &= \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{S_r}\,(-q^r)^{N_\gamma+N_\beta} \\ &= \prod_{r=\frac{1}{2}}^{\infty} \sum_{n,m=0}^{\infty} (-q^r)^{n+m} \\ &= \prod_{r=\frac{1}{2}}^{\infty} \left(\sum_{n=0}^{\infty}(-q^r)^n\right)^2 \\ &= \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^{-2} \quad ,\end{aligned}$$

where N_β (resp. N_γ) is the number operator corresponding to the $\{\beta_r\}$ (resp. $\{\gamma_r\}$) oscillators. Putting everything together we find that

$$\boxed{\mathrm{ch}_q\ H_\infty = \prod_{n=1}^{\infty}(1-q^n)^{-8} \times \prod_{r=\frac{1}{2}}^{\infty}(1+q^r)^8} \quad , \tag{VII.4.11}$$

which recovers the lightcone count as in the bosonic string.

In order to compute the signature we use the conjugation given in Appendix C. Once again the contribution to the signature now coming from the $\{a_n^\mu, b_n, c_n\}$ oscillators is $\prod_{n=1}^{\infty}(1-q^n)^{-8}$. The contribution from the Neveu-Schwarz oscillators is

$$\mathrm{Tr}_{\mathcal{F}^{(b)}}\, \mathcal{C}\, q^{\mathcal{L}^{(b)}} = \prod_{\mu=0}^{9}\prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{A_r^\mu}\, \mathcal{C}\, q^{r\, b_r^{\mu\,\dagger} b_r^\mu}$$

$$= \prod_{\mu=0}^{9} \prod_{r=\frac{1}{2}}^{\infty} \left(1 + (-1)^{\delta_{\mu,0}} q^r\right)$$

$$= \prod_{r=\frac{1}{2}}^{\infty} (1 - q^r) \cdot (1 + q^r)^9 \ .$$

Finally we compute the contribution coming from the superghosts. Notice that because of the nature of the conjugation $\mathcal{C}$ we only pick a contribution to the trace from states whose β and γ occupation numbers coincide. Therefore

$$\begin{aligned} \mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}} \mathcal{C}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{S_r} \mathcal{C}\, q^{r(N_\beta + N_\gamma)} \\ &= \prod_{r=\frac{1}{2}}^{\infty} \sum_{n=0}^{\infty} q^{2rn} \\ &= \prod_{r=\frac{1}{2}}^{\infty} \left(1 - q^{2r}\right)^{-1} \ . \end{aligned}$$

Combining all results we find

$$\mathrm{sgn}_q\ H_\infty = \prod_{n=1}^{\infty} (1 - q^n)^{-8} \times \prod_{r=\frac{1}{2}}^{\infty} (1 + q^r)^8 \ , \tag{VII.4.12}$$

which agrees with (VII.4.11), hence proving the no-ghost theorem for the Neveu-Schwarz sector.

<u>The Ramond Sector</u>

We first prove the no-ghost theorem for the Henneaux representation. We will then infer a similar result for the Schrödinger representation.

The relative subcomplex $C_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{H})$, which we abbreviate to C_∞, has the following structure

$$C_\infty = \mathcal{F}^{(a)} \otimes \mathcal{F}^{(b,c)} \otimes \mathcal{F}^{(d)} \otimes \mathcal{F}^{(\beta,\gamma)} \ , \tag{VII.4.13}$$

where $\mathcal{F}^{(a)}$ and $\mathcal{F}^{(b,c)}$ were discussed already at the beginning of this section. As for the rest

$$\mathcal{F}^{(d)} = \bigotimes_{\mu=0}^{9} \bigotimes_{n=1}^{\infty} A_n^\mu \tag{VII.4.14}$$

$$\mathcal{F}^{(\beta,\gamma)} = \bigotimes_{n=1}^{\infty} S_n \ . \tag{VII.4.15}$$

Here A_n^μ is the Hilbert space of the ${d_n^\mu}^\dagger$ oscillator and is isomorphic to the exterior algebra on one generator; and S_n is the Hilbert space of the $\{\beta_n^\dagger, \gamma_n^\dagger\}$ oscillators and is isomorphic to the polynomial algebra in two variables.

Again the contribution to the character coming from the first two terms in the above decomposition is $\prod_{n=1}^\infty (1-q^n)^{-8}$. The Ramond oscillators contribute

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}^{(d)}}\, q^{\mathcal{L}^{(d)}} &= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty} Tr_{A_n^\mu}\, q^{n\, d_n^{\mu\,\dagger} d_n^\mu} \\ &= \prod_{n=1}^{\infty} (1+q^n)^{10} \quad ,\end{aligned}$$

and the contribution from the superghosts is

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\, (-1)^{N_{\mathrm{gh}}}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n}\, (-1)^{N_\gamma - N_\beta}\, q^{n(N_\gamma+N_\beta)} \\ &= \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n}\, (-q^n)^{N_\gamma+N_\beta} \\ &= \prod_{n=1}^{\infty} \sum_{m,p=0}^{\infty} (-q^n)^{m+p} \\ &= \prod_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} (-q^n)^m\right)^2 \\ &= \prod_{n=1}^{\infty} (1+q^n)^{-2} \quad ,\end{aligned}$$

where N_β (resp. N_γ) is the number operator corresponding to the $\{\beta_n\}$ (resp. $\{\gamma_n\}$) oscillators. Putting everything together we find that

$$\boxed{\mathrm{ch}_q\; H_\infty = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^8} \quad . \tag{VII.4.16}$$

In order to compute the signature we use the conjugation given in Appendix C. The contribution coming from the $\{a_n^\mu, b_n, c_n\}$ oscillators is once again $\prod_{n=1}^\infty (1-q^n)^{-8}$. The Ramond oscillators contribute

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}^{(d)}}\, \mathcal{C}\, q^{\mathcal{L}^{(d)}} &= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty} \mathrm{Tr}_{A_n^\mu}\, \mathcal{C}\, q^{n\, d_n^{\mu\,\dagger} d_n^\mu} \\ &= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty} \left(1+(-1)^{\delta_{\mu,0}} q^n\right) \\ &= \prod_{n=1}^{\infty} (1-q^n)\cdot(1+q^n)^9 \quad .\end{aligned}$$

Finally we compute the contribution coming from the superghosts. Just as in the Neveu-Schwarz sector we only pick a contribution to the trace from states whose β and γ occupation numbers coincide. Indeed,

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\, \mathcal{C}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n}\, \mathcal{C}\, q^{n(N_\beta+N_\gamma)} \\
&= \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} q^{2nm} \\
&= \prod_{n=1}^{\infty} \left(1-q^{2n}\right)^{-1} \ .
\end{aligned}
$$

Combining all results we find

$$
\mathrm{sgn}_q\, H_\infty = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^8 \ , \tag{VII.4.17}
$$

which agrees with (VII.4.16), hence proving the no-ghost theorem.

In the Schrödinger representation Q and $\mathbb{Q}$ are hermitian and therefore the inner product in cohomology does not depend on the particular cocycle chosen to represent a given class. Let $[\Psi]$ be a class in $H_\infty(\mathfrak{R}, \mathfrak{V}_0; \mathfrak{H})$ and let $\psi \otimes 1$ denote a representative such that $T\psi = 0$. This is always possible, as shown in Appendix B. Then ψ defines a class in $H_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{H})$ in the Henneaux representation. We can normalize the inner product in the space of the superconformal ghosts' zero modes in such a way that the norm of $\psi \otimes 1$ agrees with the norm of ψ. Because $\mathcal{Q}$ is hermitian, the norm of a class in $H_\infty(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{H})$ is independent of the representative; therefore the norm of ψ is the norm of the class $[\psi]$ it represents. But by the no-ghost theorem just proven, the norm of ψ is positive. Therefore the norm of $[\Psi]$ is positive. This proves the no-ghost theorem for the Schrödinger representation.

Finally we remark that the GSO projected NSR string is also free of ghosts. This is true because modular invariance also forces the GSO projection on the superghost spectrum which goes hand in hand with the GSO projection in the spectrum of the Neveu-Schwarz and Ramond oscillators. We leave the details of this calculation as an exercise.

Appendix A. Computation of $H_\infty(\mathfrak{S}_-;\mathfrak{M})$

This is another technical appendix where we prove the superalgebra version of the theorem proven in Appendix VI.A. Although the theorem holds for any Lie superalgebra we choose to present it in the context of superconformal algebras. Throughout this appendix we let $\mathfrak{S}$ denote either $\mathfrak{N}$ or $\mathfrak{R}$. Then we have the following theorem

Theorem VII.A.1. *Let $\mathfrak{M} \cong \mathfrak{U}(\mathfrak{S}_-)$ be a free $\mathfrak{S}_-$-module. Then*

$$H^m_\infty(\mathfrak{S}_-;\mathfrak{M}) \cong \begin{cases} \mathbb{C} & \textit{for } b=0 \\ 0 & \textit{otherwise} \end{cases} . \tag{VII.A.2}$$

Proof: The Lie superalgebra $\mathfrak{S}_-$ decomposes into odd and even subspaces $\mathfrak{S}_- = \mathfrak{S}_-^{\rm even} \oplus \mathfrak{S}_-^{\rm odd}$. Let us choose a basis in each subspace and denote them by $\{e_i\}$ and $\{f_\alpha\}$, respectively. A basis for the Verma module, $\mathfrak{M}$, is then given by the highest weight vector together with the monomials

$$\{e_{i_1}e_{i_2}\cdots e_{i_n} f_{\alpha_1} f_{\alpha_2}\cdots f_{\alpha_m}\} \tag{VII.A.3}$$

where all of the subscripts are integers satisfying

$$i_1 \leq i_2 \cdots \leq i_n \quad \text{and} \quad \alpha_1 < \alpha_2 \cdots < \alpha_m \tag{VII.A.4}$$

for some positive integers m and n. Notice that we have omitted writing the highest weight vector explicitly in order to simplify the notation. A basis for the cochains $C \equiv C_\infty(\mathfrak{S}_-;\mathfrak{M})$ is given by

$$\{\beta^\dagger_{\alpha_1}\beta^\dagger_{\alpha_2}\cdots\beta^\dagger_{\alpha_k} b^\dagger_{i_1} b^\dagger_{i_2}\cdots b^\dagger_{i_l} \otimes e_{j_1}e_{j_2}\cdots e_{j_m} f_{\lambda_1} f_{\lambda_2}\cdots f_{\lambda_n}\} \tag{VII.A.5}$$

where

$$i_1 < i_2 < \cdots < i_l \ , \quad \text{and} \quad \alpha_1 \leq \alpha_2 \cdots \leq \alpha_k \ , \tag{VII.A.6}$$

$$j_1 \leq j_2 \leq \cdots \leq j_m \ , \quad \text{and} \quad \lambda_1 < \lambda_2 < \cdots < \lambda_n \ . \tag{VII.A.7}$$

It is understood that the antighosts are acting upon the usual ghost vacuum. Notice that there are no ghost creation operators since these correspond to $\mathfrak{S}_+$.

Having characterized the cochains, we proceed to define a filtration. We define the filtration degree of a cochain

$$\Omega = \beta^\dagger_{\alpha_1}\cdots\beta^\dagger_{\alpha_K} b^\dagger_{i_1}\cdots b^\dagger_{i_L}\otimes e_{j_1}\cdots e_{j_M} f_{\lambda_1}\cdots f_{\lambda_N} \tag{VII.A.8}$$

by

$$\mathrm{fdeg}\,\Omega = K+L+M+N\ . \tag{VII.A.9}$$

This allows us to define a filtration $F^pC \equiv \{\omega\in C \mid \mathrm{fdeg}\,\omega \le p\}$. In the case of the Neveu-Schwarz algebra, this is a half-integral filtration while in the case of the Ramond algebra this is an integral filtration. We shall proceed, for definiteness, as if this filtration were integral throughout the remainder of this appendix. The arguments for the case of the half-integral filtration are exactly the same. It is quite easy to see that FC is a filtration since it satisfies $F^pC\subseteq F^{p+1}C$ for all p and all of the homogeneous terms in the differential, d, have non-positive filtration degrees.

Let us compute the cohomology of the associated graded complex $\mathrm{Gr}\,C$ with respect to the induced differential, which is the part of d with zero filtration degree:

$$d = \sum_i c_i\otimes e_i + \sum_\alpha \gamma_\alpha\otimes f_\alpha\ . \tag{VII.A.10}$$

More explicitly, we can write

$$\begin{aligned} d\,\Omega =& \sum_i \beta^\dagger_{\alpha_1}\cdots\beta^\dagger_{\alpha_K} c_i b^\dagger_{i_1}\cdots b^\dagger_{i_L}\otimes e_i e_{j_1}\cdots e_{j_M} f_{\lambda_1}\cdots f_{\lambda_N}\\ &+(-1)^L\sum_\alpha \gamma_\alpha\beta^\dagger_{\alpha_1}\cdots\beta^\dagger_{\alpha_K} b^\dagger_{i_1}\cdots b^\dagger_{i_L}\otimes e_{j_1}\cdots e_{j_M} f_\alpha f_{\lambda_1}\cdots f_{\lambda_N}\ . \end{aligned} \tag{VII.A.11}$$

Two remarks are in order. First of all notice that the above sums are actually finite and second that the terms above are to be taken modulo $F^{p-1}C$. Now define a linear map $\Gamma:\mathrm{Gr}\,C\longrightarrow\mathrm{Gr}\,C$ for all $M+N>0$ by

$$\begin{aligned} \Gamma\omega =& \sum_{l=1}^{M} \beta^\dagger_{\alpha_1}\cdots\beta^\dagger_{\alpha_K} b^\dagger_{j_l} b^\dagger_{i_1}\cdots b^\dagger_{i_L}\otimes e_{j_1}\cdots\widehat{e_{j_l}}\cdots e_{j_M} f_{\lambda_1}\cdots f_{\lambda_N}\\ &+\sum_{l=1}^{N}(-1)^{L+l-1}\beta^\dagger_{\lambda_l}\beta^\dagger_{\alpha_1}\cdots\beta^\dagger_{\alpha_K} b^\dagger_{i_1}\cdots b_{i_L}\otimes e_{j_1}\cdots e_{j_M} f_{\lambda_1}\cdots\widehat{f_{\lambda_l}}\cdots f_{\lambda_N}\ , \end{aligned} \tag{VII.A.12}$$

where a $\widehat{\ }$ adorning a symbol denotes its omission. A straight forward calculation shows that this map satisfies

$$(d\,\Gamma + \Gamma\,d)\,\Omega = (K + L + M + N)\,\Omega\ . \tag{VII.A.13}$$

Therefore, if $\Omega \in C_\infty^{m>0}(\mathfrak{S}_-;\mathfrak{M})$ is a cocycle, (VII.A.13) implies that it is also a coboundary, since for $m > 0$, $K + L + M + N \neq 0$. Therefore, $H^{m>0}(\mathrm{Gr}\,C) = 0$. From the long exact sequence in cohomology associated to the exact sequence of complexes for $p \geq 1$

$$0 \longrightarrow F^{p-1}C \longrightarrow F^pC \longrightarrow \mathrm{Gr}^pC \longrightarrow 0 \tag{VII.A.14}$$

we find that $H^n(F^pC) \cong H^n(F^{p-1}C)$ for $n \neq 0$. Since $H^n(F^0C) = 0$ for $n \neq 0$ we find that $H^n(C) = 0$ for $n \neq 0$. As in Appendix VI.A, $H^0(C)$ can be computed explicitly and we see that it is one-dimensional. ■

Appendix B. Every Cohomology Class has a Singlet Cocycle

In this appendix we prove the claim stated in Section 2 that every BRST cohomology class in $H_\infty(\mathfrak{R},\mathfrak{R}_0;\mathfrak{H})$ ($\cong H_{\mathcal{Q}}(C^{F_0})$ in the notation of Section 2) has a representative cocycle which is annihilated by the operator T appearing in the decomposition of the BRST charge given in (VII.2.20). This result is a direct consequence of the vanishing theorem (VII.2.17) via the methods of Chapter V. The similar result for the open bosonic string was proven in **[60]**.

It was noticed in **[110]** (*cf.* also **[111]**)—extending to the NSR string an observation made by Siegel & Zwiebach in **[40]** for the bosonic string—that the operator T together with its adjoint T^* under the positive definite inner product induced by the conjugation $\mathcal{C}$ to be defined in Appendix C, and the ghost number operator N_{gh} obey an $\mathfrak{sl}(2,\mathbb{C})$ algebra

$$\big[T\,,\,T^*\big] = N_{\mathrm{gh}} \quad \big[N_{\mathrm{gh}}\,,\,T\big] = 2T \quad \big[N_{\mathrm{gh}}\,,\,T^*\big] = -2T^*\ . \tag{VII.B.1}$$

In fact, using other operators in the decomposition of the BRST charge and their conjugates and/or adjoints one can find a representation of one of the exceptional superalgebras $\mathfrak{Q}(3)$.

Since this action of $\mathfrak{sl}(2,\mathbb{C})$ commutes with the level operator $\mathcal{L}$ and the center of mass momentum p^μ, it stabilizes their eigenspaces, which are finite dimensional. Therefore, by Weyl's theorem, these representations are fully reducible into irreducible modules of $\mathfrak{sl}(2,\mathbb{C})$, which are obtained acting with T on vectors of lowest weight annihilated by T^*.

We now proceed to prove that each cohomology class in $H_{\mathcal{Q}}(C^{F_0})$ has a representative cocycle annihilated by T. Since it is also annihilated by N_{gh}—this is the vanishing theorem—we see that these representatives are actually $\mathfrak{sl}(2,\mathbb{C})$-invariants. This was used without proof in [**40**] as a criterion to gauge away auxiliary fields. We must also remark that singlet representatives are not necessarily unique. Indeed, in [**60**], we showed explicitly that at very low lying levels in the spectrum of the open bosonic string we already find BRST cohomology classes which contain more than one $\mathfrak{sl}(2,\mathbb{C})$-invariant representative cocycle.

The idea of the proof is the following. Let $[\psi] \in H^0_{\mathcal{Q}}(C^{F_0})$. Then we will prove that we can find a state $\widetilde{\psi}$ cohomologous to ψ but which is annihilated by T. We shall without loss of generality assume that ψ is in a particular eigenspace of p^μ and $\mathcal{L}$.

Let $\widetilde{\psi} = \psi + \mathcal{Q}\,\xi$ for some state ξ of ghost number -1. Imposing $T\,\widetilde{\psi} = 0$ we get $T\,\mathcal{Q}\,\xi = -T\,\psi$. From the fact that T and $\mathcal{Q}$ commute and the vanishing theorem we conclude that $T\,\psi = \mathcal{Q}\,\rho$ for a unique $\rho \in \text{im } \mathcal{Q}^*$ and with ghost number 1. Therefore the equation for ξ becomes $\mathcal{Q}\,(T\,\xi + \rho) = 0$. Hence all we need to do is solve the equation $T\,\xi = -\rho \pmod{\ker \mathcal{Q}}$. In fact we can do better and we can solve the equation exactly.

First of all let us break up ρ into its irreducible components. It is clear that we can restrict ourselves to each irreducible subspace at a time since T respects this. Therefore let us assume that ρ consists of exactly one such component. Then because ρ has ghost number 1 it cannot be annihilated by T^*, since the kernel of T^* consists of lowest weight vectors and these have all non-positive ghost numbers. By similar reasoning ξ cannot be annihilated by T, and hence by $T^*\,T$. Therefore we can solve for ξ as follows

$$\xi = -(T^*\,T)^{-1}\,T^*\,\rho\ , \tag{VII.B.2}$$

where the inverse of $T^*\,T$ exists in $\text{im } T^* = (\ker T)^\perp = (\ker T^*\,T)^\perp$.

Noticing that $\rho = G\,\mathcal{Q}^*\,T\,\psi$—where G is the Green's operator—we can write $\widetilde{\psi}$ as

$$\widetilde{\psi} = \big(\mathbf{1} - \mathcal{Q}\,(T^*\,T)^{-1}\,T^*\,G\,\mathcal{Q}^*\,T\big)\ \psi\ . \tag{VII.B.3}$$

The above operator turns out, after some straight forward algebra, to be a projection.

Appendix C. A Positive-Definite Inner Product

In Chapter V we proved a Poincaré duality theorem which requires two things: first that the Fock space decomposes into a direct sum of finite dimensional subspace which are stabilized by the BRST operator and second that there exists a positive definite inner product in the Fock space. The first point is obvious since there are only a finite number of states of a given level. We address the second question in this appendix, where we construct a positive-definite inner product explicitly. The inner product is defined from the original one imposed by the quantization procedure by the introduction of a self-adjoint involution $\mathcal{C}$ in such a way that the new inner product is

$$\langle\psi,\,\phi\rangle_{\mathcal{C}} = (\psi, \mathcal{C}\,\phi)\ , \tag{VII.C.1}$$

where $(,)$ is the original inner product and ψ and ϕ are vectors in the Fock space. On the ghost and anti-ghost oscillators this conjugation $\mathcal{C}$ plays the rôle of the Serre-Hodge $\bar{\star}$ operator in complex geometry[60] and therefore is consistent with the "semi-infinite" form interpretation of the ghost Fock space.

First a word of caution. Our ghost oscillators are not the natural ones but are unitarily related to them. In our conventions the mode expansion of the conformal ghost and antighost fields at $\tau = 0$ are the following:

$$\begin{aligned} b(\sigma) &= b_0 + \sum_{m>0} \sqrt{m}\left(b_m\, e^{im\sigma} + b_{-m}\, e^{-im\sigma}\right) \\ c(\sigma) &= c_0 + \sum_{m>0} \frac{1}{\sqrt{m}}\left(c_m\, e^{im\sigma} + c_{-m}\, e^{-im\sigma}\right)\ , \end{aligned}$$

and similarly for the superconformal ghosts. This seemingly unnatural choice of mode expansion turns out to be the natural one in our context. It will allow us to identify the involution $\mathcal{C}$ above with ghost conjugation when acting on ghosts and antighosts.

For the $\{a_n^\mu, b_n, c_n\}$ oscillators we define $\mathcal{C}$ as follows

$$\mathcal{C}\,p^\mu\,\mathcal{C} = p^\mu \tag{VII.C.2}$$

$$\mathcal{C}\,a_n^0\,\mathcal{C} = -a_n^0 \qquad \mathcal{C}\,a_n^i\,\mathcal{C} = a_n^i \qquad \forall\ i = 1\ldots 9 \text{ and } \forall n \neq 0 \tag{VII.C.3}$$

$$\mathcal{C}\,c_n\,\mathcal{C} = b_n \qquad \mathcal{C}\,b_n\,\mathcal{C} = c_n \qquad (\forall n \in \mathbb{Z}) \tag{VII.C.4}$$

For the Neveu-Schwarz oscillators the conjugation with the desired properties turns out to be the following

$$\mathcal{C}\,b_s^\mu\,\mathcal{C} = (-1)^{\delta_{\mu,0}}\,b_s^\mu \qquad \forall\, s \in \mathbb{Z} + \frac{1}{2} \tag{VII.C.5}$$

$$\mathcal{C}\,\gamma_r\,\mathcal{C} = \beta_r \qquad \mathcal{C}\,\beta_r\,\mathcal{C} = \gamma_r \tag{VII.C.6}$$

$$\mathcal{C}\,\gamma_{-r}\,\mathcal{C} = -\beta_{-r} \qquad \mathcal{C}\,\beta_{-r}\,\mathcal{C} = -\gamma_{-r} \qquad \forall\, r \in \mathbb{N} - \frac{1}{2}\ , \tag{VII.C.7}$$

and for the Ramond oscillators it is very similar:

$$\mathcal{C}\,d_m^\mu\,\mathcal{C} = (-1)^{\delta_{\mu,0}}\,d_m^\mu \qquad \forall\, m \in \mathbb{Z} \tag{VII.C.8}$$

$$\mathcal{C}\,\gamma_n\,\mathcal{C} = \beta_n \qquad \mathcal{C}\,\beta_n\,\mathcal{C} = \gamma_n \tag{VII.C.9}$$

$$\mathcal{C}\,\gamma_{-n}\,\mathcal{C} = -\beta_{-n} \qquad \mathcal{C}\,\beta_{-n}\,\mathcal{C} = -\gamma_{-n} \qquad \forall\, n \in \mathbb{N}\ . \tag{VII.C.10}$$

For the ghost zero modes $\{\beta_0, \gamma_0\}$ there are two possibilities depending on the choice of Hilbert space that we choose for their representation. As discussed in Section 2 we can choose the Hilbert space in which they are self-adjoint in which case we already have a positive definite inner product and therefore $\mathcal{C}$ acts leaves them inert. On the other hand, following Henneaux[37], we can treat them as annihilation and creation operators, in which case β_0 and γ_0 are mutual adjoints. It is interesting to remark that in this case there is no self-adjoint involution $\mathcal{C}$ which yields this adjointness property from the original ones for β_0 and γ_0. However these are the only operators acting in this space and hence there is no need — in order to compute adjoints — for the operator $\mathcal{C}$ itself to exist.

To show that the new inner product defined by (VII.C.1) is indeed positive-definite is completely straight-forward and is left as an exercise for the reader.

Chapter Eight:

The BRST Cohomology of the Gauged WZNW Model

This chapter is the least complete of the chapters in this dissertation, since work is still in progress to get to the interesting consequences. The main goal of this work is to give a BRST proof of the equivalence between the coset construction of Goddard, Kent, & Olive[112] (GKO) and the conformal field theory (CFT) of the gauged Wess-Zumino-Novikov-Witten (WZNW) model. The equivalence has been recently proved by Gawedzki & Kupiainen[113] using geometric quantization methods. The advantage of the BRST approach relies in its simplicity when it comes to computations: a successful outcome of this program will provide resolutions of the CFTs constructed via the GKO mechanism. Let us first review their construction. We assume the reader has at least a working knowledge of CFT techniques.

Let $\mathfrak{g}$ be, for simplicity, a simple $d_{\mathfrak{g}}$-dimensional complex Lie algebra and let $\widehat{\mathfrak{g}}$ denote its affinization. Let ${}^{\mathfrak{g}}\mathfrak{M}_\lambda$ be a unitary irreducible Verma module over $\widehat{\mathfrak{g}}$ at level k. We can assemble the $\widehat{\mathfrak{g}}$ generators into currents

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n\, z^{-n-1} \text{ for } a = 1, \ldots, d_{\mathfrak{g}} \tag{VIII.0.1}$$

whose operator product expansion (OPE) is given by

$$J^a(z)J^b(w) = \frac{k\gamma^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c}{z-w} J^c(w) + \text{reg} \;, \tag{VIII.0.2}$$

where $\gamma^{ab} = (J^a_0, J^b_0)$ are the values in this basis of a fixed invariant symmetric bilinear form $(,)$ on $\mathfrak{g}$. Throughout we will make use of the Einstein summation convention. The Sugawara construction (*cf.* **[114]** and references therein) allows us to give ${}^{\mathfrak{g}}\mathfrak{M}_\lambda$ the structure of a $\mathfrak{V}$ (Virasoro) module in a canonical way. The $\mathfrak{V}$ generators are the modes of the energy momentum tensor $T^{\mathfrak{g}}(z)$ which is constructed out the $\widehat{\mathfrak{g}}$ currents as a suitably regularized bilinear form

$$T^{\mathfrak{g}}(z) = \frac{\gamma_{ab}}{2k + c_{\mathfrak{g}}} {:} J^a(z) J^b(z) {:} \ . \tag{VIII.0.3}$$

This energy momentum tensor satisfies the Virasoro ($\mathfrak{V}$) algebra

$$T^{\mathfrak{g}}(z)T^{\mathfrak{g}}(w) = \frac{c(\mathfrak{g},k)}{2(z-w)^4} + \frac{2}{(z-w)^2}T^{\mathfrak{g}}(w) + \frac{1}{z-w}\partial T^{\mathfrak{g}}(w) + \mathrm{reg} \ , \tag{VIII.0.4}$$

with central charge

$$c(\mathfrak{g},k) \equiv \frac{2kd_{\mathfrak{g}}}{2k + c_{\mathfrak{g}}} \ , \tag{VIII.0.5}$$

where γ_{ab} is the inverse of γ^{ab}, and $c_{\mathfrak{g}}$ is the eigenvalue of the quadratic Casimir operator $\gamma_{ab}J_0^a J_0^b$ in the adjoint representation of $\mathfrak{g}$. In particular, one can show that $d_{\mathfrak{g}} \geq c(\mathfrak{g},k) \geq \operatorname{rank}\mathfrak{g} \geq 1$. The $\widehat{\mathfrak{g}}$ currents are primary fields with respect to $\mathfrak{V}$:

$$T^{\mathfrak{g}}(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} + \mathrm{reg} \ . \tag{VIII.0.6}$$

Now suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra and suppose also, for convenience, that the index of embedding (*i.e.*, the ratio of the lengths of the longest roots) is one. Then by restriction, ${}^{\mathfrak{g}}\mathfrak{M}_\lambda$ becomes an $\widehat{\mathfrak{h}}$-module at the same level. The Sugawara construction applied to the $\widehat{\mathfrak{h}}$ currents $\{J^i(z)\}_{i=1}^{d_{\mathfrak{h}}}$ allows us to give another Virasoro module structure to ${}^{\mathfrak{g}}\mathfrak{M}_\lambda$ with generators given by the modes of $T^{\mathfrak{h}}(z)$

$$\widetilde{T}^{\mathfrak{h}}(z) = \frac{\gamma_{ij}}{2k + c_{\mathfrak{h}}} {:} \widetilde{J}^i(z)\widetilde{J}^j(z) {:} \tag{VIII.0.7}$$

which obey the Virasoro algebra with central charge given by $c(\mathfrak{h},k)$. Again one verifies that

$$T^{\mathfrak{h}}(z)J^i(w) = \frac{J^i(w)}{(z-w)^2} + \frac{\partial J^i(w)}{(z-w)} + \mathrm{reg} \ , \tag{VIII.0.8}$$

whence $T^{\mathrm{GKO}}(z) \equiv T^{\mathfrak{g}}(z) - T^{\mathfrak{h}}(z)$ has regular OPE with the $\widehat{\mathfrak{h}}$ currents and, therefore, with $T^{\mathfrak{h}}(z)$. $T^{\mathrm{GKO}}(z)$ obeys a Virasoro algebra with central charge $c^{\mathrm{GKO}} = c(\mathfrak{g},k) - c(\mathfrak{h},k)$. In particular, $c^{\mathrm{GKO}} \geq 0$.

Since $T^{\rm GKO}$ commutes with the $\widehat{\mathfrak{h}}$ currents we can break ${}^{\mathfrak{g}}\mathfrak{M}_\lambda$ into irreducible $(\widehat{\mathfrak{h}} \times \mathfrak{V})$-modules as follows

$$ {}^{\mathfrak{g}}\mathfrak{M}_\lambda = \bigoplus_{\mu,h} {}^{\mathfrak{h}}\mathfrak{M}_\mu \otimes {}^{\mathfrak{V}}\mathfrak{M}^{\mu}_{\lambda,h} \ , \tag{VIII.0.9} $$

where ${}^{\mathfrak{h}}\mathfrak{M}_\mu$ are irreducible $\widehat{\mathfrak{h}}$-modules and ${}^{\mathfrak{V}}\mathfrak{M}^{\mu}_{\lambda,h}$ are irreducible $\mathfrak{V}$-modules. The holomorphic sector of the coset CFT has as Hilbert space

$$ \mathfrak{H}_\lambda = \bigoplus_{\mu,h} {}^{\mathfrak{V}}\mathfrak{M}^{\mu}_{\lambda,h} \ , \tag{VIII.0.10} $$

which is a finite direct sum if and only if[**115**] $c^{\rm GKO} < 1$. Of course, by extending $\mathfrak{V}$ to a larger chiral algebra $\mathfrak{A}$ (essentially by tacking on primary fields which commute with the $\widehat{\mathfrak{h}}$ currents) we may be able to obtain a finite decomposition (*i.e.*, a rational CFT)

$$ \mathfrak{H}_\lambda = \bigoplus_{\mu,a} {}^{\mathfrak{A}}\mathfrak{M}^{\mu}_{\lambda,a} \ . \tag{VIII.0.11} $$

Notice that by construction these representations are unitary.

We can then do the same for the antiholomorphic sector. To make a CFT we must then glue the holomorphic and antiholomorphic sectors in a modular invariant fashion. However, if we can show equivalence of the holomorphic sectors of two theories then we are essentially done since whichever way we glue the two sectors in one theory can be done in the other once an isomorphism has been set up at the level of the holomorphic sectors. A necessary condition for the holomorphic sectors to agree is for their Virasoro characters to agree, since otherwise the two theories would not have the same (holomorphic) primary fields. However this may not be sufficient since there is no information about the operator algebra. We should probably demand that the maximal chiral algebras agree and that their representations also agree. However as a first step (and since the maximal chiral algebra is not trivial to find in most cases) we will concentrate on the Virasoro characters.

In Section 1 we will describe the CFT which arises from the gauged WZNW model after some straight-forward manipulations in the usual Faddeev-Popov prescription, and making use of the Polyakov-Wiegmann identities. Our starting point will be the CFT itself and not its derivation from path integrals. For a lucid explanation of this derivation, we refer the reader to [**116**] and [**113**]. We see that the CFT is obtained from three uncoupled CFTs after imposing a first class constraint which couples them. Since the constraint is first class we can treat it à la BRST, and in this vein, we define the Hilbert space of the gauged WZNW CFT as the BRST cohomology at zero ghost number. We prove that this cohomology inherits a Virasoro module structure where the Virasoro generators are induced from the GKO generators on the cochains.

In Section 2 we identify the BRST complex of the gauged WZNW model with a particular relative subcomplex of semi-infinite forms, for whose cohomology we can prove half of a vanishing theorem. In the case of $\mathfrak{h}$ abelian, we can use duality to prove the other half. I am still investigating the possibility of extending this to a full vanishing theorem in general.

In Section 3 we prove the no ghost theorem for the BRST cohomology in the abelian case and we compute the chiral partition function (the Virasoro character) of the theory leaving it ready to compare with the Virasoro characters of the parafermionic theories. I am still in the comparison process.

There is little to conclude so far from the work in this chapter; although there are clear indications that there is light at the end of the tunnel.

1. The CFT of the Gauged WZNW Model

In this section we review the conformal field theory to which the gauged WZNW model gives rise. As described in **[117]**, the CFT of the gauged WZNW model consists of the following ingredients. First we have a WZNW CFT with group G and level k. This is described by the current algebra corresponding to the affine Lie algebra $\widehat{\mathfrak{g}}$ at level k; where $\mathfrak{g}$ is the Lie algebra of G. That is we have a set of currents[26] $\{J^a(z)\}_{a=1}^{d_\mathfrak{g}}$ whose modes are given by (VIII.0.1) and whose OPE is given by (VIII.0.2). The energy momentum tensor has the standard Sugawara form given by (VIII.0.3) and obeys the OPE given by (VIII.0.4).

The next ingredient is a WZNW CFT with group $H \subset G$ and level $-(k+c_\mathfrak{h})$, where, for simplicity, we assume that the index of embedding $\mathfrak{h} \subset \mathfrak{g}$ is one. This is defined by a set of currents

$$\widetilde{J}^i(z) = \sum_{n\in\mathbb{Z}} \widetilde{J}^i_n\, z^{-n-1} \tag{VIII.1.1}$$

obeying the OPE

$$\widetilde{J}^i(z)\widetilde{J}^j(w) = \frac{(k+c_\mathfrak{h})\gamma^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k}{z-w}\widetilde{J}^k(w) + \text{reg}\ , \tag{VIII.1.2}$$

where γ^{ij} is the restriction of γ^{ab} to $\mathfrak{h}$. The energy momentum tensor again has the standard

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[26] We only consider the holomorphic sector of the CFT. The treatment of the antiholomorphic sector is completely analogous.

Sugawara form

$$\widetilde{T}^{\mathfrak{h}}(z) = \frac{-\gamma_{ij}}{2k + c_{\mathfrak{h}}} :\widetilde{J}^i(z)\widetilde{J}^j(z): \qquad \text{(VIII.1.3)}$$

and obeys the usual OPE

$$\widetilde{T}^{\mathfrak{h}}(z)\widetilde{T}^{\mathfrak{h}}(w) = \frac{c(\mathfrak{h}, -k - c_{\mathfrak{h}})}{2(z-w)^4} + \frac{2}{(z-w)^2}\widetilde{T}^{\mathfrak{h}}(w) + \frac{1}{z-w}\partial\widetilde{T}^{\mathfrak{h}}(w) + \text{reg} , \qquad \text{(VIII.1.4)}$$

where γ_{ij} is the inverse of γ^{ij}.

The third ingredient is a set of $d_{\mathfrak{h}}$ free (b, c) systems of spins (1,0) respectively with mode expansions:

$$b^i(z) = \sum_{n\in\mathbb{Z}} b^i_n\, z^{-n-1} \qquad \text{(VIII.1.5)}$$

$$c_i(z) = \sum_{n\in\mathbb{Z}} c_{i,n}\, z^{-n} \qquad \text{(VIII.1.6)}$$

and with OPE given by

$$b^i(z)c_j(w) = \frac{\delta^i_j}{z-w} + \text{reg} = c_j(z)b^i(w) \ . \qquad \text{(VIII.1.7)}$$

Their energy momentum tensor has the standard form

$$T^{\text{gh}}(z) = -:b^i\partial c_i: \qquad \text{(VIII.1.8)}$$

and obeys the standard OPE

$$T^{\text{gh}}(z)T^{\text{gh}}(w) = \frac{-d_{\mathfrak{h}}}{(z-w)^4} + \frac{2}{(z-w)^2}T^{\text{gh}}(w) + \frac{1}{z-w}\partial T^{\text{gh}}(w) + \text{reg} \ . \qquad \text{(VIII.1.9)}$$

The final ingredient of the theory is the constraint which couples these three independent CFTs:

$$J_{\text{tot}}{}^i(z) = J^i(z) + \widetilde{J}^i(z) + J^i_{\text{gh}}(z) \ , \qquad \text{(VIII.1.10)}$$

with

$$J^i_{\text{gh}}(z) \equiv f^{ij}{}_k :b^k(z)c_j(z): \ . \qquad \text{(VIII.1.11)}$$

The OPEs obeyed by these currents can be easily read from the ones given above and are given by (VIII.1.2) together with

$$J^i(z)J^j(w) = \frac{k\gamma^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k}{z-w} J^k(w) + \text{reg} \ , \tag{VIII.1.12}$$

$$J^i_{\text{gh}}(z)J^j_{\text{gh}}(w) = \frac{c_{\mathfrak{h}}\gamma^{ij}}{(z-w)^2} + \frac{f^{ij}{}_k}{z-w} J^k_{\text{gh}}(w) + \text{reg} \ . \tag{VIII.1.13}$$

Adding the central charges we see that in fact they cancel so that the constraint is first class. This just reiterates the fact that we gauged an anomaly free subgroup. Because of this fact we can build a BRST charge and we are guaranteed that it is square-zero. We define the BRST operator as the contour integral of the BRST current. That is,

$$Q = \oint_{C_0} \frac{dz}{2\pi i} \, j_{\text{BRST}}(z) \ , \tag{VIII.1.14}$$

where

$$j_{\text{BRST}}(z) = : c_i(z)[J^i(z) + \widetilde{J}^i(z) + \frac{1}{2} J^i_{\text{gh}}(z)] : \ . \tag{VIII.1.15}$$

Therefore to quantize the holomorphic part of this CFT we merely look for suitable representations of the relevant operator algebras. In this case this involves representations of two affine Lie algebras: $\widehat{\mathfrak{g}}$ at level k and $\widehat{\mathfrak{h}}$ at level $-(k + c_{\mathfrak{h}})$; whereas the ghosts—being free fields—are quantized trivially. The physical subspace is then defined as the (relative) cohomology of the BRST operator at zero ghost number.

We will leave the detailed analysis of the representations until the next section and, hence we conclude this section by studying the Virasoro algebras appearing in this construction. The total energy momentum tensor $T(z)$ is given by the sum of three commuting terms:

$$T(z) = T^{\mathfrak{g}}(z) + \widetilde{T}^{\mathfrak{h}}(z) + T^{\text{gh}}(z) \ . \tag{VIII.1.16}$$

Adding up the central charges we notice that the total central charge is

$$\begin{aligned} c &= \frac{2kd_{\mathfrak{g}}}{2k + c_{\mathfrak{g}}} + \frac{2(-k - c_{\mathfrak{h}})d_{\mathfrak{h}}}{2(-k - c_{\mathfrak{h}}) + c_{\mathfrak{h}}} - 2d_{\mathfrak{h}} \\ &= \frac{2kd_{\mathfrak{g}}}{2k + c_{\mathfrak{g}}} - \frac{2kd_{\mathfrak{h}}}{2k + c_{\mathfrak{h}}} \ , \end{aligned} \tag{VIII.1.17}$$

which coincides with the central charge of the G/H coset CFT[112]. However the energy momentum tensor of the coset CFT is not the same as $T(z)$. In fact, it is given by

$$T^{\rm GKO}(z) = T^{\mathfrak{g}}(z) - T^{\mathfrak{h}}(z) \ , \tag{VIII.1.18}$$

where

$$T^{\mathfrak{h}}(z) = \frac{\gamma_{ij}}{2k + c_{\mathfrak{h}}} \colon J^i(z) J^j(z) \colon \ . \tag{VIII.1.19}$$

We can therefore split $T(z)$ as a sum of two commuting terms $T^{\rm GKO}(z) + T'(z)$ where

$$T'(z) = T^{\mathfrak{h}}(z) + \widetilde{T}^{\mathfrak{h}}(z) + T^{\rm gh}(z) \ . \tag{VIII.1.20}$$

Notice that $T'(z)$ has zero central charge.

Now, the BRST charge can be checked to commute with both $T^{\rm GKO}(z)$ and $T'(z)$. Hence they are physical operators; that is, they induce operators in the physical space. One can show that $T'(z)$ induces the zero operator on physical states. This is done by showing that there exists an operator $\Theta(z)$ such that

$$\left\{Q \, , \, \Theta(z)\right\} = T'(z) \ . \tag{VIII.1.21}$$

The operator in question is simply given by

$$\Theta(z) = \frac{\gamma_{ij}}{2k + c_{\mathfrak{h}}} b^i(z) (J^j(z) - \widetilde{J}^j(z)) \ . \tag{VIII.1.22}$$

To check (VIII.1.21) we can make use of the following identities:

$$\left\{Q \, , \, b^i(z)\right\} = J_{\rm tot}{}^i(z) \ ; \tag{VIII.1.23}$$

$$\left[Q \, , \, J^i(z)\right] = k\gamma^{ij}\partial c_j(z) - f^{ij}{}_k c_j(z) J^k(z) \ ; \tag{VIII.1.24}$$

$$\left[Q \, , \, \widetilde{J}^i(z)\right] = -(k + c_{\mathfrak{h}})\gamma^{ij}\partial c_j(z) - f^{ij}{}_k c_j(z) \widetilde{J}^k(z) \ . \tag{VIII.1.25}$$

That $T'(z)$ induces the zero operator would also follow from the positive definiteness of the physical scalar product. It is a fact—proved, for instance, in **[118]**— that any highest weight unitary representation of the Virasoro algebra with zero central charge is necessarily trivial. Therefore since $T'(z)$ has zero central charge and since the representations we will consider are highest weight, all we need to show is that the scalar product of the physical states is positive-definite. This is precisely the approach followed by Karabali and Schnitzer in **[116]**, where they prove it for the special case of $\mathfrak{h}$ abelian using the quartet mechanism of Kugo and Ojima[**25**]. Of course, in the abelian case the ghost theory decouples since the constraint does not involve the ghosts, *i.e.*, $J^i_{\rm gh}(z) = 0$; and the decoupling strategy of the Kugo-Ojima mechanism may just reflect this fact.

In the next sections we will rederive this result using the homological methods of Chapter VI. We feel that these methods are better suited for proving the general no-ghost theorem; although, as we will see, we run into technical difficulties which we only know how to resolve in the abelian case. It is interesting to note how two seemingly different approaches stumble on precisely the same hurdle.

2. BRST Quantization of the Gauged WZNW CFT

In this section we establish the Lie algebraic objects appearing in the quantization of the theory described in the previous section. We will first quantize the three independent CFTs separately and subsequently impose the constraints à la BRST.

We first take a look at the ghost Fock space. From equations (VIII.1.5), (VIII.1.6), and (VIII.1.7) we can read off the canonical anticommutation relations of the modes:

$$\left\{c_{i,m}\,,\,c_{j,n}\right\} = \left\{b^i_m\,,\,b^j_n\right\} = 0 \qquad \left\{b^i_m\,,\,c_{j,n}\right\} = \delta^i_j\delta_{m+n,0}\ . \tag{VIII.2.1}$$

The energy momentum tensor has the following mode expansion

$$T^{\mathrm{gh}}(z) = \sum_{n\in\mathbb{Z}} L^{\mathrm{gh}}_n\, z^{-n-2}\ , \tag{VIII.2.2}$$

where, from (VIII.1.8),

$$L^{\mathrm{gh}}_n = \sum_{m\in\mathbb{Z}} (n-m)\!:\!b^i_m c_{i,n-m}\!:\ . \tag{VIII.2.3}$$

The zero modes $\{b^i_0, c_{i,0}\}$ obey a Clifford algebra of signature $(d_\mathfrak{h}, d_\mathfrak{h})$. Therefore it has a unique irreducible representation of dimension $2^{d_\mathfrak{h}}$, which makes the vacuum degenerate. The degeneracy is eliminated by demanding that the true vacuum be $\mathfrak{sl}(2,\mathbb{C})$ invariant. The unique $\mathfrak{sl}(2,\mathbb{C})$ invariant vacuum is given (up to a phase) by the unit norm state $|0\rangle_{\mathrm{gh}}$ satisfying

$$c_{i,n}\,|0\rangle_{\mathrm{gh}} = 0 \quad \forall i,\ \forall n \geq 1 \tag{VIII.2.4}$$

$$b^i_n\,|0\rangle_{\mathrm{gh}} = 0 \quad \forall i,\ \forall n \geq 0\ . \tag{VIII.2.5}$$

This vacuum corresponds to the semi-infinite form

$$\epsilon_{-1} \wedge \epsilon_{-2} \wedge \cdots\ , \tag{VIII.2.6}$$

where ϵ_{-n} is short for the volume form

$$j^{1\prime}_{-n} \wedge j^{2\prime}_{-n} \wedge \cdots \wedge j^{d_\mathfrak{h}\prime}_{-n} \tag{VIII.2.7}$$

where $\{j^{i\prime}_n\}$ is a basis for $\widehat{\mathfrak{h}}'$ which is canonically dual to the basis $\{j^i_n\}$ for $\widehat{\mathfrak{h}}$.

The ghost Fock space $\bigwedge_\infty$ is then generated by repeated application of the creation operators $\{c_{i,-n}, b^i_{-n} \mid n > 0\}$ on the ghost vacuum $|0\rangle_{\text{gh}}$. The ghost Fock space $\bigwedge_\infty$ is graded by ghost number which is defined as the eigenvalues of the ghost number operator

$$\frac{1}{2}(c_{i,0}b^i_0 - b^i_0 c_{i,0}) + \sum_{n\in\mathbb{Z}}(c_{i,-n}b^i_n - b^i_{-n}c_{i,n}) \ . \tag{VIII.2.8}$$

Notice that the $\mathfrak{sl}(2,\mathbb{C})$ invariant vacuum has ghost number $-\frac{d_\mathfrak{h}}{2}$. Therefore it will be convenient in what follows to use the **relative ghost number** operator—denoted N_{gh}—which is the ghost zero mode independent piece of the ghost number operator. This makes the relative ghost number of $|0\rangle_{\text{gh}}$ zero.

Next we consider the representations of $\widehat{\mathfrak{g}}$ at level k. The OPE of the G currents given by (VIII.0.2) translates into the following mode algebra

$$\left[J^a_m \, , \, J^b_n\right] = f^{ab}{}_c J^c_{m+n} + km\gamma^{ab}\delta_{m+n,0} \ . \tag{VIII.2.9}$$

Highest weight representations of $\widehat{\mathfrak{g}}$ are obtained as follows. We first choose an irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight λ: $\mathbb{V}(\lambda)$. We then act on $\mathbb{V}(\lambda)$ with the negative modes $\{J^a_{-n} \mid n > 0\}$ to generate the representation $\mathfrak{M}_\lambda$. We may grade $\widehat{\mathfrak{g}}$ by the adjoint action of L_0 as follows

$$\widehat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{g}}_n \ ; \tag{VIII.2.10}$$

and let us define

$$\widehat{\mathfrak{g}}_\pm = \bigoplus_{\pm n>0} \widehat{\mathfrak{g}}_n \ . \tag{VIII.2.11}$$

Then we see that $\mathfrak{M}_\lambda$ is precisely the free $\widehat{\mathfrak{g}}_-$-module generated by $\mathbb{V}(\lambda)$.

We can define a hermitian conjugation in $\mathfrak{M}_\lambda$ as follows: $(J^a_n)^\dagger = -J^a_{-n}$. With this conjugation, a necessary and sufficient condition for $\mathfrak{M}_\lambda$ to be irreducible and unitary is[**114**],[**119**]

$$k \geq \frac{2(\lambda,\theta)}{(\theta,\theta)} \geq 0 \ , \tag{VIII.2.12}$$

where θ is the highest root of $\mathfrak{g}$ and where $(,)$ stands for the induced form in the root space. For a fixed k there are just a finite number of dominant weights λ satisfying the above equation. Notice that $\mathfrak{M}_\lambda$ is also graded by finite dimensional eigenspaces of L_0:

$$\mathfrak{M}_\lambda = \bigoplus_{n=0}^{\infty} \mathfrak{M}_\lambda^n \ , \tag{VIII.2.13}$$

where the eigenvalue of L_0 on $\mathfrak{M}_\lambda^n$ is $\frac{c_\lambda}{2k+c_\mathfrak{g}} + n$, where c_λ is the value of the quadratic Casimir in the representation $\mathbb{V}(\lambda)$. In particular we see that L_0 is bounded below. These remarks simply imply that $\mathfrak{M}_\lambda$ is a $\widehat{\mathfrak{g}}$–module in the category $\mathcal{O}_o$.

Finally we discuss the representations of $\widehat{\mathfrak{h}}$ at level $-(k+c_\mathfrak{h})$. From (VIII.1.2) we obtain the mode algebra

$$\left[\widetilde{J}^i_m \ , \ \widetilde{J}^j_n\right] = f^{ij}{}_k \widetilde{J}^k_{m+n} - (k+c_\mathfrak{h}) m \gamma^{ij} \delta_{m+n,0} \ . \tag{VIII.2.14}$$

To build a highest weight representation we again choose a finite dimensional irreducible representation $\widetilde{\mathbb{V}}(\mu)$ of $\mathfrak{h}$ with heighest weight μ. We then define $\widetilde{\mathfrak{M}}_\mu$ to be the free $\widehat{\mathfrak{h}}_-$–module generated by $\widetilde{\mathbb{V}}(\mu)$. Since the level is negative we will always have negative norm states in $\widetilde{\mathfrak{M}}_\mu$. Therefore the choice of μ is no longer restricted by unitarity.

Karabali & Schnitzer[**116**] impose an *ad hoc* restriction on μ. Let

$$|\psi\rangle \equiv |\phi\rangle \otimes |\widetilde{\phi}\rangle \otimes |0\rangle_{\mathrm{gh}} \in \mathbb{V}(\lambda) \otimes \widetilde{\mathbb{V}}(\mu) \otimes \bigwedge\nolimits_\infty$$

be a possible vacuum state. Requiring that $|\psi\rangle$ be physical (*i.e.*, annihilated by the BRST operator) and since b^i_0 annihilates the ghost vacuum we see that $|\psi\rangle$ must be annihilated by $J^i_{\mathrm{tot},0}$. Since $J^i_{\mathrm{gh},0}\,|0\rangle_{\mathrm{gh}} = 0$ this is equivalent to

$$(J^i_0 + \widetilde{J}^i_0)(|\phi\rangle \otimes |\widetilde{\phi}\rangle) = 0 \ . \tag{VIII.2.15}$$

Let us split $\mathbb{V}(\lambda)$ into irreducible representations of $\mathfrak{h}$:[27]

$$\mathbb{V}(\lambda) = \bigoplus_\alpha \widetilde{\mathbb{V}}(\alpha) \ . \tag{VIII.2.16}$$

Then (VIII.2.15) is possible if and only if for some α appearing in the above decomposition we have

$$\widetilde{\mathbb{V}}(0) \subset \widetilde{\mathbb{V}}(\alpha) \otimes \widetilde{\mathbb{V}}(\mu) \ . \tag{VIII.2.17}$$

This is equivalent to the existence of an $\mathfrak{h}$-invariant map $\varphi : \widetilde{\mathbb{V}}(\alpha) \to \widetilde{\mathbb{V}}(\mu)^*$ which, by Schur's lemma, must be an isomorphism. Hence if and only if $\widetilde{\mathbb{V}}(\mu)^*$ is isomorphic to $\widetilde{\mathbb{V}}(\alpha)$ for some

[27] That this can always be done can be shown as follows. Since $\mathfrak{h} \subset \mathfrak{g}$ any representation of $\mathfrak{g}$ restricts to a representation of $\mathfrak{h}$. Since $\mathbb{V}(\lambda)$ is finite dimensional and $\mathfrak{h}$ is the Lie algebra of a compact group, we may apply Weyl's theorem to fully reduce $\mathbb{V}(\lambda)$ into $\mathfrak{h}$ irreducibles.

α appearing in (VIII.2.16), do they consider $\widetilde{\mathfrak{M}}_\mu$. In particular, their allowed values of μ are in bijective correspondence with the allowed values of α. Since there are just a finite number of allowed λ's (by unitarity), there are a finite number of allowed α's and hence a finite number of allowed μ's. However there is no good reason *a priori* to require that $|\psi\rangle$ be physical. In fact, as we shall show, at least when $\mathfrak{h}$ is abelian there is plenty of cohomology inheriting a positive definite scalar product, without fixing the representations $\widetilde{\mathbb{V}}(\mu)$. Therefore we will not restrict ourselves at the outset to any particular values of μ.

For a fixed λ and μ, the space of quanta becomes

$$\mathfrak{F} \equiv \mathfrak{F}_{\lambda,\mu} = \mathfrak{M}_\lambda \otimes \widetilde{\mathfrak{M}}_\mu \otimes \bigwedge\nolimits_\infty \ . \tag{VIII.2.18}$$

We can decompose $\mathfrak{F}$ as follows. By construction $\widetilde{\mathfrak{M}}_\mu$ is a free $\widehat{\mathfrak{h}}_-$–module and $\mathfrak{M}_\lambda$ is a free $\widehat{\mathfrak{g}}_-$-module. But, by the Poincaré-Birkhoff-Witt theorem**[119]**, $\mathfrak{U}(\widehat{\mathfrak{g}}_-)$ is a free $\widehat{\mathfrak{h}}_-$-module itself. Therefore $\mathfrak{M}_\lambda$ is a free $\widehat{\mathfrak{h}}_-$-module. Let us then decompose $\mathfrak{F}$ as follows

$$\mathfrak{F} = \bigoplus_i \mathfrak{M}_i \otimes \bigwedge\nolimits_\infty \ , \tag{VIII.2.19}$$

where each $\mathfrak{M}_i$ is a free $\widehat{\mathfrak{h}}_-$-module.

We claim that we don't need all of $\mathfrak{F}$ to compute the BRST cohomology but only the $\mathfrak{h}$–invariants $\mathfrak{F}^{\mathfrak{h}}$. The proof runs as follows. First of all the BRST cocycles in $\mathfrak{F}$ are automatically in $\mathfrak{F}^{\mathfrak{h}}$. This is because if $Q\,|\psi\rangle = 0$ and $b_0^i\,|\psi\rangle = 0$ then, taking the anticommutator, $J^i_{\mathrm{tot},0}\,|\psi\rangle = 0$ as well. Hence the only thing that could possibly go wrong in considering $\mathfrak{F}^{\mathfrak{h}}$ from the beginning is that a cocycle may be non-trivial (*i.e.*, not a coboundary) yet if we allowed ourselves to look at all cochains we would find that it is actually trivial. This however cannot be the case as we now see. Using Weyl's theorem let us split $\mathfrak{F}$ as follows[28]

$$\mathfrak{F} = \mathfrak{F}^{\mathfrak{h}} \oplus \mathfrak{h}\mathfrak{F} \ , \tag{VIII.2.20}$$

where $\mathfrak{h}\mathfrak{F}$ is the image of $\mathfrak{F}$ under the $\mathfrak{h}$ action. Now let $|\psi\rangle$ be an (invariant) cocycle and suppose that it is also a coboundary: $|\psi\rangle = Q\,|\chi\rangle$. Then according to the above

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[28] We have been too quick. Weyl's theorem only applies to finite dimensional representations and $\mathfrak{F}$ is infinite dimensional. We must first decompose $\mathfrak{F}$ into L_0 eigenspaces. Each such eigenspace is finite dimensional and is stabilized by $\mathfrak{h}$. Hence we apply Weyl's theorem to each one in turn and then collate the different eigenspaces to obtain the formula given.

decomposition we can write $|\chi\rangle = |\chi_1\rangle + |\chi_2\rangle$ where $|\chi_1\rangle$ is invariant. Since Q commutes with the $\mathfrak{h}$ action we find that $Q\,|\chi_2\rangle = 0$, whence $|\psi\rangle = Q\,|\chi_1\rangle$. In fact, we notice *a fortiori* that $|\chi_2\rangle = 0$ since all cocycles are invariant.

Therefore, the space $\mathfrak{F}^{\mathfrak{h}}$ is nothing but the subcomplex of semi-infinite forms relative to $\mathfrak{h}$[29]

$$C_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \bigoplus_i \mathfrak{M}_i)\ . \tag{VIII.2.21}$$

This complex is graded by the relative ghost number operator introduced above and its cohomology (the relative semi-infinite cohomology)

$$H_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \bigoplus_i \mathfrak{M}_i) \tag{VIII.2.22}$$

is also graded by relative ghost number. Semi-infinite cohomology behaves well under direct sums

$$H_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \mathfrak{M} \oplus \mathfrak{N}) \cong H_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \mathfrak{M}) \oplus H_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \mathfrak{N})\ , \tag{VIII.2.23}$$

and the above decomposition respects the grading. Repeated application of the above equation yields

$$H_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \bigoplus_i \mathfrak{M}_i) \cong \bigoplus_i H_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \mathfrak{M}_i)\ . \tag{VIII.2.24}$$

Hence we only need to work with one such $\mathfrak{M}_i$ at a time, call it $\mathfrak{M}$.

Since $\mathfrak{M}$ is $\widehat{\mathfrak{h}}_-$-free, (VI.1.50) implies that the relative cohomology vanishes for negative relative ghost number. However in order to extend this result to a full vanishing theorem we need to be able to exhibit some sort of duality. This is guaranteed if the Hermitian structure of $\mathfrak{M}$ is non-degenerate. Notice that this Hermitian structure is nothing but the contravariant (Šapovalov) form, whose non-degeneracy implies the irreducibility of the representation. A sufficient and necessary condition for the non-degeneracy of the contravariant form has been given by Kac and Kazhdan in [**120**] (although see also [**119**], [**114**]). However, a case by case inspection of the Kac-Kazhdan formula does not look promising unless we could find a counterexample. In the special case of $\mathfrak{h}$ abelian the scalar product is non-degenerate

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[29] Strictly speaking is relative to the subalgebra consisting of $\mathfrak{h}$ and the center. But as usual we drop mention of the center.

since the modes J_n^i correspond to the modes of $d_{\mathfrak{h}}$ free bosons and thus $\mathfrak{M}$ is a Fock space. Therefore, *if* $\mathfrak{h}$ *is abelian*, (VI.1.36) guarantees the full vanishing theorem:

$$H^n_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \mathfrak{M}) \cong 0 \quad \forall\, n \neq 0 \ . \tag{VIII.2.25}$$

It should be remarked that the vanishing theorem only depends on the properties of the $\widehat{\mathfrak{h}}$-module $\widetilde{\mathfrak{M}}_\mu$, since the $\widehat{\mathfrak{h}}$-modules appearing in $\mathfrak{M}_\lambda$ are hermitian and $\widehat{\mathfrak{h}}_-$ free. We can make this observation precise by generalizing an argument of Zuckerman[**121**] for the case of the Virasoro algebra. Let π_λ and $\widetilde{\pi}_\mu$ denote the representations of $\widehat{\mathfrak{h}}$ afforded by $\mathfrak{M}_\lambda$ and $\widetilde{\mathfrak{M}}_\mu$, respectively. According to the filtration defined by (VI.1.43) the filtration degrees of π_λ and $\widetilde{\pi}_\mu$ are the same. The observation of Zuckerman is that we can essentially filter π_λ away at first by redefining the filtration degree of π_λ to be $\operatorname{fdeg} \pi_\lambda(\widehat{\mathfrak{h}}_n) = |n|$. Therefore the E_1 term of the spectral sequence associated to this new filtration is just

$$E_1^{b,c} = \Big(H^c(\widehat{\mathfrak{h}}+) \otimes H^b_\infty(\widehat{\mathfrak{h}}_-; \widetilde{\mathfrak{M}}_\mu) \Big)^{\mathfrak{h}} \ . \tag{VIII.2.26}$$

Therefore, if $\widetilde{\mathfrak{M}}_\mu$ is $\widehat{\mathfrak{h}}_-$ free, $E_1^m = 0$ for $m < 0$ and this vanishing propagates to the limit term. If, furthermore, $\widetilde{\mathfrak{M}}_\mu$ is a hermitian module, we can use (VI.1.36) to prove the full vanishing theorem. We can therefore, summarize these results in the following theorem:

Theorem VIII.2.27. *If* $\widetilde{\mathfrak{M}}_\mu$ *is a free hermitian* $\widehat{\mathfrak{h}}_-$*-module—e.g., if* $\mathfrak{h}$ *is abelian—*

$$\boxed{H^n_\infty(\widehat{\mathfrak{h}}, \mathfrak{h}; \mathfrak{M}_\lambda \otimes \widetilde{\mathfrak{M}}_\mu) \cong 0 \ \forall\, n \neq 0} \ . \tag{VIII.2.28}$$

3. The No-Ghost Theorem for $\mathfrak{h}$ Abelian

In this section we shall prove the no-ghost theorem for the special case of $\mathfrak{h}$ abelian using the vanishing theorem for the relative BRST cohomology proven in the last section. As a bonus we will obtain the chiral partition function of the theory.

As usual, the only hard part is to define the conjugation $\mathcal{C}$. In the case of $\mathfrak{h}$ abelian, however, the modes of the currents agree with the modes of $d_{\mathfrak{h}}$ free bosons and therefore $\mathfrak{M}$ is a Fock module. In this case it is easy to define $\mathcal{C}$: it leaves the Fock vacuum invariant and up to a canonical transformation it acts as ghost conjugation on the ghost part and it anticommutes with the $\widehat{\mathfrak{h}}_-$ currents.

The character and the signature are now easily computed. First of all notice that we don't have to worry about the positive level Kac-Moody representations since these are unitary and the trace is multiplicative over tensor products. So we just have to worry about the ghosts and the negative level Kac-Moody representation. Let's first compute the traces for the ghost system. These calculations are identical to the ones performed for the super Virasoro algebras: only the number of species of ghosts is different ($d_{\mathfrak{h}}$) and also the lowest eigenvalue of $L_0^{\rm gh}$ is different since these ghosts have different conformal weight. Hence we simply get

$$\mathrm{sgn}_q \bigwedge\nolimits_\infty = \prod_{n=1}^{\infty} \left[(1-q^n)(1+q^n)\right]^{d_{\mathfrak{h}}} \ ; \tag{VIII.3.1}$$

$$\mathrm{ch}_q \bigwedge\nolimits_\infty = \prod_{n=1}^{\infty} (1-q^n)^{2d_{\mathfrak{h}}} \ . \tag{VIII.3.2}$$

For the character and signature for $\widetilde{\mathfrak{M}}_\mu$ we can also make use of the results of the super Virasoro calculations. We notice that these are $d_{\mathfrak{h}}$ time-like bosons (because the level is negative) and therefore we obtain

$$\mathrm{sgn}_q \widetilde{\mathfrak{M}}_\mu = q^{d_\mu} \prod_{n=1}^{\infty} (1+q^n)^{-d_{\mathfrak{h}}} \ ; \tag{VIII.3.3}$$

$$\mathrm{ch}_q \widetilde{\mathfrak{M}}_\mu = q^{d_\mu} \prod_{n=1}^{\infty} (1-q^n)^{d_{\mathfrak{h}}} \ ; \tag{VIII.3.4}$$

where $d_\mu = \frac{-c_\mu}{2k}$ is the eigenvalue of L_0 on $\widetilde{\mathbb{V}}(\mu) \subset \widetilde{\mathfrak{M}}_\mu$. Notice that since $\mathfrak{h}$ is abelian $\widetilde{\mathbb{V}}(\mu)$ is one dimensional. Multiplying these power series we find for the characters

$$\begin{aligned} \mathrm{ch}_q \widetilde{\mathfrak{M}}_\mu \otimes \bigwedge\nolimits_\infty &= \mathrm{ch}_q \widetilde{\mathfrak{M}}_\mu \cdot \mathrm{ch}_q \bigwedge\nolimits_\infty \\ &= q^{d_\mu} \prod_{n=1}^{\infty} (1-q^n)^{d_{\mathfrak{h}}} \ , \end{aligned} \tag{VIII.3.5}$$

and, similarly, for the signatures

$$\begin{aligned} \mathrm{sgn}_q \widetilde{\mathfrak{M}}_\mu \otimes \bigwedge\nolimits_\infty &= \mathrm{sgn}_q \widetilde{\mathfrak{M}}_\mu \cdot \mathrm{sgn}_q \bigwedge\nolimits_\infty \\ &= q^{d_\mu} \prod_{n=1}^{\infty} (1-q^n)^{d_{\mathfrak{h}}} \ ; \end{aligned} \tag{VIII.3.6}$$

which proves the no-ghost theorem.

We can use these calculations to compute the physical partition function (*i.e.*, the formal q-character) of the gauged WZNW model whose space of quanta is given by $\mathfrak{F}_{\lambda,\mu}$ in (VIII.2.18). We first compute the character of the representation of the positive level Kac-Moody algebra $\mathfrak{M}_\lambda$. This gives

$$\mathrm{ch}_q\,\mathfrak{M}_\lambda = \dim \mathbb{V}(\lambda)\cdot q^{d_\lambda}\prod_{n=1}^{\infty}(1-q^n)^{-d_{\mathfrak{g}}}\ , \tag{VIII.3.7}$$

where $d_\lambda = \frac{c_\lambda}{2k+c_{\mathfrak{g}}}$ is the eigenvalue of L_0 on $\mathbb{V}(\lambda)\subset\mathfrak{M}_\lambda$. Finally using the multiplicative nature of the character we get

$$\boxed{\mathrm{ch}_q\,\mathfrak{F}_{\lambda,\mu} = \dim \mathbb{V}(\lambda)\cdot q^{d_\mu+d_\lambda}\prod_{n=1}^{\infty}(1-q^n)^{-(d_{\mathfrak{g}}-d_{\mathfrak{h}})}}\ , \tag{VIII.3.8}$$

which manifestly shows that we have $\dim\mathfrak{g}/\mathfrak{h}$ degrees of freedom. This observation can be further substantiated by the results in **[116]** where, via the quartet mechanism, the BRST cohomology is explicitly computed and found to be generated by $\mathfrak{g}/\mathfrak{h}$ degrees of freedom.

Finally we must remark that if $\widetilde{\mathfrak{M}}_\mu$ is a free hermitian $\widehat{\mathfrak{h}}_-$ module then we can also compute the chiral partition function of the theory, although we cannot prove the no-ghost theorem unless we can figure out how to compute the signature. The calculation of the signature is work in progress. For the chiral partition function we have

$$\boxed{\mathrm{ch}_q\,\mathfrak{F}_{\lambda,\mu} = \dim \mathbb{V}(\lambda)\cdot\dim\widetilde{\mathbb{V}}(\mu)\cdot q^{d_\mu+d_\lambda}\prod_{n=1}^{\infty}(1-q^n)^{-(d_{\mathfrak{g}}-d_{\mathfrak{h}})}}\ . \tag{VIII.3.9}$$

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